PASSIVE TOMOGRAPHY OF THE TWO-DIMENSIONAL STRUCTURE OF INHOMOGENEOUS MEDIA

V.P. Yakubov and D.V. Losev

Tomsk State University

Received October 11, 1996

Based on the analysis of many foreshortening projections of incoherently emitting inhomogeneities, the tomography problem of their structure is reduced to an integral equation of Abel type. New solution based on a convenient variable substitution, transforming the problem to convolution equation, and on the use of the Fourier transform. Efficiency and stability of the solution have been confirmed by the results of simulation.

1. INTRODUCTION

Recently, there appeared a considerable interest in tomography for remote diagnostics of various media. Numerical tomographic methods based on the allowance for the attenuation effects in a material are widely employed.[1] The density of a material is usually the main parameter, sought and the attenuation of radiation at multi-aspect radioscopy, e.g., X-raying, is the measured quantity. The attenuation is supposed to vary in proportion to integral density of a material at the wave paths. Considerable success of the theory and tomographic technology has become possible due to efficient inversion methods of the wave projections by use of transforms of Radon and Abel-type and the so-called convolution methods.[1–6] The majority of tomographic methods, use an outside of radiation source with given parameters to expose the media. This corresponds to active tomography. Natural emission of a material is used in passive tomography. It arises, for instance, under heating or other excitation. The action of external radiation often leads to irreversible changes in the material studied, so passive methods attract more interest. In particular, they are ecologically more safe.

One of the well-known examples of passive tomography is the thermography aimed at determining temperature inside a medium. It is used for solving some technical and biomedical problems.[7] In particular, information about the temperature change of internal organs helps early diagnostics of various deseases, the control of internal temperature is also necessary in hyperthermia, oncology, etc. Radioemission and IR radiation are widely used in thermomapping. Thermomapping can be applied to remote sensing of fires. Methods of passive tomography will also be useful for making radiometric analysis of microwave radiation accompanying the appearance of radioactive elements in the atmosphere under radioactive contamination. The problems of incoherent scattering of radiowaves in the ionosphere and troposphere can be reduced to application of passive tomography methods. The number of possible applications of passive tomography can significantly be increased.

In spite of the importance of this kind of tomography, mathematical apparatus for it is not so well developed as for active tomography. It is often impossible to write the exact solution, and the problem is to be reduced to an ill-posed system of linear algebraic equations.[1,7,9] The arising difficulties are connected first of all with the fact that, in contrast to active tomography, the spatial intensity distribution of the radiation is unknown and it is to be sought. Besides, in passive methods, one has to use incoherent low-intensity radiation irregularly varied in time what excludes the possibility of using *a priori* information about radiation and makes it difficult to reliably reconstruct the structure of the object sounded.

However, it should be noted that the distinguishing between passive and active tomography is rather schematic because the case of passive tomography can be considered as the case with radiation from active sources distributed with unknown density inside the object studied. These problems can be generalized in some form. A generalized procedure for obtaining the basic equation for different tomographic problems is presented in Ref. 9. The similarity of the integral relations causes possible similarity of the methods used for their solution. Modern tendencies in the development of passive methods is to generalize well-developed methods of active tomography.

A step in this direction is made in this paper by an example of reconstructing structure of an object without axial symmetry. The initial equation obtained is shown to be similar to Abel equation widely used in active tomography. We analyze a possibility of generalizing the well-known solutions. We also propose an alternative method for solving the inverse sounding problem based on reduction of an integral equation to an equation of the convolution type. The equation obtained makes it possible to use fast numerical algorithms and well-developed regularization methods.

0235-6880/97/02 110-04 $02.00

2. THE INTEGRAL EQUATION OF PASSIVE TOMOGRAPHY

Let us consider a two-dimensional problem of passive tomography for an object with incoherent distribution of the intensity of natural radiation of a weakly absorbing medium.

It is necessary to reconstruct the distribution from the results of scanning by measuring radiation intensity as a function of azimuthal angle ψ (Fig. 1). The weak absorption approximation means that the attenuation of radiation occurs mainly because of spherical divergence. The position of the reception point with respect to the volume studied is set by angle θ determining the direction of observations.

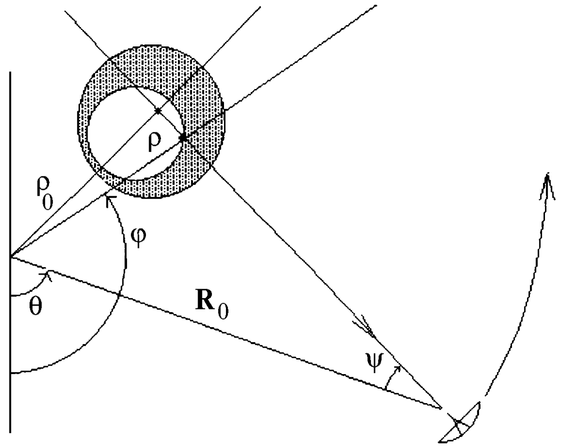

FIG. 1.

The value of full power P of radiation coming to the observation point from a volume with spatial intensity distribution $I(\rho, \varphi, z)$ is defined by the following expression in cylindrical coordinates

$$P = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{2\pi} \frac{I(\rho, \varphi, z)\ \rho\ d\rho\ d\varphi\ dz}{|\mathbf{R}_0 - \rho|^2},$$

where R_0 is the distance from a center of the volume to the reception point; $\rho_0 = R_0 \sin\psi$ is the sighting distance. Considering the radiation intensity constant with respect to the z coordinate and taking the integral over ψ instead of the integral over φ, we obtain the following expression for the power received from the direction ψ:

$$P(\rho_0, \theta) = \int_{\rho_0}^{\infty} \frac{\rho\ d\rho}{\sqrt{\rho^2 - \rho_0^2}} \left[I(\rho, \varphi_1) + I(\rho, \varphi_2)\right], \qquad (1)$$

where φ_1 and φ_2 are two different solutions of the equation $R_0 \sin\psi = \rho \sin(\varphi - \theta + \psi)$. They correspond to the angle between the current radius of the point in the medium and the direction from which the angle θ is counted off (see Fig. 1): $\varphi_1 < \pi/2 - \psi$, $\varphi_2 > \pi/2 - \psi$. The expression (1) is the Volterra integral equation of the first kind for the density distribution function $I(\rho, \varphi)$.

To reconstruct the two-dimensional structure of the object in the cylindrical coordinates, let us expand the measured and unknown functions into a Fourier series over circular harmonics. Then Eq. (1) takes the form

$$P_n(\rho_0) = 2\int_{\rho_0}^{\infty} I_n(\rho) \exp(i\,n(\pi/2 - \psi)) \frac{d\rho}{\sqrt{1 - (\rho_0/\rho)^2}} T_n(\rho_0/\rho), \qquad (2)$$

where $T_n(x) = \cos(n \arccos x)$ is the Chebyshev polynomial of the first kind of the nth order. After inversion of Eq. (2) the unknown two-dimensional distribution of radiation intensity can be obtained by inversion of the Fourier series for the coefficients I_n

$$I(\rho, \varphi) = \sum_{n=-\infty}^{\infty} I_n(\rho)\, e^{i\,n\,\varphi}.$$

When determining the internal structure of a medium with axial symmetry, only one direction of observation is sufficient. This corresponds to the case $n = 0$, and Eq. (2) becomes the well-known Abel equation.

The above derived expression (2) is similar to the integral equation arising in active tomography when determining the unknown density of a material.[1,10] The difference is in the factor $e^{in\psi}$ under the integral sign. It takes into account the curvature of the wave front caused by finiteness of the distance from the reception point to the radiation volume. The similarity of the equations confirms the similarity of active and passive tomography.

3. SOLUTION OF THE TOMOGRAPHIC PROBLEM BY THE METHOD OF CONVOLUTION EQUATION

At present, three solutions of the integral equation (2) are known. Two of them are called causal and noncausal. They are reduced to an integral transform of a given function with a kernel containing Chebyshev polynomials of the first or second kind.[10] The noncausal solution has some advantages in comparison with the causal one as its kernel has no singularity near the integration boundary. Algorithmization of these solutions is connected with the necessity to choose a variable integration step with more detailed calculation of the integrand near the coordinate origin. This involves a hidden instability of the solution to measurement errors and does not allow one to accelerate the process of internal structure reconstruction significantly. The third method based on Mellin's transform has similar shortcomings but there is an effective regularization for it. However the rate of calculations does not increase.

Another method significantly accelerating the calculations can be applied to the solution of the equation (2). Let us make the substitution of variables

$$\rho_0 = a\, e^{\tau_0}\,, \quad \rho = a\, e^{\tau}\,, \qquad (3)$$

where a is a constant. The equation will take the form

$$\tilde{P}(\tau_0) = \int_{-\infty}^{\infty} \tilde{I}(\tau_0)\, Q(\tau_0 - \tau)\, d\tau\,. \qquad (4)$$

Here

$$\tilde{I}(\tau) = a\, e^{3/2\tau}\, I_n(a\, e^{\tau})\,;$$

$$\tilde{P}(\tau_0) = e^{\tau_0/2}\, P_n(a\, e^{\tau_0})\, e^{-i\, n(\pi/2 - \psi)};$$

$$Q(t) = \frac{\sqrt{2}\, T_n(e^t)}{\sqrt{\sinh(-t)}}\, \chi(-t)$$

and $\chi(t)$ is the Heaviside step-wise function.

The equation (4) is an integral convolution equation for which regularization tools are well-developed.[9] For instance, by use of the Fourier transform, the solution can be written in the form[11]

$$\tilde{I}(\omega) = \tilde{P}(\omega)\, Q^*(\omega)\, [Q^*(\omega)\, Q(\omega) + \alpha(\omega^2 + 1)]^{-1}\,, \quad (5)$$

where $I(\omega)$, $P(\omega)$, and $Q(\omega)$ are Fourier transforms of the corresponding functions, α is the regularization parameter, and * denotes the complex conjugation. The function $P(\omega)$ can be obtained by the Fourier transformation applied to experimental data. The expression for the function $Q(\omega)$ can be obtained analytically[12]:

$$Q(\omega) = 2 \int_0^1 x^{i\,\omega - 1/2}\, \frac{T_n(x)\, dx}{\sqrt{1 - x^2}} =$$

$$= \frac{\sqrt{2}}{2^{i\,\omega}}\, \frac{\Gamma\left(\frac{1}{2} + i\,\omega\right)}{\Gamma\left(\frac{3}{4} + \frac{i\,\omega + n}{2}\right)\Gamma\left(\frac{3}{4} + \frac{i\,\omega - n}{2}\right)}\,.$$

Here $\Gamma(\omega)$ is the gamma function. To obtain the solution, it is sufficient to take the inverse Fourier transform of the function $\tilde{I}(\omega)$ and make the substitution of variables inverse to Eqs. (3).

The advantage of the method of convolution equation proposed is the possibility of using the Fourier transform for which there is an effective algorithm known as the fast Fourier transform (FFT). The analytical representation of the Fourier transform of the kernel of the integral equation also favors the decrease of the level of reconstruction errors. Besides, the proposed method realizes the change of discretization step in a way more convenient for exact integration. An adaptive step with respect to the variable ρ_0 is provided by the equidistant step of integration. Near the coordinate origin, the integral is calculated with a smaller step; the step increases with the increase of the sighting parameter. Finally, the use of regularization in Eq. (5) allows one to avoid additional efforts on improving solution stability under the presence of measurement noise.

4. IMITATION SIMULATION

In order to verify the solution proposed, the angular distribution of radiation intensity of a cylindrical source was imitatively simulated. The cylinder of radius R was displaced by a distance b from the coordinate origin at an angle β. If the distribution density of the sources is uniform inside the cylinder, the received power function of the angle is expressed, according to Eq. (1), analytically in the form

$$P(\rho_0, \theta) = 2\,\sqrt{a^2 - c^2\,\sin^2(y - a)}\;\chi(a - c\,\sin(\psi - \alpha))\,,$$

where $c^2 = b^2 + R_0^2 - 2bR_0\sin(\theta - \beta)$; $\sin\alpha = b/c\,\cos(\theta - \beta)$. A combination of enclosed cylinders of different intensity makes it possible to simulate different variants of the density distribution of radiation sources. Under real conditions, function $P(\rho_0, \theta)$ will be given by measurements.

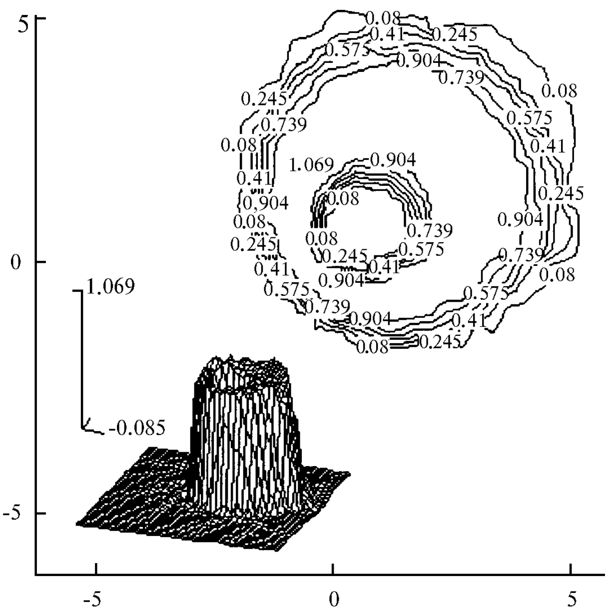

FIG. 2.

Figures 2 and 3 present examples of intensity distribution reconstruction for a source with decreased ($I = 0$) and increased ($I = 2$) concentration inside the cylinder with $I = 1$. The calculation was performed for $R_0 = 9$. The parameters of the external and the internal cylinders were chosen as follows: $a_1 = 3$, $b_1 = 2$, $\beta_1 = -45°$, $a_2 = 1$, $b_2 = 1$, $\beta_2 = -45°$, respectively. The reconstruction was performed using 32 observation directions and 64 samples over the azimuthal angle ψ. The algorithm was simple enough to realize it in the MathCAD integrated system. The shape of the distribution is reconstructed completely. The errors of solution manifest themselves by blurring of boundaries with different radiation source intensity in the representation of its distribution by isolines.

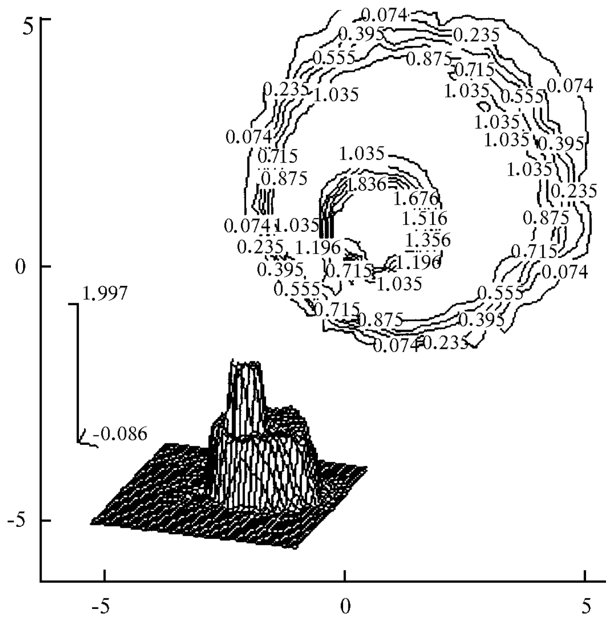

FIG. 3.

It should be noted that the accuracy of reconstruction of the image can be essentially improved by increasing the number of observation directions and samples; finally, it is determined only by measurement noise. This noise did not exceed –30 dB in the imitative simulation performed.

5. CONCLUSION

An integral equation similar to the equation in the active tomography problem with the use of an external radiation source was obtained on the basis of multi-aspect analysis of spatial distribution of the natural incoherent radiation of the medium in the case of weak absorption. A possibility of using known solutions is discussed and another method of inversion of the initial equation is proposed. The method is based on reduction to an integral equation of a convolution type. The algorithm of its solution is distinguished by structural simplicity and possibility to increase the realization rate by using fast Fourier transform. Its high accuracy is confirmed by results of imitative simulation. The new method can be applied for a more exact and fast reconstruction of the internal structure of objects in the problems of both passive and active tomography.

REFERENCES

1. G.A. Fedorov and S.A. Tereshchenko, *Computational Emission Tomography* (Energoizdat, Moscow, 1990), 184 pp.
2. J. Radon, in: S. Helgason, *The Radon Transform* (Birkhauser, Boston, 1980).
3. A.M. Cormack, J. Appl. Phys. **34**, No. 9, 2722–2727 (1963).
4. A.M. Cormack, International Journal of Imaging System and Technology **6**, 2–5 (1995).
5. R. Gordon, in: *Image Formation from Coherence Functions in Astronomy* (Dordrecht, 1978).
6. V.P. Yakubov and D.V. Losev, Atmos. Oceanic Opt., **9**, No. 10, 867–870 (1996).
7. A.A. Anosov, M.G. Isrefilov, and V.I. Pasechnik, Radiotekhnika, No. 9, 65–68 (1995).
8. A.G. Sel'skii, A.M. Fisher et al., Radiotekhnika, No. 9, 85–89 (1995).
9. A.N. Tikhonov, V.Ya. Arsenin, and A.A. Timonov, *Mathematical Problems of Computer Tomography* (Nauka, Moscow, 1987), 160 pp.
10. E.W. Hansen, J. Opt. Soc. Amer. **71**, No. 3, 304–308 (1981).
11. A.N. Tikhonov, A.V. Goncharskii, V.V. Stepanov, and A.G. Yagola, *Numerical Methods for Solving Ill-Posed Problems* (Nauka, Moscow, 1990), 232 pp.
12. A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series. Special Functions* (Nauka, Moscow, 1983), 753 pp.