DISPERSION RELATIONS AND THE MINIMAL PHASE EXISTENCE AT PROPAGATION AND INTERFERENCE OF LIGHT WAVES

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The subject of this paper is formulation of the conditions under which a functional relation exists and keeps between the logarithm of wave function amplitude and its phase. The dispersion relations, describing this relationship, are extended to two–dimensional case. The interference pattern and real zeroes of a light wave in an inhomogeneous medium are studied to demonstrate the approach application.

1. INTRODUCTION

If an analytical function $G(x)$ has no zeroes, and its logarithm singular points in the upper half–plane of the complex variable $\zeta = x + i\eta$, $\eta \geq 0$, then there exists a relation between $\chi(x) = \text{Re}\ln G(x)$ and $\varphi(x) = \text{Im}\ln G(x)$, involving Hilbert transform. The foundation theorem for the establishment of this interrelation in the class of quadratically integrable functions is the Titchmarsh theorem, Refs. 1, 3, and 8. But the condition of quadratic integrability does not allow one to consider the functions, that do not diminish at $x \to \pm\infty$, for example, the periodic functions. Therefore we address to Ref. 1, page. 306, where the theorem is proved setting the following expression

$$\chi(\zeta) = \ln|G(\zeta)| = \frac{\eta}{\pi}\int\limits_{-\infty}^{\infty} \frac{\ln|G(s)|}{(s-x)^2+\eta^2}\,\mathrm{d}s + c\eta, \qquad (1)$$

where $c = \lim\limits_{\eta\to\infty} \sup \dfrac{\ln|G(0+i\eta)|}{\eta}$ and s is the real variable. In so doing it was assumed, that $G(x)$ is an analytical function of the finite degree of the growth, when $\eta > 0$. This is possible, if $G(x)$ is an entire function of the exponential type, or if $G(x)$ has causal Fourier transform being an analytic signal. Besides the Paley–Wiener condition holds[3,10,11]

$$\int\limits_{-\infty}^{\infty} \frac{\ln|G(x)|}{1+x^2}\,\mathrm{d}x < \infty.$$

Having substituted the Cauchy–Riemann condition for function $\ln G\,(x + i\eta)$, namely, $\dfrac{\partial\chi}{\partial x} = \dfrac{\partial\varphi}{\partial\eta}$, $\dfrac{\partial\chi}{\partial\eta} = -\dfrac{\partial\varphi}{\partial x}$, into the expression for total differential of the function $\text{Im}\ln G(x + i\eta)$, we have

$$\mathrm{d}\varphi(x,\eta) = \frac{\partial\chi(x,\eta)}{\partial\eta}\,\mathrm{d}x - \frac{\partial\chi(x,\eta)}{\partial x}\,\mathrm{d}\eta. \qquad (2)$$

After taking derivatives of Eq. (1) with respect to x and η, we find

$$\frac{\partial\chi(x,\eta)}{\partial x} = \frac{2\eta}{\pi}\int\limits_{-\infty}^{\infty}(x-s)\,\chi(s)\,[(s-x)^2+\eta^2]^{-2}\,\mathrm{d}s,$$

$$\frac{\partial\chi(x,\eta)}{\partial\eta} = \frac{2\eta^2}{\pi}\int\limits_{-\infty}^{\infty}\chi(s)\,[(s-x)^2+\eta^2]^{-2}\,\mathrm{d}s +$$

$$+\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\chi(s)\,[(s-x)^2+\eta^2]^{-1}\,\mathrm{d}s + c. \qquad (3)$$

In the region of the function analyticity the integral of its total differential does not depend on the integration path. Having substituted derivatives from Eq. (3) in Eq. (2), taking $\eta = 0$, and integrating over x, we obtain

$$\varphi(x) = \frac{1}{2\pi}\,\text{v.p.}\int\limits_{-\infty}^{\infty}\frac{\chi(s)}{x-s}\,\mathrm{d}s + l(x) = \underset{x}{\mathbf{H}}\,\chi(s) + l(x), \qquad (4)$$

where $\underset{x}{\mathbf{H}}$ is the operator of Hilbert transform over variable x and $l(x) = cx + \text{const}$.

Thus determined phase we call the minimal phase, and the wave with such a phase is the minimal–phase wave. The expression (4) is called the logarithmic Hilbert transformation,[7] or the dispersion relation.[6,12] This notation is traditional. First similar relation was obtained by Kramers and Kronig. It connects the real and imaginary parts of the complex refractive index of a medium, that causes the wave dispersion, Refs. 2, 8. In that case the argument of the Hilbert transform was the optical frequency.

The functions, vanishing at infinity, do not tend to a finite limit after taking their logarithm, therefore

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the improper integral, Eq. (4), may not exist. The integral divergence is usually eliminated by differentiation of the function transformed, and by multiplying it by a suitable window, using the so called subtractions,[8] thus making transition to the region of Fourier transform. In any case there is a need to analyze the class of functions to be transformed.

The class of periodic functions is of special interest since it is just for such functions of the fast Fourier transform (FFT) applies. This algorithm is the basis for numerical analysis of signals and realizes the Hilbert transform.

Let us transform Eq. (4), assuming without any loss of generality, that the functions $\varphi(x)$ and $\chi(x)$ have the 2π period. Then the relation $(x \to x + 2\pi) \Rightarrow (s \to s - 2\pi)$ is valid and substitution of the variables $x = \exp i\vartheta$, $s = \exp i\theta$ is possible, which results[4] in $\frac{\mathrm{d}s}{x - s} \to \frac{1}{2}\cot\frac{\vartheta - \theta}{2}\,\mathrm{d}\theta + \frac{i}{2}\,\mathrm{d}\theta$.

Then the expression (4) takes the form

$$\varphi(\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} \chi(\theta)\cot\frac{\vartheta - \theta}{2}\,\mathrm{d}\theta + \frac{1}{2\pi}\int_0^{2\pi} \chi(\theta)\,\mathrm{d}\theta + l(\vartheta). \quad (5)$$

The singular Hilbert integral, entering into this dispersion relation, has only one singular point in contrast to the integral in expression (4).

Let us give, as an example, two minimal-phase solutions to the wave equation. The first one is the plane wave $\exp i\alpha x$, but it is the degenerate case, here the phase consists only of one linear component. The second one is a more interesting paraxial Gaussian beam

$$G(x, y, z) = \frac{a}{1 + i\gamma z}\exp\left(ikx - \frac{k\,\gamma}{2}\frac{x^2 + y^2}{1 + i\gamma z}\right),$$

where z is the longitudinal coordinate; x, y are the transverse coordinates; a and k are constants. Having substituted the complex value γ in the form $\gamma = (p + iq)/(p^2 + q^2)$ into the expression for $G(x, y, z)$ and having executed transformation, we can write the following expressions:

$$\ln|G(x, y, z)| = \chi(x, y, z) =$$

$$= \text{const} - \frac{k}{2}\frac{p(x^2 + y^2)}{(z - q)^2 + p^2} - \int \frac{(z - q)\,\mathrm{d}z}{(z - q)^2 + p^2},$$

$$\arg G(x, y, z) = \varphi(x, y, z) =$$

$$= kz + \frac{k}{2}\frac{(z - q)\,(x^2 + y^2)}{(z - q)^2 + p^2} - \int \frac{p\,\mathrm{d}z}{(z - q)^2 + p^2}.$$

Comparing the logarithm of the module and argument of the Gaussian beam and using Hilbert transform tables,[5] we have

$$\varphi(x, y, z) = \underset{z}{\mathbf{H}}\,\chi(x, y, z) + kz.$$

Let us introduce the oblique cross section of the beam $G(x, y, z)$ cut by the plane $z = z_0 + \theta\,x$, $\theta > 0$, $y = y$ and find $\varphi(x, y, \theta) = \underset{x}{\mathbf{H}}\,\chi(x, y, \theta) + l(x, y)$, where the last summand is some plane.

The generalization of the dispersion relation (4) to the case, when $G(\zeta)$ has zeroes and $\ln G(\zeta)$ has singularities in the upper half-plane of the complex variable ζ, one can obtain using the Blaschke factor,[6,7,8] $B(\zeta) = \prod_k \frac{\zeta - \zeta_k}{\zeta - \zeta_k^*}$, which is purely phase function on the real axis

$$B(x) = \exp i\,2\sum_k \arctan\frac{x_k - x}{\eta_k}. \quad (6)$$

The product $G(\zeta) \cdot B(\zeta)$ will have zeroes at the points $\zeta_k = x_k + i\eta_k$ of the upper half-plane, that appear here as the conjugated points ζ_k^* of the bottom half-plane due to multiplication by the Blaschke factor.

Because the wave function is multidimensional we may rewrite Eq. (4) for the general case as follows:

$$\varphi(x, \bar{x}) = \underset{x}{\mathbf{H}}\ln|G(x, \bar{x})| + l(x, \bar{x}) +$$

$$+ 2\sum_k \arctan\frac{x_k(\bar{x}) - x}{\eta_k(\bar{x})}, \quad (7)$$

where $\bar{x}$ denotes all variables, except for x, $l(x, \bar{x})$ is an arbitrary function of $\bar{x}$ and a linear one of x.

At present no effective methods of finding the coordinates of complex zeroes are known for Eq. (7) to be used. Nevertheless it is possible to identify the conditions for the minimal-phase signal to exist, when no zeroes in the complex half-plane occur. The real zeroes of the wave function also create difficulties. They result in the logarithmic singularities when using Eq. (4) and in breaks of the phase function. That means that the so-called dislocations appear. There is also the problem of the constructive generalization of the dispersion relations to the two-dimensional case. But first we shall consider the robustness of the phase determination from the dispersion relations and the criterion of its correctness.

2. ROBUSTNESS OF THE WAVE APPROXIMATION BY A FUNCTION WITH A FINITE FOURIER TRANSFORM

The dispersion relation (4) has been derived for the approximation of the wave function,

$$G(x, \bar{x}) = U(x, \bar{x}) + iV(x, \bar{x}),$$

by a function with a finite spectrum. But, in reality the spectrum may be infinite, therefore it is necessary to investigate the convergence of the wave model components, when extending their spectra of spatial frequencies.

We shall consider one–dimensional cross section of the real part of the wave function $U(x)$ and its finite model $U_s(x)$. Let $U(x)$ have absolutely integrable spectrum $S(\alpha)$, then

$$U(x)=\int_{-\infty}^{\infty} S(\alpha)\, e^{i\alpha x}\, d\alpha, \qquad U_s(x) = \int_{-s}^{s} S(\alpha)\, e^{i\alpha x}\, d\alpha,$$

$$\int_{-\infty}^{\infty} |S(\alpha)|\, d\alpha = M.$$

According to the Weierstrass ratio test the improper integral for $U_s(x)$ converges uniformly to the function $U(x)$ at $s \to \infty$, that is starting with some s $|U(x) - U_s(x)| < \varepsilon$ for all x, where ε is an arbitrarily small constant.

The uniform convergence of the imaginary part of the model $V_s(x)$ follows from the property of the Fourier transform, according to which the majorant of the integrand does not vary

$$V_s(x) = \int_{-\infty}^{\infty} \frac{U_s(y)}{x - y}\, dy = \int_{-\infty}^{\infty} \frac{dy}{x - y} \int_{-s}^{s} S(\alpha)\, e^{i\alpha y}\, d\alpha =$$

$$= \int_{-s}^{s} i \,\mathrm{sgn}\, \alpha\, S(\alpha)\, e^{i\alpha x}\, d\alpha < M.$$

The squares of the functions considered also uniformly converge. It follows from the inequality

$$|U^2(x)-U_s^2(x)| = |U(x)-U_s(x)|\, |U(x)+U_s(x)| < 2M\varepsilon,$$

the right-hand side of which vanishes as vanishes ε at $s \to \infty$.

We shall consider the convergence of the logarithm of the module $\chi_s = \ln|G_s(x)|$ to its limit at $|G_s(x)| > 0$. We have

$$\frac{1}{2}\, |\ln \chi^2 - \ln \chi_s^2| = \frac{1}{2} \left| \ln \left(1 - \frac{\chi^2 - \chi_s^2}{\chi^2} \right) \right| ,$$

but $|\chi^2 - \chi_s^2| < 4M\varepsilon$, than the uniform convergence is provided.

Finally, the convergence of the model phase φ_s follows from the lack of the singularity points of the function χ_s on the real axis, $|G_s(x)| > 0$, and from the existence of the Cauchy principal value for the Hilbert integral, (4).

The inverse statement is also true, i.e., from the uniform convergence of the sequences φ_s and χ_s follows the uniform convergence of the wave function, which is calculated from these sequences. Actually, for all x

$$\lim_{s \to \infty} \frac{\exp(\chi + i\varphi)}{\exp(\chi_s + i\varphi_s)} = \exp \lim_{s \to \infty} [\chi - \chi_s + i(\varphi - \varphi_s)] = 1.$$

Thus the uniform convergence of the finite model components of the wave function to their limits allows one to apply the dispersion relation (4) when determining the wave function phase as the phase of the entire function of exponential type that approximates the wave function. The criterion correctness of such a procedure is an arbitrarily exact coincidence of the initial wave function with the wave function, the phase of which is determined by the dispersion relation.

3. CONDITIONS OF THE CAUSALITY EXISTENCE OF THE WAVE FUNCTION LOGARITHM FOURIER TRANSFORM

We shall first consider necessary conditions. Let $R(x) = \ln G(x)$ be a complex function limited on the real axis, being an element of A_0 set with the causal spectrum functions, the lower frequency of which is $\alpha_0 \geq 0$.

Let us consider the power series

$$G(x) = \exp R(x) = 1 + \sum_{k=1}^{\infty} \frac{R^k(x)}{k!} = 1 + W(x) \qquad (8)$$

in order to find out the properties of $G(x)$ function.

From the limitation of $R(x)$ follows uniform convergence of this series at all x, determining the function $W(x)$. Fourier spectrum of the product of n functions is equal to n–fold convolution of these functions spectra. We shall require the existence of the convolution for a spectrum of $R(x)$, then the Fourier transform of the power series in (8) will also exist. The convolution of the causal spectra, which is by definition the case with the spectrum of the function $R(x)$, will also be causal. Therefore the lower frequency of the spectrum of $R^k(x)$ will be $k\alpha_0 \geq 0$.

Besides, from the limitation of $R(x)$ function follows the condition $|G(x)| > 0$. Then from the inequality

$$|G(x)|^2 = 1 + |W(x)|^2 + W(x) + W^*(x) > 0$$

we obtain $|W(x)| < 1$.

Let us now define sufficient conditions. Let $G(x) = 1 + W(x)$, $|G(x)| > 0$, and $|W(x)| < 1$. Then the power series

$$\ln G(x) = \ln [1 + W(x)] = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{W^k(x)}{k!} \qquad (9)$$

uniformly converges along the entire real axis. Let also $W(x) \in A_0$ with the lower frequency of the spectrum $\alpha_0 \geq 0$. Then by virtue of the convolution causality of the spectra of the analytic signals the function $\ln G(x)$ is an element of A_0 set with the lower frequency of the spectrum α_0.

Thus, the existence of the dispersion relations (4) is determined by the following relationship:

$$\begin{bmatrix} \ln G(x) \in A_0,\ \alpha_0 > 0, \\ |G(x)| < \infty \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 + W(x) = G(x) \in A_0,\ |G(x)| < 0. \\ W(x) \in A_0,\ |W(x)| < 1,\ \alpha_0 > 0 \end{bmatrix}, \qquad (10)$$

i.e., it is necessary and sufficient for $G(x)$ to be the sum of a constant and an analytic signal with the amplitude less than this constant. This result can also be obtained by the Rouche theorem, Ref. 9, page 287.

Let us now consider the consequence of the conditions (10). Let $G(x)$ be the function with the period T being presented by a segment of a Fourier series with the frequencies from the interval $[n, N]$. Let us make the transformations

$$G(x) = \sum_{k=n}^{N} c_k\, e^{(i2\pi k/T)x} \to \underset{\substack{\uparrow \\ e^{(-i2\pi n/T)x}}}{\otimes} \to c_n + \sum_{k=1}^{N-n} c_{k+n}\, e^{(i2\pi k/T)x} = c_n + W(x). \qquad (11)$$

If the inequality $|W(x)| < c_n$ holds, then the final result in Eq. (11) satisfies to conditions (10) and $G(x)$ is the minimal–phase function with some linear phase component.

In experiment only intensity is accessible for analysis, and the conclusions should be drawn from *a priopi* information concerning the wave properties. Possibly, the following formulation of the conditions for minimal phase existence will be more practically. The periodic analytic signal is the minimal–phase at *a priopi* prevalence of an oscillation of main frequency and absence of constant component.

4. DISPERSION RELATIONS FOR THE WAVE FUNCTION LOGARITHM IN A TWO–DIMENSIONAL CASE

In a two–dimensional case we shall require the uniqueness of the phase $\varphi(x, y)$ for the function $G(x, y)$. This requirement is provided by the fact that the dispersion relation, in which the Hilbert transform over argument x is executed, should coincide with the dispersion relation, in which the argument of Hilbert transform is y. This is presented in the form

$$\varphi(x, y) = \underset{x}{\mathbf{H}}\, \chi(x, y) + xa(y) + b(y) + \text{const} =$$

$$= \underset{y}{\mathbf{H}}\, \chi(x, y) + yc(x) + d(x) + \text{const}. \qquad (12)$$

The one–dimensional functions entering this expression have to be defined. For this purpose we shall present without any loss of generality the logarithm of the amplitude in the form

$$\chi(x, y) = {}_{+}\chi(x, y) + {}_{|}\chi(y) + {}_{-}\chi(x). \qquad (13)$$

Here the first component has no one–dimensional additive components, they are set by two other components. Let us now substitute Eq. (13) into Eq. (12) and write the following equalities:

$$c(x) = c = \text{const}, \qquad a(y) = a = \text{const},$$

$$d(x) = \underset{x}{\mathbf{H}}\, {}_{-}\chi(x) + ax, \qquad b(y) = \underset{y}{\mathbf{H}}\, {}_{|}\chi(y) + cy,$$

$$\underset{x}{\mathbf{H}}\, {}_{+}\chi(x, y) = \underset{y}{\mathbf{H}}\, {}_{+}\chi(x, y). \qquad (14)$$

Now substitute these equalities into Eq. (12) and, then using the function $\chi(x, y)$ again, we shall obtain the dispersion relations between the phase and logarithm of the amplitude in the two–dimensional case

$$\varphi(x, y) =$$

$$= l(x, y) + \underset{x}{\mathbf{H}}\, \chi(x, y) + [\underset{y}{\mathbf{H}}\, \chi(x, y) - \underset{x}{\mathbf{H}}\, \chi(x, y)]_{x=\text{const}} =$$

$$= l(x, y) + \underset{y}{\mathbf{H}}\, \chi(x, y) + [\underset{x}{\mathbf{H}}\, \chi(x, y) - \underset{y}{\mathbf{H}}\, \chi(x, y)]_{y=\text{const}},$$

$$l(x, y) = ax + cy + \text{const}. \qquad (15)$$

Here the two–dimensional functions are related the one–dimensional Hilbert transform. This result is the consequence of the physically reasonable requirement of the two–dimensional phase to be unique, it was published in Refs. 13, 14.

In other works[12,17] exist, there were also considered generalizations of the dispersion relations to a two–dimensional case. In our earlier work[12] we studied the function $\ln G(x_1 + i\eta_1, x_2 + i\eta_2)$ in the space of two complex variables. Such approaches gave no positive results because of the impossibility to solve the integrated equations concerning the logarithm of the module or the argument of the function $G(x_1, x_2)$. Only in the specific case, when a two–dimensional function can be represented as the product of the two one–dimensional functions, enables one to obtain the final result in the form of a sum of two one–dimensional Hilbert transforms of these functions.

If the two–dimensional dispersion relations are applied to the function $G(x, y)$, satisfying the existence conditions, (10), of the spectrum causality of the logarithm of both coordinates

$$G(x, y) = 1 + W(x, y),\ |W(x, y)| < 1,$$

$$W(x_0, y) \in A_0,\ W(x, y_0) \in A_0, \text{ then}$$

$$W(x, y) = U(x, y) + iV(x, y),$$

$$\underset{x}{\mathbf{H}}\, U(x, y) = V(x, y) = \underset{y}{\mathbf{H}}\, U(x, y). \qquad (16)$$

From Eq. (16) it follows that $W(x, y)$ has no additive one–dimensional components, and the function $G(\zeta_1, \zeta_2)$ at $\zeta_1 = x + i\eta_1$, $\zeta_2 = y + i\eta_2$ is a limited function at $\eta_1 > 0$, $\eta_2 > 0$. As a result one can find the constants from Eq. (15)

$$c = \lim_{\substack{\eta_1 \to \infty \\ \eta_2 = 0 \\ x = 0}} \sup \frac{\ln|G(\zeta_1, \zeta_2)|}{\eta_1} = 0 = \lim_{\substack{\eta_2 \to \infty \\ \eta_1 = 0 \\ y = 0}} \sup \frac{\ln|G(\zeta_1, \zeta_2)|}{\eta_2} = a.$$

The expression (15) is simplified and takes the form

$$\arg[1 + W(x, y)] = \underset{x}{\mathbf{H}} \ln|1 + W(x, y)| + \text{const} =$$

$$= \underset{y}{\mathbf{H}} \ln|1 + W(x, y)| + \text{const}. \qquad (17)$$

Localization of the function spectrum, satisfying the two–dimensional dispersion relations, can be established proceeding from the property of Hilbert transform, according to which it is reduced to multiplication by the sign function in the frequency region. Let us designate by the capital letters a Fourier image of the values, designated by the appropriate lower case letters, and consider the two–dimensional spectrum of the function $\ln G(x, y) = \chi(x, y) + i\varphi(x, y)$ in the plane of α and β frequencies. From Eq. (15) we have

$$\mathrm{X}(\alpha, \beta) + i\Phi(\alpha, \beta) =$$

$$= L(\alpha, \beta) + (1 + \operatorname{sgn}\alpha)\,[{}_{+}\mathrm{X}(\alpha, \beta) +$$

$$+ {}_{-}\mathrm{X}(\alpha)\,\delta(\beta)] + (1 + \operatorname{sgn}\beta)\,{}_{|}\mathrm{X}(\beta)\,\delta(\alpha). \qquad (18)$$

It follows from these relations, that the two–dimensional Fourier spectrum of the function $\ln G(x, y)$ is located in one quadrant of the frequency plane $\alpha\beta$, including the singularities on the coordinate axes. In this case this is the first quadrant. Similarly for expression (17) it is possible to write the following expression:

$$(\operatorname{sgn}\alpha - \operatorname{sgn}\beta)\,\mathrm{X}(\alpha, \beta) = i\delta(\alpha, \beta). \qquad (19)$$

As is seen, no singularities on the coordinate axes occur.

The application to interferometry. Let $|G(x,y)|^2$ be the interferogram of a phase object, that is $|W(x, y)| = \text{const}$, and for it the expressions (10) and (16) are valid. Define the rectilinear or curvilinear cross section in the plane XY by parametric equations $x = x(l)$, $y = y(l)$. Let the interference fringes have in such a cross section a full profile and the object wave phase $W(l)$ is monotonic. Then $W(l)$ will be an analytic signal[19] in this cross section and the conditions (10) hold. We assume also, that $W(l)$ occupies, in the frequency region the interval $[b, e]$ at the Nyquist frequency equal to N. This is enough to write the relations for defining the object phase in the interferogram cross section

$$\arg W(l) = \arctan \frac{|G(l)| \sin \underset{l}{\mathbf{H}} \ln |G(l)|}{|G(l)| \cos \underset{l}{\mathbf{H}} \ln |G(l)| - 1} =$$

$$= \arg \mathbf{F}_{be}\,[\exp 2\, \mathbf{F}_{bN} \ln|G(l)|]. \qquad (20)$$

Here $\mathbf{F}_{be}$, $\mathbf{F}_{bN}$ are the filtration operators, which make all the spectrum values zero outside the interval.

The fraction in Eq. (20) is copied out from Ref. 6, the expression in the right hand side of Eq. (20) has an advantage because it does not require knowledge of the reference wave value and can be applied to interferograms of a general form, at which the amplitudes of reference and object waves differ from a constant. This expression provides for homomorphic interferogram filtration and suppress the multiplicative noise in the interval $(-b, b)$, and the additive noise outside the interval $[b, e]$. These opportunities were used for processing the data of a field experiment.[15]

5. THE DISPERSION RELATION WITH A WEIGHTING FUNCTION

This relation may be derived, Ref. 16, in the same way as conditions (10). Let $|W(x)| = c = \text{const} < 1$ in Eq. (10), then for the function $|G(x)|^2 \ln G(x)$ the dispersion relation (21) is valid. One can find that

$$|G(x)|^2 \ln G(x) =$$

$$= [1 + W(x)]\,[1 + W^*(x)] \ln[1 + W(x)] = [1 +$$

$$+ |W(x)|^2 + W(x) + W^*(x)] \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} W^k(x) =$$

$$= [1 + W(x)]\,[c^2 + W(x)] \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} W^{k-1}(x).$$

The Fourier transform of this expression and as well as of Eq. (9) is causal with the lower frequency $\alpha_0 = 0$, and therefore the dispersion relation with the weighting function

$$\arg G(x) = |G(x)|^{-2} \underset{x}{\mathbf{H}} |G(x)|^2 \ln|G(x)| + l(x) \qquad (21)$$

is valid.

It seems so that this expression is similar to the property of the Hilbert transform

$$\underset{x}{\mathbf{H}}\, \Omega(x)\, U(x) = \Omega(x)\, \underset{x}{\mathbf{H}}\, U(x),$$

that the low–frequency function can be removed out from the operator sign. Here $\Omega(x)$, $U(x)$ are real functions and their Fourier transforms do not overlap in the frequency domain. But it is not so. The Fourier transforms of functions $|G(x)|^2$, $\ln|G(x)|$ overlap on the frequency axis, because both contain $W(x)$ as a summand.

The condition $|W(x)| = \text{const}$ holds in the interferometry of purely phase objects. Then the expression (21) allows one to smoothen the logarithmic singularities, when $|G(x)| \to 0$.

6. TRANSFORMATIONS OF THE MINIMAL PHASE SIGNALS

Let us now consider the transformations, which conserve the minimal phase of the functions transformed. Proceeding from the properties of the logarithmic function, for example let it be the raising into a power. But first the transformations are interesting, which are characteristic of optical systems.

The extension–compression transformation. Let the function $G(x) \in L_2(T)$ satisfy the sufficient conditions of the minimal phase existence and be presented by a monotonic function $\tau(x) \in [0, T]$, mapping the interval $[0, T]$ on itself. Then the ranges of $G(x)$ and $G(\tau)$ values coincide, $|G(\tau)| > 0$ and there exists $\ln G(\tau)$.

Having considered the Fourier series

$$G[\tau(x)] = 1 + \sum_{k=1}^{\infty} a_k \exp i \frac{2\pi k}{T} \tau(x),$$

we shall notice, that each term of the series has the dispersion causal Fourier transform[19] because of the monotony of the function $\tau(x)$. Therefore the spectrum of the function $G[\tau(x)]$ is causal in the dispersion sense, and its phase is minimal in the same sense. It should be noted that the width of the spectrum is the second moment of the square of its module.

The optical feedback according to the scheme (22) allows one to form a signal with a minimal phase, inverse to that which follows from the conditions (10)

$$\begin{array}{c} W(x,\, y) \\ \downarrow\!\longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\uparrow \\ 1 \mapsto \oplus \xrightarrow[1]{\quad\quad} \circ\!\to G(x,\, y). \end{array} \qquad (22)$$

From this scheme we shall find that $G(x, y) = 1 + G(x, y) \cdot W(x, y)$, then the function $G(x, y) = [1 - W(x, y)]^{-1}$. If $W(x, y)$ is a two–dimensional analytic signal and the condition $|W(x, y)| < 1$ holds, the function $\ln G(x, y) = -\ln [1 - W(x, y)]$ will have the minimal phase based on the necessary and sufficient conditions (10).

7. DISPERSION RELATIONS FOR THE WAVE FUNCTION ZERO

Let us present the wave, in an arbitrarily small vicinity of its zero point by the linear terms of the power series

$$G_0(x, y) = (p_R\, x + q_R\, y) + i\, (p_I\, x + q_I\, y). \qquad (23)$$

Having chosen a suitable coordinate system and having made the normalization, we shall obtain the function of a zero (23) in the form

$$G_0(x, y) = x + (a + ib)\, y = \\ = \sqrt{(x + ay)^2 + b^2 y^2} \exp i \arctan \frac{by}{x + ay}. \qquad (24)$$

In these expressions p_R, p_I, q_R, q_I, a, b are the real valued constants.

Consider now the operation sequence over the variable x at $y = \text{const}$

$$\ln |G_0(x, y)| = \ln \sqrt{(x + ay)^2 + b^2 y^2} \xrightarrow{\partial/\partial x} \frac{x + ay}{(x + ay)^2 + b^2 y^2} \xrightarrow{\mathbf{H}} \frac{by}{(x + ay)^2 + b^2 y^2} \xrightarrow{\int \mathrm{d}x} \arctan \frac{by}{x + ay} + \text{const}.$$

The comparison of the result of this operations with the Eq. (24) shows, that the function of zero is the minimal–phase in a rectilinear cross section not passing through zero in the XY plane. That is logarithm of the module of zero function and its phase are connected by Hilbert transform in this cross section accurate to a constant.

Let us present the function of zero in the polar coordinate system ρ, ϑ and make the following operations:

$$G_0(x, y) = (p_R x + q_R y) + i\, (p_I x + q_I y) \xrightarrow[y = \rho \sin \vartheta]{x = \rho \cos \vartheta} \rho\, (p e^{i\vartheta} + q e^{-i\vartheta}) \to \underset{\underset{(q\rho)^{-1}}{\uparrow}}{\otimes} \to c e^{i\vartheta} + e^{-i\vartheta} \to \underset{\underset{e^{i\vartheta}}{\uparrow}}{\otimes} \to 1 + e^{i2\vartheta} = G_0(\vartheta), \qquad (25)$$

where p, q, c are complex constants, here $|q| > |p|$, and $|c| < 1$.

From the result of (25) it follows, that the function $G_0(\vartheta) = \exp \chi_0(\vartheta) + i\varphi_0(\vartheta)$ satisfies the necessary and sufficient conditions (10) of the minimal phase existence $\ln G_0(\vartheta) \in A_0$. Therefore, the logarithm of a module and the phase of the function $G_0(\vartheta)$ are connected by the Hilbert transform on an arbitrarily small circle with a center at the zero point accurate to a linear function $l(\vartheta) \in [\text{const}, 2\pi \pm \text{const}]$

$$\varphi_0(\vartheta) = \underset{\vartheta}{\mathbf{H}}\, \chi_0(\vartheta) + l(\vartheta). \qquad (26)$$

Let the zero point be at the center of the Gaussian beam

$$W(x, y) = [x + (a + ib)\, y] \exp [- (x^2 + y^2)/2\, r^2]. \qquad (27)$$

Using the relations

$$\exp - \frac{x^2}{2\, r^2} \xrightarrow{\mathbf{F}} \sqrt{2\pi}\, r \exp - \frac{r^2\, \alpha^2}{2},$$

$$x \exp - \frac{x^2}{2\, r^2} \xrightarrow{\mathbf{F}} i \sqrt{2\pi}\, r^3\, \alpha \exp - \frac{r^2\, \alpha^2}{2},$$

where **F** is the one–dimensional Fourier transform operator. Let us perform the two–dimensional Fourier transform of the expression (27) and find the representation of the spatial spectrum of the Gaussian beam with a zero at it center

$$S_0(\alpha, \beta) =$$

$$= i\, 2\pi\, r^4\, [\alpha + (a + ib)\, \beta] \exp [- (\alpha^2 + \beta^2)\, r^2/2]. \qquad (28)$$

Here a, b, and r are constants.

In a polar coordinate system the functions (27) and (28) have the minimal phase on circles around a zero at a fixed polar radius as in the case (25).

The Gaussian beam, Eq. (27), and its spatial spectrum, Eq. (28), has a zero at the coordinate origin. From this it follows, that both these functions, in particular, have no constant components. If the zero of a spatial spectrum is placed at the coordinate origin of the frequency area, the wave acquires linear phase shift, which in polar coordinates ρ, ϑ on a circle $\rho = \text{const}$ will be transformed into a sum of a sine and a cosine. The amplitudes a, and b of these harmonics will be determined by the circle radius and the phase inclination value. In this case more general dispersion relation is valid

$$\varphi_0(\vartheta) = \underset{\vartheta}{\mathbf{H}}\, \chi_0(\vartheta) + a \cos \vartheta + b \sin \vartheta + l(\vartheta). \qquad (29)$$

The absence of a constant component was found out also in the numerical experiment, Ref. 18, where zero and focal spots of a light wave, propagating in an inhomogeneous medium were studied. There, the wave function zero was placed at the center of the circular subaperture without apodization. The radius of the subaperture could be changed, but did not exceed the size of the vortex area, where the wave phase had the screw structure. There is no constant component on any concentric circle in this subaperture, if zero is at the coordinate origin in the spatial frequencies area. This is one of the conditions for the minimal phase existence.

Other condition, i.e., the existence of oscillation at the prevailing main frequency, holds because of the presence of the wave function zero in the subaperture. Going round the first order zero, the phase monotonically changes from 0 up to $\pm 2\pi$, the length of this interval is equal to the period of the main frequency of the azimuth oscillation. A larger period does not exist for the first order zero.

Under these conditions any wave function should have minimal phase on concentric circles around a zero in the subapertures and within the area occupied by a vortex, if it were the analytic signal. To use Eq. (10) the information about the azimuth wave spectrum localization on these circles is also needed. Analytically this question is not investigated.

However in the numerical statistical experiment the expression (26) held highly accurate. For a wave with the maximal spatial frequency, equal to 0.04 from the Nyquist frequency, the characteristic diameter of the vortex area around a zero was 0.2 of the size of the reads–out matrix. On the circles within this area, including the displaced concerning zero, the degree of causality[19] was above 0.97, and the rms error for the minimal phase calculated by (26) did not exceed $4\pi \cdot 10^{-3}$ rad.

The occurrence of the wave function zeroes and of the phase dislocations connected with the results in disintegration of the wave on separate uncorrelated parts at its propagation. It is especially interesting, that the rigid functional connection, established by the dispersion relation, exists just in these conditions.

8. CONCLUSIONS

The wave function phase with the absolutely integrable spectrum, determined by the dispersion relation, differs arbitrarily little from the phase of the entire function of the exponential type, the spectrum of which coincides with the wave function spectrum on the finite interval, at increasing of the interval width.

Two–dimensional dispersion relations, which set the interrelation between a phase and a logarithm of the amplitude of a two–dimensional wave function are obtained. These relations generalize the known interrelations in one–dimensional case.

The dispersion relation with the weight, equal to the interferogram of the phase object, which sets the interrelation between the phase and the logarithm of the amplitude of interference field, smoothing the logarithmic singularities in the interferogram zeroes is obtained.

In this paper were specified the transformations, which are performed by optical systems, that conserve the minimal phase of the wave function.

We have also established, that the wave function has a minimal phase on a circle of an arbitrarily small radius around a zero point. This property is valid at increasing of the radius in the area, occupied by a vortex, if the circle lies in the plane, normal to the direction of the wave propagation.

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