CLOUD OPTICAL THICKNESS ESTIMATION FROM GROUND-LEVEL MEASUREMENTS

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Two methods for estimating the unknown optical thickness of a homogeneous cloudy atmosphere using a ground-level downward radiance or irradiance measurement are presented. An iteration method incorporates an analytically-computed first derivative of the unknown that is obtained from the F_N method of transport theory; sample results from a sensitivity analysis for the iteration scheme are included. A noniterative method derived from two analytical solutions of the equation of radiative transfer is also given – an asymptotic radiative transfer algorithm and a transport-corrected diffusion algorithm.

1. INTRODUCTION

In current models used to analyze global warming, one of the greatest sources of uncertainty arises from the effects of clouds. A major contribution to this uncertainty is due to the absorption of radiant energy within the clouds, and this is directly a function of the total effective thickness of the cloud layer. Thus if an optical thickness of clouds can be estimated from radiation measurements, then this source of uncertainty can be reduced. We shall consider only a source-free, homogeneous, anisotropic scattering cloud layer with known albedo of single scattering and angular scattering (phase) function.

In 1987 King proposed a one-detector algorithm that uses the backscattered radiance above an optically thick cloud layer to estimate the optical thickness when the ground albedo is known[1]; this algorithm was derived using an approximate (i.e., asymptotic) direct transport theory solution. Then in 1990 the use of the backscattered irradiance (i.e., the angle-integrated partial current passing through a flat collector) was proposed for the same purpose[2]; separate algorithms were developed for above-cloud, middle-cloud, and below-cloud detector locations.

In contrast with such explicit inverse methods that do not require iterative transport calculations to reproduce measured values, one can combine – in an iterative manner – the solutions of direct radiative transfer problems with measurements to yield simultaneous estimates of the cloud optical thickness and the albedo of the underlying surface.[3]

Here we summarize a portion of the work done at the University of Washington[2,3] on such inverse radiative transfer problems and focus on the estimation of only a single parameter, the optical thickness of a cloud above a surface of known albedo. It will be assumed that either the downward radiance or irradiance is measured at the ground-level. This portion of Refs. 2 and 3 was selected because it is likely to be of most use in a ground-based atmospheric radiation monitoring program.

Section 2 introduces the physical model and the basic iterative strategy for the solution of the inverse problem. Two algorithms for initiating the iteration procedure are given in Sec. 3; these algorithms also can be used without radiative transfer calculations to obtain good initial estimates of the optical thickness. A numerical sensitivity analysis is given in Sec. 4 for a model cloud. Some conclusions about the use of such an inverse procedure are provided in Sec. 5.

2. THE ITERATIVE INVERSE METHOD

Let $I(\tau, \mu, x)$ be the time-independent radiance at position τ that is integrated over the azimuthal angle ϕ,

$$I(\tau, \mu; x) = \int_0^{2\pi} I(\tau, \mu, \phi; x)\, d\phi.$$

The illuminated surface of the cloud is at $\tau = 0$, and its thickness $\tau = x$ is in optical mean free paths; μ is the cosine of the polar angle with respect to the τ-axis. A vacuum is assumed between the cloud bottom and where the ground-level measurements are made, so the albedo of the ground is effectively that at the lower surface of the cloud. Other Fourier moments over the

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azimuthal angle of the radiative transfer equation could be considered, but those moments of I are harder to measure and there is no resulting simplification in the algorithm. Use of the wavelength dependence of the radiation would represent an additional variable that could be used in the inversion algorithm, but we shall consider only a single wavelength (not explicitly denoted).

Integration over the azimuthal angle yields an azimuthally-symmetric radiative transfer equation for the radiance that can be written as[4]

$$(\mu\,\partial_\tau + 1)\, I(\tau, \mu; x) = \frac{\omega}{2} \sum_{\ell=0}^{L} (2\,\ell + 1)\, f_\ell\, P_\ell(\mu) \int_{-1}^{1} P_\ell(\mu')\, I(\tau, \mu'; x)\, d\mu', \qquad 0 \le \tau \le x, \tag{1}$$

where ω, with $\omega < 1$, is the albedo of single scattering and the phase function has been expanded in a series of $(L + 1)$ Legendre polynomials with expansion coefficients f_ℓ, $\ell = 0$ to L and $f_0 = 1$. The ω and all coefficients f_ℓ are presumed known; the most important of the higher-order coefficients is the scattering asymmetry factor $f_1 \equiv g$.

The monodirectional incident radiation striking the top of the cloud layer can be written as

$$I(0, \mu; x) = I_0\, \delta(\mu - \mu_0), \quad 0 \le \mu \le 1, \quad 0 < \mu_0 \le 1, \tag{2}$$

where the magnitude of the radiance I_0 is assumed known. The boundary condition at the surface located at distance $\tau = x$ is taken to be that of isotropic (Lambertian) reflection,

$$I(x, -\mu; x) = 2\rho \int_{0}^{1} \mu'\, I(x, \mu'; x)\, d\mu', \; 0 \le \mu \le 1, \tag{3}$$

where the albedo of the ground ρ satisfies the constraint $0 \le \rho \le 1$ and also is presumed known.

The problem can be stated as follows: For either the downward, normally-directed radiance $I(x, 1; x)$ or irradiance $\int_0^1 I(x, \mu; x)\, \mu\, d\mu$ at the ground, let I^{m} be the measured value and $I^{\mathrm{c}} = I(x, \mathbf{p})$ be the corresponding calculated value, where $\mathbf{p}$ represents the known parameters for the problem: the ω, f_ℓ and ρ values. We wish to obtain the value of x such that the corresponding computed "measurement" $I^{\mathrm{c}} = I(x, \mathbf{p})$ is close to the experimental value I^{m}.

In order to solve this problem we define a least-squares functional $\tilde{F}$ of x

$$\tilde{F}(x, \mathbf{p}, I^{\mathrm{m}}) = \frac{1}{2}\, [I^{\mathrm{c}}(x, \mathbf{p}) - I^{\mathrm{m}}]^2. \tag{4}$$

The solution of the inverse problem is obtained by minimizing the value of the functional with respect to the unknown x. This is solved by an iteration procedure using $\partial_x \tilde{F} = 0$. At the start of the nth iteration the radiative transfer equation is solved using the known value $x^{(n)}$ and then the corresponding value for the functional $\tilde{F}^{(n)}$ is computed. If the convergence criteria are not satisfied, then a new value $x^{(n+1)}$ is estimated by a second-order local approximation to the functional at $x = x^{(n)}$,

$$\tilde{F}(x + \Delta x) \approx \tilde{F}(x) + \Delta x\, \partial_x \tilde{F} + \frac{1}{2}\, (\Delta x)^2\, \partial_x^2 \tilde{F}.$$

The minimization of this functional yields an algebraic equation for Δx

$$H\, \Delta x = -\, h, \tag{5}$$

where

$$h = \partial_x \tilde{F} = [I^{\mathrm{c}}(x, \mathbf{p}) - I^{\mathrm{m}}]\, \partial_x I^{\mathrm{c}}; \tag{6}$$

$$H = \partial_x^2 \tilde{F} = (\partial_x I^{\mathrm{c}})^2 + [I^{\mathrm{c}}(x, \mathbf{p}) - I^{\mathrm{m}}]\, \partial_x^2 I^{\mathrm{c}} \approx (\partial_x I^{\mathrm{c}})^2, \tag{7}$$

if the second-order derivative term is neglected in Eq. (7) so that H contains only the term of the form $(\partial_x I^{\mathrm{c}})^2$. Notice that this approximation becomes better as the iterate gets closer to the solution.

The iterations can be assumed to have converged if the measurement value is obtained within a given relative error ε_1^*,

$$\varepsilon_1 = \left| \frac{I^{\mathrm{c}} - I^{\mathrm{m}}}{I^{\mathrm{m}}} \right| < \varepsilon_1^*, \tag{8}$$

or if the functional $\tilde{F}$ is becoming stationary, i.e., if for two consecutive iterations

$$\varepsilon_2 = \left| \frac{\tilde{F}^{(n)} - \tilde{F}^{(n-1)}}{\tilde{F}^{(n-1)}} \right| < \varepsilon_2^*. \tag{9}$$

The iteration procedure is initiated by using either of the two one-dimensional explicit inverse algorithms discussed in the next section to obtain $x^{(0)}$; these algorithms are based on asymptotic radiative transfer and transport-corrected diffusion approximations and give a value for $x^{(0)}$ knowing ρ and the transmission of the cloud. The transmission $t(x)$ is the ratio of the downward irradiance at the surface where the measurements are made to that incident at the top of the atmosphere, i.e.,

$$t(x) = \frac{\int_0^{2\pi} d\phi \int_0^1 d\mu\mu\, I(x, \mu, \phi; x)}{\int_0^{-2\pi} d\phi \int_0^1 d\mu\mu\, I(0, \mu, \phi; x)} =$$

$$= \frac{\int_0^{2\pi} d\phi \int_0^1 d\mu\mu\, I(x, \mu, \phi; x)}{I_0\, \mu_0}\,. \qquad (10)$$

In the usual iteration procedure one numerically evaluates the derivatives $\partial_x I^c$ from two consecutive iterations and uses the results with the latest I^c values to estimate x, but such evaluations are often very sensitive to inaccuracies in the calculations of the derivatives. A better approach, when possible, is to develop an analytical procedure for computing the derivatives. For the azimuthally-independent problem that we consider here, the F_N method[5–8] is especially well suited for this purpose since it minimizes the required numerical calculations in the iteration scheme. This is because the matrix needed to calculate the emerging intensities and their derivatives, which depends only on the properties of the cloud, is independent of the iterations, thus leaving only new source terms to be computed with each iteration. Details about the use of the F_N method to compute these derivatives are given in Ref. 3 and summarized in the Appendix.

3. THE INITIAL ESTIMATE

This section summarizes two explicit algorithms for estimating the optical thickness of a homogeneous cloud layer of uniform depth from the transmittance irradiance ratio.[2] The first algorithm is based on asymptotic radiative transfer theory and follows from the work of King[1] who used the bidirectional reflection function; the second is based on transport-corrected diffusion theory.

The formula for estimating x using either algorithm is[2]

$$x = 1/k \ln\left[\frac{\beta}{2t(x)}\left\{1 + \left[1 + \gamma\, G(\rho)\left(\frac{2t(x)}{\beta}\right)^2\right]^{1/2}\right\}\right] \qquad (11)$$

Here k is the diffusion exponent that for a weakly absorbing atmosphere is correlated with the similarity parameter s, defined in terms of the single scattering albedo ω and the scattering asymmetry factor g as

$$s = [(1 - \omega)/(1 - \omega g)]^{1/2}. \qquad (12)$$

King and Harshvardhan[10] give an approximation for k as

$$\frac{k}{1 - \omega g} = 3^{1/2}\, s - \frac{(0.985 - 0.253\, s)\, s^2}{(6.464 - 5.464\, s)}\,. \qquad (13)$$

The parameters γ, β and $G(\rho)$ all depend on the cloud single scattering albedo and the scattering phase function, and are different for the asymptotic radiative transfer and the transport-corrected diffusion approximation algorithms.

For the asymptotic radiative transfer algorithm,[2]

$$\gamma = \ell; \qquad (14a)$$

$$\beta = mn\, K(\mu_0)/(1 - \rho\rho^*); \qquad (14b)$$

$$G(\rho) = \ell - mn^2\rho/(1 - \rho\rho^*), \qquad (14c)$$

where ℓ, m, n and ρ^*, that are defined in Ref. 9, depend on the cloud optical properties. Approximate expressions for these functions also have been correlated with the similarity parameter s. King and Harshvardhan[10] determined that

$$\ell \approx \frac{(1 - 0.681s)\,(1 - s)}{1 + 0.792s}\,; \qquad (14d)$$

$$m \approx (1 + 1.537s) \ln\left[\frac{1 + 1.800s - 7.087\, s^2 + 4.740s^3}{(1 - 0.819s)\,(1 - s)^2}\right]; \qquad (14e)$$

$$n \approx \left[\frac{(1 + 0.414s)\,(1 - s)}{1 + 1.888\, s}\right]^{1/2} \qquad (14f)$$

while van de Hulst (Ref. 9, p. 369) found that

$$\rho^* \approx (1 - 0.139\, s)\,(1 - s)/(1 + 1.170\, s). \qquad (14g)$$

All of these functions have been shown to be relatively insensitive to the higher-order coefficients in the phase function, f_n for $n \geq 2$, and so the above approximations should be good for most clouds.

The escape function $K(\mu)$ in Eq. (14b), defined in Ref. 9, also is relatively insensitive to the higher order coefficients in the phase function. For this reason it can be estimated by a fit based on the Henyey-Greenstein phase function[11] that depends on only ω and g. Using the Dlugach and Yanovitskii[12] tables of the escape functions for the Henyey-Greenstein scattering function with $\omega \geq 0.8$; $0.8 \leq g \leq 0.9$, and $\mu \geq 0.5$, the following polynomial fit was developed[2]:

$$K(\mu) \approx A(\mu) + B(\mu)\,(1 - s) + C(\mu)\,(1 - s)^2 + D(\mu)\,(1 - s)^3 \tag{14h}$$

where the coefficients are

$$A(\mu) = -\,1.1130 + 5.3924\,\mu - 9.1658\,\mu^2 + 5.4673\,\mu^3;$$
$$B(\mu) = 5.9551 - 26.488\,\mu + 46.782\,\mu^2 - 25.743\,\mu^3;$$
$$C(\mu) = -\,8.7748 + 43.229\,\mu - 73.949\,\mu^2 + 39.059\,\mu^3;$$
$$D(\mu) = 4.3639 - 21.230\,\mu + 36.285\,\mu^2 - 18.799\,\mu^3.$$

This fit agrees with the van de Hulst values for Henyey-Greenstein phase functions to < 3 % for $\omega \geq 0.8$, < 1.5 % for $\omega \geq 0.9$, and < 0.6 % for $\omega \geq 0.95$ for the given ranges of g and μ.

For the transport-corrected diffusion algorithm, on the other hand, the factors in Eq. (11) are the same as in Eq. (14) except that now

$$\gamma = r(\infty); \tag{15a}$$

$$\beta = [1 - r^2(\infty)]/[1 - \rho r(\infty)]; \tag{15b}$$

$$G(\rho) = \frac{r(\infty) - \rho}{1 - \rho r(\infty)}\,. \tag{15c}$$

Here the reflected irradiance ratio $r(\infty)$ is the calculated ratio of the upward irradiance to the downward one at the top of a semi-infinite cloud,

$$r(\infty) = \frac{\int_0^{2\pi} \mathrm{d}\phi \int_0^1 \mathrm{d}\mu\mu\, I(0, -\mu, \phi; \infty)}{\int_0^{2\pi} \mathrm{d}\phi \int_0^1 \mathrm{d}\mu\mu\, I(0, \mu, \phi; \infty)} = \frac{\int_0^{2\pi} \mathrm{d}\phi \int_0^1 \mathrm{d}\mu\mu\, I(0, -\mu, \phi; \infty)}{I_0\, \mu_0}\,. \tag{15d}$$

An approximate polynomial fit of $r(\infty)$ to the van de Hulst[9] values for a Henyey-Greenstein phase function for $\omega \geq 0.8$, $0.75 \leq g \leq 0.875$ and $\mu_0 \geq 0.5$ gave[2]

$$r(\infty) \approx a(\mu) + b(\mu)\,(1 - s) + c(\mu)\,(1 - s)^2 + d(\mu)\,(1 - s)^3, \tag{15e}$$

where the coefficients are

$$a(\mu) = 0.02101 - 0.18048\,\mu + 0.20185\,\mu^2 - 0.09142\,\mu^3;$$
$$b(\mu) = 1.5734 - 2.6548\,\mu + 2.0400\,\mu^2 - 0.41333\,\mu^3;$$
$$c(\mu) = -\,1.1479 + 3.6698\,\mu - 4.2307\,\mu^2 + 1.0236\,\mu^3;$$
$$\mathrm{d}(\mu) = 0.55283 - 0.83289\,\mu + 1.9855\,\mu^2 - 0.51758\,\mu^3.$$

This fit agrees with the van de Hulst values to < 3 % for $s \leq 0.6$ and < 1 % for $s \leq 0.3$.

The formula in Eq. (11) is useful for estimating the optical thickness x even if no iterations are performed. Both the asymptotic radiative transfer algorithm and the transport-corrected diffusion algorithm give good results for optically thick clouds; the transport-corrected diffusion algorithm tends to be better for optically thin clouds but is less accurate for intermediate cloud thicknesses than the asymptotic algorithm.

4. SENSITIVITY ANALYSIS

Let **y** stand for the known parameters ω, f_ℓ, and ρ and the measured value I^m. The impact of uncertainties in the known parameters and of measurement errors on the values of the estimated parameters is given to first-order in Δ**y** by

$$\left|\frac{\Delta x}{x}\right| \approx \sum_j s_j \left|\frac{\Delta y_j}{y_j}\right|\,. \tag{16}$$

The sensitivity coefficients s_j,

$$s_j = |(y_j/x)\,(\partial_{y_j} x)|, \tag{17}$$

give the multiplying factor that allows one to compute the relative error in x from the relative error in y_j. Since we presume the values of ω, f_ℓ and ρ are known, we are interested in the sensitivity coefficient for the optical thickness due to errors in the detector measurement I^{m}. We shall investigate these sensitivities for measurements of the radiance $I(x, 1; x)$ and the irradiance $\int_0^1 I(x, \mu; x)\, \mu \mathrm{d}\mu$.

Numerical tests were done for the Haze–L phase function for clouds[13] for a wavelength of 0.7 μm, a real refractive index of 1.33, and a particle size distribution of $4.9757 \times 10^6\, r^2 \exp(-\,15.116\, r^{1/2})$ with a mean radius of $r_c = 0.07$ μm. For this function, the 83 expansion coefficients reported by Benassi et al.[14] were used, for which the scattering asymmetry factor is $f_1 = 0.8042$. The albedo of single scattering was taken to be $\omega = 0.99999$ and the incident radiation at the top of the cloud was normally directed so that $\mu_0 = 1$. The convergence criteria in Exprs. (8) and (9) were set to $\varepsilon_1^* = 10^{-4}$ and $\varepsilon_2^* = 10^{-8}$.

The tests of the implicit iterative procedure were done for a set of selected points for the cloud by performing an F_N-calculation to obtain the simulated measurement I^{m}. Then the convergence of the functional $\tilde{F}$ was tested for different assumed initial values $x^{(0)}$. All the initial estimates were obtained by using the asymptotic algorithm to obtain the value of $x^{(0)}$.

A variety of calculations were done to explore the convergence of the iterations for different selected values of x and for the two different detectors. In all cases the iterations converged according to the criteria of Eqs. (8) and (9) within a few iterations so we conclude that the iteration scheme is stable when searching for the single variable x.

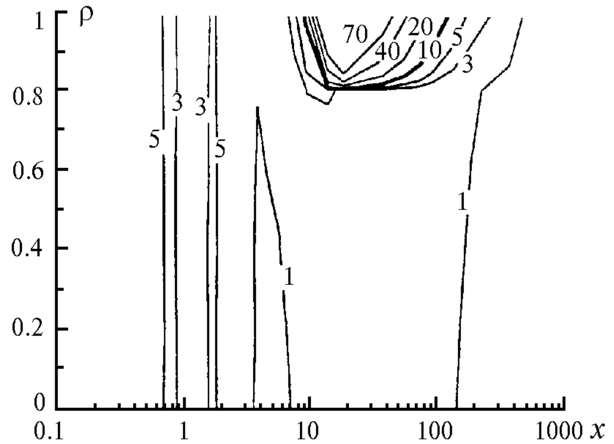

FIG. 1. Optical thickness sensitivity coefficients for a radiance detector measuring $I(x,\ 1;\ x)$ for different surface albedo values ρ and optical thicknesses x. (From Ref. 3.)

Figures 1 and 2 show the sensitivities s for different values of the surface albedo ρ for the two detectors. (The figures were constructed from linear interpolation of calculated results for a grid of 24 x and 6 ρ values, so the curves are only approximately correct.) Such graphs can be used to define the useful retrieval region in the x space in terms of the maximum admissible error in the estimated value of x, $\varepsilon_x = \Delta x/x$, and the maximum error of the detector, $\varepsilon_I = \Delta I^{\mathrm{m}}/I^{\mathrm{m}}$. For instance, if we want to estimate x within 20%, $\varepsilon_x = 0.2$, and if the precision in the detector is 2%, $\varepsilon_I = 0.02$, then the retrieval region is defined by the area under the curve for sensitivity coefficient $s = 10$ shown in bold.

From the figures it is seen that for the same precision in the measurements, the retrieval region for measurements with the normally-directed radiance detector is much larger than that for the irradiance detector. However, the transmitted signal for the radiance detector will be much smaller, which could actually lead to larger errors in the measurement.

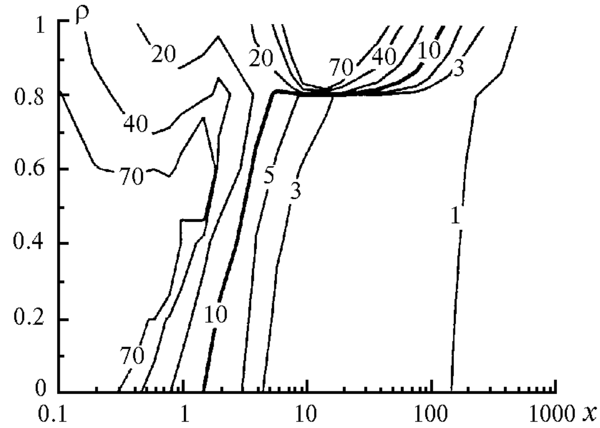

FIG. 2. Optical thickness sensitivity coefficients for an irradiance detector $\int_0^1 I(x,\ \mu;\ x)\ \mu d\mu$ for different surface albedo values ρ and optical thicknesses x. (From Ref. 3.)

5. CONCLUSIONS

An implicit inverse method has been presented for estimating the optical thickness x of a cloud layer and the feasibility of the proposed inversion technique has been demonstrated; this is a special case of an iteration approach to estimate x and ρ. An azimuthally-independent plane-parallel problem was assumed by assuming the top of the cloud layer to be uniformly illuminated and the detector to measure the ground-level, normally-directed radiance or irradiance. For such a problem the iteration procedure can be implemented using the F_N method so that the least squares minimization can be done with analytically computed derivatives obtained with the F_N method. For an azimuthally-dependent problem the F_N method becomes more time consuming, but the iteration procedure could still be used with a different method for solving the direct problem, such as the discrete ordinates method; however, then the derivatives would need to be computed numerically.

Both the asymptotic radiative transfer algorithm and the transport-corrected diffusion algorithm give good approximate results for optically thick clouds; the transport-corrected diffusion algorithm tends to be better for optically thin clouds but is less accurate for intermediate cloud thicknesses than the asymptotic algorithm.

APPENDIX: THE F_N METHOD

We follow closely the work of Devaux, Siewert, and Yuan[15] and express the outgoing radiances at the surfaces of the medium as series expansions of the form

$$I(0, -\mu) = \mathbf{p}(\mu) \bullet \mathbf{a} + \rho\Phi(x) \exp(-x/\mu), \quad 0 \le \mu \le 1, \tag{A1a}$$

$$I(x, \mu) = \mathbf{p}(\mu) \bullet \mathbf{b} + F(\mu) \exp(-x/\mu), \quad 0 \le \mu \le 1, \tag{A1b}$$

where

$$\mathbf{p}(\mu) = \{P_n(2\mu - 1),\ n = 0, N\};$$

$$\mathbf{a} = \{(\omega/2)\, a_n,\ n = 0, N\};\ \mathbf{b} = \{(\omega/2)\, b_n,\ n = 0, N\}$$

and

$$\Phi(\tau) = 2\int_0^1 \mu\, F(\mu) \exp(-\tau/\mu)\, d\mu. \tag{A2}$$

The F_N equations follow from the system of integral equations

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\, I(0, -\mu;x)\, d\mu + \exp(-x/\xi) \times$$
$$\times \int_{-1}^{1} \mu\phi(-\xi,\mu)\, I(x, \mu;x)\, d\mu = 0, \tag{A3a}$$

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\, I(x, \mu;x)\, d\mu + \exp(-x/\xi) \times$$
$$\times \int_{-1}^{1} \mu\phi(-\xi,\mu)\, I(0, -\mu;x)\, d\mu = 0, \tag{A3b}$$

where $\phi(\xi, \mu)$ are the elementary solutions of the radiative transfer equation[16,17] and ξ is either in the interval $[0, 1]$ or a discrete eigenvalue ν_i, $i = 0, 1, 2, \ldots, k - 1$, that can be calculated using a procedure such as that in Ref. 18. If Eqs. (A1) are substituted into Eqs. (A3) there follows[19,20] a set of linear equations for the $2(N + 1)$ unknowns a_n and b_n, that are solved by using $\xi = \nu_i$, $i = 0, 1, \ldots, k - 1$, plus the remaining collocation points $i = k, k + 1, \ldots, N$, which are chosen to be the zeros of the Chebyshev polynomial of the second kind $U_{N+1-k}(2x - 1)$.

The F_N equations can be written in the form

$$R\, \mathbf{z} = \mathbf{q}, \tag{A4}$$

where $\mathbf{z} = (\mathbf{a}, \mathbf{b})$; R is the matrix

$$R = \begin{bmatrix} B & DA \\ DA & B \end{bmatrix} - \rho \begin{bmatrix} 0 & D\hat{B} \\ 0 & \hat{A} \end{bmatrix} \tag{A5}$$

and D is the diagonal matrix $\{\exp(-x/\xi_i)\}$. The matrices A and B have elements

$$A_n(\xi_i) = -B_n(-\xi_i) =$$
$$= (2/\xi_i)\int_0^1 \mu\, P_n(2\mu - 1)\, \phi(-\xi_i, \mu)\, d\mu,$$

which can be calculated using recursion relations[15]; also

$$\hat{B}_n(\xi_i) = (\delta_{n\,0} + \delta_{n1}/3)\, B_0(\xi_i);$$

$$\hat{A}_n(\xi_i) = (\delta_{n\,0} + \delta_{n1}/3)\, A_0(\xi_i).$$

The source vector $\mathbf{q} = (\mathbf{q}_a, \mathbf{q}_b)$ has elements consisting of the nonsingular integrals

$$q_a(\xi_i) = e^{-x/\xi_i}\, q_b(-\xi_i) =$$

$$2\,(\omega\, \xi_i)^{-1}\int_0^1 \{F(\mu)\, \phi(-\xi_i, \mu)\, [1 - e^{-x/\xi_i}\, e^{-x/\mu}] +$$
$$+ \rho\, \Phi(x)\, \phi(\xi_i, \mu)\, [e^{-x/\xi_i} - e^{-x/\mu}]\}\, \mu d\mu. \tag{A6}$$

To perform the minimization procedure we need to compute the values of $\partial_x I^c$. For example, from Eq. (A1) these values are obtained for the detector $\int_0^1 I(x, \mu; x)\, \mu d\mu$ as

$$\partial_x I^c = \partial_x \mathbf{a} \left[\int_0^1 \mathbf{p}(\mu)\, \mu d\mu\right] + \rho\partial_x \left[\phi(x) \int_0^1 e^{-x/\mu}\, \mu d\mu\right],$$
$$\tau = 0 \tag{A7a}$$

and

$$\partial_x I^c = \partial_x \mathbf{b} \left[\int_0^1 \mathbf{p}(\mu)\, \mu d\mu\right] + \partial_x \left[\int_0^1 F(\mu)\, e^{-x/\mu}\, \mu d\mu\right],$$
$$\tau = x. \tag{A7b}$$

The values of $\partial_x\, \mathbf{a}$ and $\partial_x\, \mathbf{b}$ are obtained from the system of equations

$$R\, \partial\mathbf{z}/\partial x_k = \partial\mathbf{q}/\partial x_k - \partial R/\partial x_k\, \mathbf{z}, \tag{A8}$$

where

$$\partial R/\partial x = \begin{bmatrix} 0 & D'A \\ D'A & 0 \end{bmatrix} - \rho \begin{bmatrix} 0 & D'\hat{B} \\ 0 & 0 \end{bmatrix}; \quad \text{(A9a)}$$

$$\partial R/\partial \rho = -\begin{bmatrix} 0 & D\hat{B} \\ 0 & A' \end{bmatrix} \quad \text{(A9b)}$$

and $D' = \text{diag}\,\{-\exp(-x/\xi_i)/\xi_i\}$.

Since the inverse matrix R^{-1} is also required for the computation of **z**, only the relatively small additional numerical effort to compute $\partial \mathbf{q}/\partial x_k$ will be necessary to estimate a new iterate.

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