`SYNTHESIS OF THE OPTIMAL BASIS TO RECONSTRUCT RANDOM WAVE FIELDS

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We present algorithms for calculating the transformation matrixes that relate the coefficients of the optimal Karhunen-Loeve-Obukhov basis to those of the Walsh and Haar bases in the series expansions of the optical wave phase. In so doing, we have made numerical simulations for Kolmogorov spectrum of turbulence and round receiving aperture. Analysis of the numerical results shows that the algorithms developed allow the transformation matrixes to be calculated with the accuracy that could suit the control of the wave fronts.

In the turbulent atmosphere we normally observe distortions of optical radiation propagated through it because of the random optical properties of the medium. To study such random wave fields the series expansion of a wave phase is often used over basis functions.[1,2]

The orthogonal normalized system of functions that obeys the Karhunen-Loeve-Obukhov (KLO) theorem is considered to be optimal for representing the waves propagated through a random medium.[3,4] The KLO basis is a solution to the variation problem by minimizing the norm of error in the series expansion of the random phase, within a receiving aperture with the *a priori* information in the form of the phase correlation function. In our earlier papers[2,5] we have solved the problem on obtaining the KLO functions for the medium with Kolmogorov turbulence.

But, the KLO basis, while being optimal has no properties of a fast transform that allows the wave dynamics to be observed in real time. Therefore we have derived analytical relations[6] enabling one to pass, in the expansions of the optical wave phase, from the bases of Walsh functions and Haar wavelets, whose series expansions are of the fast type, to the statistically optimal KLO basis.

It is more suitable for the round aperture to present the KLO functions in factored form by separating the radial and azimuth components

$$\psi_k(\boldsymbol{\rho}) = R_j(\rho)\, \Theta^l(\theta)\ , \qquad (1)$$

where $\boldsymbol{\rho} = \{x, y\} = (\rho, \theta)$. Walsh functions $Wal(\boldsymbol{\rho})$ and Haar functions $H(\boldsymbol{\rho})$ are presented in a similar way

$$Wal_{nm}(\boldsymbol{\rho}) = Wal_n(\rho) Wal_m(\theta),$$

$$H_{nm}(\boldsymbol{\rho}) = H_n(\rho) H_m(\theta).$$

The azimuth component of the KLO function $\Theta^l(\theta)$ has the following form:

$$\Theta^l(\theta) = \exp\,(il\theta),\ l \in \mathbf{Z}\ . \qquad (2)$$

Expansion of the functions $\Theta^l(\theta)$ into a series over Walsh functions yields the azimuth transformation matrix $\mathbf{b}^l = (b_1^l, b_2^l, \ldots, b_N^l)$

$$\Theta^l(\theta) = \sum_{n=0}^{N} b_n^l\, Wal_n(\theta). \qquad (3)$$

The relation of the exponential functions to Walsh functions determined by the matrix $\mathbf{b}^l$ is widely used in the fast transforms and may be readily found in the literature.[7,8]

Transformation matrixes for azimuth components of the KLO functions in terms of the Haar functions $H_m(\theta)$ are calculated similarly.

The radial component of the KLO functions may be presented, in terms of the Walsh functions, in the following form:

$$R_j^l(\rho) = \sum_{n=0}^{N} d_{jn}^l\, Wal_n(\rho), \qquad (4)$$

where the coefficients $\mathbf{d}_j^l = (d_{j1}^l, d_{j2}^l, \ldots, d_{jN}^l)$ are the eigenvectors of the Gram matrix with the elements

$$A_{ps}^l = \frac{1}{N^2} \int_0^1 \int_0^1 \rho M_l(\rho, \rho') \times Wal_p(\rho)\, Wal_s(\rho')\, d\rho\, d\rho'. \qquad (5)$$

Here $M_l(\rho, \rho')$ is the kernel of the homogeneous Fredholm integral equation of the second kind (the formula (10) in Ref. 6), the coordinate ρ is normalized by the receiving aperture radius.

We have made use of the Jacobi method[9] to obtain the eigenvalues and eigenvectors of the Gram matrix when making numerical simulations.

Numerical simulation of the KLO functions through the Walsh and Haar functions was performed

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for the Kolmogorov turbulence spectrum. Assuming this model the structure function has the following form[1]:

$$D(\rho) = \frac{6.88}{r_0^{5/3}}\,\rho^{5/3}\,,$$

where r_0 is the Fried radius. The form of the kernel $M_l(\rho, \rho')$ for the model of Kolmogorov turbulence may be found in Ref. 6.

Figure 1 shows the view of the radial components for the first KLO functions expanded over 8 (Fig. 1*a*) and 32 (Fig. 1*b*) Walsh functions. One can see from the figure that for $N = 32$ the radial components of the KLO functions $R_j^l(\rho)$ practically coincide with the radial components of the KLO functions calculated with high accuracy in Refs. 5 and 10, and much exceed the accuracy of the optimal function calculated in Ref. 11.

Below we present an explicit form of the transformation matrixes for radial components of the KLO functions $R_j^l(\rho)$ in terms of the Walsh functions for $j = \overline{1, N}$, $N = 8$, and several first azimuth indexes l

$$\begin{pmatrix} R_1^{\pm1} \\ R_2^{\pm1} \\ R_3^{\pm1} \\ R_4^{\pm1} \\ R_5^{\pm1} \\ R_6^{\pm1} \\ R_7^{\pm1} \\ R_8^{\pm1} \end{pmatrix} = \begin{pmatrix} 0.850 & -0.461 & -0.228 & -0.029 & -0.114 & -0.014 & -0.006 & 0.002 \\ 0.340 & 0.584 & 0.144 & -0.632 & 0.086 & -0.306 & -0.109 & -0.106 \\ 0.127 & 0.501 & -0.528 & 0.414 & -0.148 & 0.121 & 0.009 & -0.496 \\ -0.052 & -0.096 & 0.166 & -0.313 & -0.329 & 0.259 & 0.740 & -0.374 \\ -0.017 & -0.223 & 0.027 & 0.110 & 0,723 & -0.377 & 0.226 & -0.470 \\ -0.027 & -0.097 & -0.087 & -0,336 & 0.374 & 0.747 & -0.347 & -0.227 \\ -0.134 & 0.051 & -0.698 & -0.283 & 0.250 & -0.017 & 0.381 & 0.454 \\ 0.354 & 0.354 & 0.355 & 0.354 & 0.353 & 0.354 & 0.353 & 0.353 \end{pmatrix} \times \begin{pmatrix} Wal_1 \\ Wal_2 \\ Wal_3 \\ Wal_4 \\ Wal_5 \\ Wal_6 \\ Wal_7 \\ Wal_8 \end{pmatrix},$$

$$\begin{pmatrix} R_1^{\pm2} \\ R_2^{\pm2} \\ R_3^{\pm2} \\ R_4^{\pm2} \\ R_5^{\pm2} \\ R_6^{\pm2} \\ R_7^{\pm2} \\ R_8^{\pm2} \end{pmatrix} = \begin{pmatrix} 0.798 & -0.550 & -0.219 & 0.010 & -0.107 & 0.008 & 0.010 & 0.050 \\ -0.346 & -0.389 & -0.183 & 0.753 & -0.126 & 0.334 & -0.015 & -0.029 \\ -0.178 & -0.478 & 0.575 & -0.133 & 0.131 & -0.051 & -0.359 & 0.493 \\ 0.082 & 0.236 & -0.440 & 0.280 & 0.399 & -0.358 & -0.523 & 0.324 \\ 0.112 & 0.003 & 0.084 & -0.131 & 0.398 & 0.544 & -0.504 & -0.506 \\ 0.017 & 0.259 & -0.230 & -0.221 & -0.524 & 0.522 & -0.282 & 0.454 \\ -0.252 & -0.257 & -0.450 & -0.374 & 0.498 & 0.262 & 0.382 & 0.257 \\ 0.360 & 0.360 & 0.364 & 0.362 & 0.341 & 0.348 & 0.344 & 0.348 \end{pmatrix} \times \begin{pmatrix} Wal_1 \\ Wal_2 \\ Wal_3 \\ Wal_4 \\ Wal_5 \\ Wal_6 \\ Wal_7 \\ Wal_8 \end{pmatrix},$$

$$\begin{pmatrix} R_1^{\pm3} \\ R_2^{\pm3} \\ R_3^{\pm3} \\ R_4^{\pm3} \\ R_5^{\pm3} \\ R_6^{\pm3} \\ R_7^{\pm3} \\ R_8^{\pm3} \end{pmatrix} = \begin{pmatrix} 0.754 & -0.604 & -0.212 & 0.071 & -0.110 & 0.032 & -0.003 & 0.071 \\ -0.340 & -0.202 & -0.286 & 0.776 & -0.135 & 0.354 & -0.051 & -0.117 \\ -0.189 & -0.344 & 0.411 & 0.056 & 0.178 & -0.019 & -0.612 & 0.517 \\ -0.111 & -0.288 & 0.567 & -0.234 & -0.429 & 0.494 & 0.225 & -0.225 \\ -0.208 & -0.409 & 0.275 & 0.229 & 0.218 & -0.573 & 0.531 & -0.060 \\ -0.123 & 0.086 & -0.165 & -0.010 & -0.615 & -0.144 & 0.284 & 0.687 \\ 0.254 & 0.269 & 0.356 & 0.358 & -0.485 & -0.428 & -0.328 & -0.286 \\ 0.382 & 0.384 & 0.392 & 0.392 & 0.305 & 0.312 & 0.321 & 0.326 \end{pmatrix} \times \begin{pmatrix} Wal_1 \\ Wal_2 \\ Wal_3 \\ Wal_4 \\ Wal_5 \\ Wal_6 \\ Wal_7 \\ Wal_8 \end{pmatrix}.$$

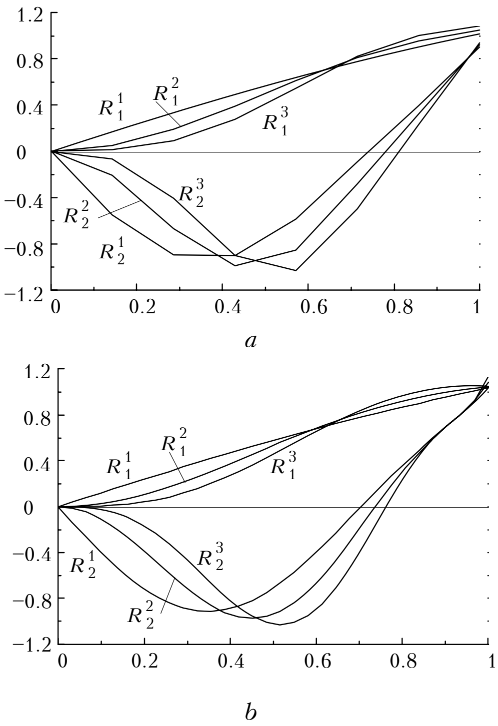

FIG. 1. The view of radial components of the KLO functions in the expansion over N Walsh functions: N = 8 (a), N = 32 (b).

It should be noted that the sum of squares of the elements in every line and column is the norm of the function $R_j^l(\rho)$. This value is constant and in this case it equals to unit because the basis is orthonormal

$$\sum_{n=1}^{N} (d_{jn}^{\,l})^2 = 1.$$

It is clear that the closeness to unit can serve as the precision criterion of the calculations of the KLO basis expansion coefficients.

Similarly were obtained the coefficients of $R_j^l(\rho)$ expansion over the Haar wavelets.

But, if we have the expansion coefficients of the radial component $R_j^l(\rho)$ over the Walsh functions the expansion coefficients of $R_j^l(\rho)$ in the Haar basis can be obtained in different way using a close relation between the Walsh and Haar functions written by the transformation matrix.[12]

Figure 2 presents a 3-D view of the first KLO functions $\psi_k(\rho)$ represented using a limited number of the Haar functions. One can see from this figure that these KLO functions practically coincide with the KLO functions calculated highly accurate using Zernike polynomials in Ref. 5.

In the papers devoted to the choice optimal basis for the wave phase series expansion the KLO functions are obtained from the numerical solution of the integral equation[10] or using approximations, for their calculation.[11,13] We have developed an effective method to calculate the functions of an optimal basis. Using this method we have calculated, in the numerical experiment, the KLO functions through the Walsh functions and Haar wavelets for the Kolmogorov atmospheric turbulence within a round receiving aperture to a high degree of accuracy.

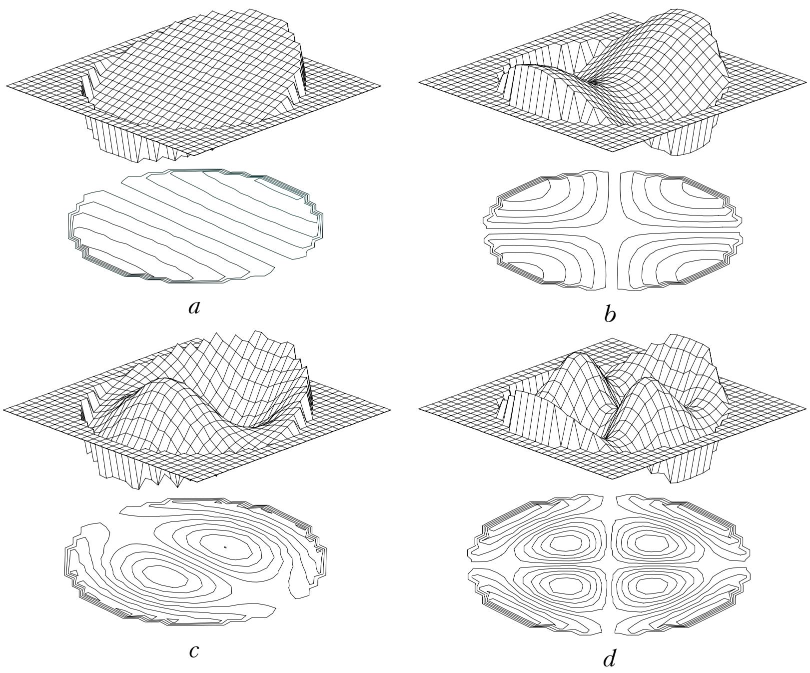

FIG. 2. Spatial form of the KLO functions $\psi_k(\rho)$ represented in the basis of Haar functions $H_{nm}(\rho)$, N = 32, k equals to 2, 4, 8, and 13 for a, b, c, and d, respectively.

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