APPROXIMATE METHOD FOR SOLVING THE PARABOLIC EQUATION FOR A MONOCHROMATIC ELECTROMAGNETIC WAVE PROPAGATION THROUGH A NONLINEAR MEDIUM

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A method is proposed for solving the parabolic equation for propagation of a monochromatic electromagnetic wave through a nonlinear medium. Two cases are discussed, namely, the absence and the presence of the wind in the atmosphere. In the former case the use of an axially symmetric beam model seems to be justifiable and the parabolic equation reduces to the Bernoulli equation. In the second case (in the presence of wind) the equation reduces to an integro–differential equation, which can be solved by the iteration method.

Traditionally all the methods for solving the parabolic equation dealt with linear media.[1, 2] That means that the dielectric constant of a medium is taken to be independent of the electromagnetic wave intensity.

This paper presents an approximate method for solving the parabolic equation for the case of nonlinear media. This task is very important when solving problems in atmospheric optics[3–5] and in the studies of the Earth's crust.[6,7] However, in contrast to atmospheric optics, the propagation of electromagnetic waves in the nonlinear (but homogeneous) earth's crust has been only poorly studied.

We take the parabolic equation for the case of an axially symmetric problem as the basis for our study. As shown in the monograph,[4] the model of an axially symmetric beam is not always valid. For instance, in the presence of wind, as it follows from the calculations in the approximation of a thin lens, the beam, while propagating in the medium, will not keep its axial symmetry (see Ref. 4). In this case we use the other approach that is based on the results obtained in the monograph by V.I. Klyatskin.[12]

Let us first assume that axial symmetry holds (there is no wind in the atmosphere). Then the initial parabolic equation is as follows:

$$\frac{\partial^2\tilde{E}}{\partial r^2} + \frac{2}{r}\frac{\partial\tilde{E}}{\partial r} + 2i\,\tilde{k}_0\frac{\partial\tilde{E}}{\partial z} + \tilde{k}_0^2\,\tilde{\tilde{\varepsilon}}(r, z)\,\tilde{E} = 0. \qquad (1)$$

Here

$$\tilde{k}_0^2 = \langle\varepsilon(r)\rangle\, k_0^2;\;\; k_0 = \frac{w_0}{c};\;\; \tilde{\tilde{\varepsilon}}(r) = \frac{\varepsilon(r) - \langle\varepsilon(r)\rangle}{\langle\varepsilon(r)\rangle}.$$

We assume that for a randomly inhomogeneous medium the dielectric constant is

$$\varepsilon(r) = \langle\varepsilon(r)\rangle + \tilde{\varepsilon}(r),$$

where $\langle\varepsilon(r)\rangle$ is the average value of the constant, and $\tilde{\varepsilon}(r)$ is its fluctuating part. But, in a nonlinear medium we have

$$\tilde{\tilde{\varepsilon}}(r, z) = \tilde{\tilde{\varepsilon}}_1(r, z) + \tilde{\tilde{\varepsilon}}_2\,(r, z)\,|E|^2. \qquad (2)$$

Then Eq. (1) takes the form

$$\frac{\partial^2\tilde{E}}{\partial r^2} + \frac{2}{r}\frac{\partial\tilde{E}}{\partial r} + 2i\,\tilde{k}_0\frac{\partial\tilde{E}}{\partial z} + \tilde{k}_0^2\,\tilde{\tilde{\varepsilon}}_1(r, z)\,\tilde{E} +$$

$$+ \tilde{k}_0^2\,\tilde{\tilde{\varepsilon}}_2(r, z)\,|\tilde{E}|^2\,\tilde{E} = 0. \qquad (3)$$

At $\tilde{\tilde{\varepsilon}}_1(r, z) = 0$, $\tilde{\tilde{\varepsilon}}_2(r, z) = \beta_0$ this equation was first studied by A.M. Prokhorov and V.N. Lugovoy[8] and at $\tilde{\tilde{\varepsilon}}_1(r, z) = 0$, $\tilde{\tilde{\varepsilon}}_2(r, z) = i\beta_0$ this equation was used for studying the influence of nonlinearity of an active medium on the spatial structure of electromagnetic field in a resonator.[9] In this case the two-dimensional problem is to be considered.

Let us next represent the scalar wave field in Eq. (3) as

$$\tilde{E}(r, z) = A(r, z)\exp\,[i\Phi(r, z)], \qquad (4)$$

where $A(r, z)$ is the wave amplitude; $\Phi(r, z)$ is the wave phase; in this case we believe that $A(r, z)$ and $\Phi(r, z)$ are real numbers. Omitting calculation of the derivatives and separating real and imaginary parts, taking into account Eq. (4), the initial equation may be presented in the following form. For the real part we have

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$$\frac{\partial^2 A}{\partial r^2} + \frac{2}{r}\frac{\partial A}{\partial r} + \varkappa(r, z)\, A + \beta(r, z)\, A^3 = 0, \quad (5)$$

where

$$\varkappa(r, z) = \tilde{k}_0^2\, \approx{\varepsilon}_1(r, z) - \left(\frac{\partial \Phi}{\partial r}\right)^2 - 2\,\frac{\partial \Phi}{\partial z}\,; \quad (6)$$

$$\beta(r, z) = \tilde{k}_0^2\, \approx{\varepsilon}_2(r, z).$$

For the imaginary part the following equation is obtained:

$$2\,\frac{\partial A}{\partial r}\frac{\partial \Phi}{\partial r} + 2\,\tilde{k}_0\,\frac{\partial A}{\partial z} + \left(\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r}\right) A = 0. \quad (7)$$

First we consider Eq. (5). This equation will be solved using the substitution

$$r = a_0\,\tau^k;\;\; A = b_0\,\tau^c\,M^p\;(\tau > 0;\, M > 0). \quad (8)$$

Here a_0, b_0, k, c, and p are the real parameters.

At $p = 1$ such a substitution was made in the Ref. 10. The author of this paper generalized it for the case of arbitrary p values. Here τ is the new argument; M is the new function. Substituting Eq. (8) into the Eq. (5) we bring this equation to the form

$$\frac{\mathrm{d}^2 M}{\mathrm{d}\tau^2} + \frac{(p-1)}{M}\left(\frac{\mathrm{d}M}{\mathrm{d}\tau}\right)^2 + \frac{(2c+k+1)}{\tau}\frac{\mathrm{d}M}{\mathrm{d}\tau} +$$

$$+ \frac{c\,(c+k)}{p\tau^2}\,M + \frac{a_0^2\,k^2}{p}\,\varkappa(\tau, z)\,\tau^{2(k-1)}\,M +$$

$$+ \frac{a_0^2\,k^2\,b_0^2}{p}\,\beta(\tau, z)\,\tau^{2(k+c-1)}\,M^{2p+1} = 0. \quad (9)$$

No exact solution of Eq. (9) can be achieved, in the general case. Therefore Eq. (9) will be solved using an approximate method. Let us suppose that the following inequality holds:

$$\frac{\mathrm{d}^2 M}{\mathrm{d}\tau^2} \ll \frac{(p-1)}{M}\left(\frac{\mathrm{d}M}{\mathrm{d}\tau}\right)^2. \quad (10)$$

When solving such an inequality we obtain

$$\frac{M^{2-p}}{2-p} \ll C_1\,\tau + C_2, \quad (11)$$

where C_1, C_2 are the constant values determined from the boundary conditions. The constants C_1 and C_2 may be chosen so that the inequality (11) takes place. Then we obtain the quadratic equation

$$\left(\frac{\mathrm{d}M}{\mathrm{d}\tau}\right)^2 + \frac{M}{(p-1)}\frac{(2c+k+1)}{\tau}\frac{\mathrm{d}M}{\mathrm{d}\tau} +$$

$$+ \frac{c(c+k)}{p(p-1)\,\tau^2}\,M^2 + \frac{a_0^2\,k^2\,\varkappa(\tau,z)}{p(p-1)}\,\tau^{2(k-1)}\,M^2 +$$

$$+ \frac{a_0^2\,k^2\,\beta(\tau, z)\,b_0^2}{p(p-1)}\,\tau^{2(k+c-1)}\,M^{2p+2} = 0. \quad (12)$$

If this equation is solved as a quadratic one, relative to $\mathrm{d}M/\mathrm{d}\tau$, and the radicand is expanded into a power series, the equation reduces to the Bernoulli differential equation, which can easily be integrated.

When solving this quadratic equation we obtain

$$\frac{\mathrm{d}M}{\mathrm{d}\tau} = -\,\frac{M\,(2c+k+1)}{2(p-1)\,\tau} \pm \frac{M}{2}\times$$

$$\times\sqrt{f(\tau, z) - \beta(\tau, z)\,g(\tau, z)\,M^{2p}}. \quad (13)$$

The notations used here are as follows:

$$f(\tau, z) = \frac{(2c+k+1)^2}{(p-1)^2\,\tau^2} - \frac{4c\,(c+k)}{p(p-1)\,\tau^2} -$$

$$-\,4\varkappa(\tau, z)\,\frac{a_0^2\,k^2}{p(p-1)}\,\tau^{2(k-1)}; \quad (14)$$

$$g(\tau, z) = \frac{4a_0^2\,k^2\,b_0^2}{p(p-1)}\,\tau^{2(k+c-1)}. \quad (15)$$

Assume that $a_0 > 0$, $b_0 > 0$, $p > 1$. Then the inequality $g(\tau, z) > 0$ always holds. The inequality

$$f(\tau, z) = \frac{(2c+k+1)^2}{(p-1)^2\,\tau^2} - \frac{4c\,(c+k)}{p(p-1)\,\tau^2} - 4\,\tilde{k}_0^2\,\approx{\varepsilon}_1(r, z)\times$$

$$\times\frac{a_0^2\,k^2}{p(p-1)}\,\tau^{2(k-1)} + 4\left(\frac{\partial \Phi}{\partial r}\right)^2\frac{a_0^2\,k^2}{p(p-1)}\,\tau^{2(k-1)} +$$

$$+\,2\,\frac{\partial \Phi}{\partial z}\,\frac{a_0^2\,k^2}{p(p-1)}\,\tau^{2(k-1)} \quad (16)$$

can be either positive or a negative function.

If $f(\tau, z) < 0$, that at $\beta(\tau, z) > 0$, $g(\tau, z) > 0$ the radicand is negative and the equation becomes complex that contradicts the accepted condition that the function M is real. Therefore we shall assume that it is possible to select such parameters when $f(\tau, z) > 0$. The radicand at the integer $p = p_0 > 1$ transforms to the following expression:

$$\sqrt{f(\tau, z) - \beta(\tau, z)\,g(\tau, z)\,M^{2p_0}} =$$

$$= \sqrt{f(\tau, z)}\left[1 - \frac{\beta(\tau, z)\,g(\tau, z)}{f(\tau, z)}\,M^{2p_0}\right]^{1/2}. \quad (17)$$

Assuming that

$$\left|\frac{\beta(\tau, z)\,g(\tau, z)}{f(\tau, z)}\,M^{2p_0}\right| \ll 1, \quad (18)$$

Eq. (17) can be expanded into a series. Then we have

$$\left[1-\frac{\beta(\tau, z)\, g(\tau, z)}{f(\tau, z)} M^{2p_0}\right]^{1/2} \approx 1-\frac{\beta(\tau, z)\, g(\tau, z)}{2\, f(\tau, z)} M^{2p_0}$$
.(19)

Therefore, taking account of Eq. (19), Eq. (13) reduces to the Bernoulli equation:

$$\frac{\mathrm{d}\, M}{\mathrm{d}\tau} + \frac{(2c + k + 1)}{2(p - 1)\,\tau} M \mp \frac{M}{2} \sqrt{f(\tau, z)} \pm$$

$$\pm \frac{1}{4\sqrt{f(\tau, z)}}\, \beta(\tau, z)\, g(\tau, z)\, M^{2p_0+1}. \tag{20}$$

This equation can be reduced to an ordinary linear differential equation of the first order by making the substitution[11]

$$y = M^{1-n}; \quad n = 2\, p_0 + 1. \tag{21}$$

Then Eq. (21) takes the form

$$\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{(2c + k + 1)}{2(p - 1)} \frac{(1 - n)}{\tau}\, y \mp \frac{1}{2} (1 - n) \sqrt{f(\tau, z)}\; y \pm$$

$$\pm \frac{\beta(\tau, z)\,(1 - n)}{4\sqrt{f(\tau, z)}}\, g(\tau, z) = 0. \tag{22}$$

Having solved Eq. (22) with the boundary condition

$$y(\tau, z)\,|_{\tau\, =\, \tau_0} = y(\tau_0, z), \tag{23}$$

we obtain the following expression:

$$y(\tau, z) = \exp\left[-\int\limits_{\tau_0}^{\tau} C_0(\tau, z)\, \mathrm{d}\tau\right] \times$$

$$\times \left\{ y(\tau_0, z) - \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau'\, D_0(\tau', z) \times \right.$$

$$\left. \times \exp\left[\int\limits_{\tau_0}^{\tau'} C_0(\tau'', z)\, \mathrm{d}\tau''\right] \right\}. \tag{24}$$

Here the following designations are introduced:

$$C_0(\tau, z) = \frac{1}{2} \frac{(1 - n)}{\tau(p_0 - 1)} \times$$

$$\times \{(2c + k + 1) \mp (p_0 - 1)\, \tau \sqrt{f(\tau, z)}\}; \tag{25}$$

$$D_0(\tau, z) = \frac{\beta(\tau, z)}{4\sqrt{f(\tau, z)}}\, (1 - n)\, g(\tau, z). \tag{26}$$

Equation (24) can be rewritten in the old variables

$$A(r, z) = b_0\, \tau^c \exp\left[-\frac{1}{2}\int\limits_{\tau_0}^{\tau} C_0(\tau, z)\, \mathrm{d}\tau\right] \times$$

$$\times \left[\{A^{-2}(\tau_0, z)\, b_0^2\, \tau_0^{2c}\} - \int\limits_{\tau_0}^{\tau} \mathrm{d}\tau\, D_0(\tau, z) \times \right.$$

$$\left. \times \exp\left[\int\limits_{\tau_0}^{\tau} \mathrm{d}\tau'\, C_0(\tau', z)\right] \right]^{-1/2}, \tag{27}$$

where

$$\tau = [a_0^{-1}\, r]^{1/k}; \;\; \mathrm{d}\tau = \frac{1}{k}\, a_0^{-1/k}\, r^{1/k-1}\, \mathrm{d}r.$$

Unfortunately, we do not know the law that describes the wave phase. For simplicity, we assume that the phase is given by the parabolic law

$$\Phi(r, z) = \Phi_0(z) + \frac{r^2}{2a^2}\, \Phi_1(z), \tag{28}$$

whence it follows that

$$\frac{\partial\Phi}{\partial r} = \frac{r}{a^2}\, \Phi_1(z), \quad \frac{\partial^2\Phi}{\partial r^2} = \frac{1}{a^2}\, \Phi_1(z).$$

From Eq. (7) we have for the imaginary part that

$$2\, \frac{\partial A}{\partial r} \frac{r}{a^2}\, \Phi_1(z) + 2\, \tilde{k}_0 \frac{\partial A}{\partial z} + \left[\frac{1}{a^2}\, \Phi_1(z) + \frac{2}{a^2}\, \Phi_1(z)\right] A = 0,$$

from that we find

$$\Phi_1(z) = -\frac{2\, \tilde{k}_0 \dfrac{\partial A}{\partial z}}{\dfrac{3}{a^2}\, A + \dfrac{\partial A}{\partial r} \dfrac{r}{a^2}}. \tag{29}$$

Given A, $\frac{\partial A}{\partial r}$, and $\frac{\partial A}{\partial z}$, using this formula, we find $\Phi_1(z)$. At $r = 0$, $\Phi(r, z) = \Phi_0(z)$.

Now we consider the case when there is a wind in the atmosphere and the axial symmetry cannot be used. Let us consider the parabolic equation[12]

$$\Delta_\perp U + 2i\, \tilde{k}\, \frac{\partial U}{\partial z} + \tilde{k}^2\, \tilde{\tilde{\varepsilon}}(z, \rho)\, U(z, \rho) = 0. \tag{30}$$

In this case $\Delta_\perp U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}$; $\rho = \{x, y\}$. We use the V.I. Klyatskin method[12] and write this equation with the boundary condition

$$U(0,\rho)=U_0(\rho) \tag{31}$$

in the form of the integro-differential equation

$$U(z,\rho)=U_0(\rho)\exp\left[i\frac{\tilde{k}}{2}\int_0^z \mathrm{d}\xi\,\tilde{\tilde{\varepsilon}}(\xi,\rho)\right]+$$

$$+\frac{i}{2\tilde{k}}\int_0^z \mathrm{d}\xi\,\exp\left[i\frac{\tilde{k}}{2}\int_\xi^z \mathrm{d}\eta\,\tilde{\tilde{\varepsilon}}(\eta,\rho)\right]\Delta_\perp U(\xi,\rho). \tag{32}$$

And now assume that

$$\tilde{\tilde{\varepsilon}}(z,\rho)=\tilde{\tilde{\varepsilon}}_1(z,\rho)+\tilde{\tilde{\varepsilon}}_2(z,\rho)\,|\,U(z,\rho)\,|^2. \tag{33}$$

Substituting Eq. (33) into Eq. (32) we obtain a very complicated integro–differential equation

$$U(z,\rho)=U_0(\rho)\exp\left[i\frac{\tilde{k}}{2}\int_0^z \mathrm{d}\xi\,\{\tilde{\tilde{\varepsilon}}_1(\xi,\rho)+\right.$$

$$\left.+\tilde{\tilde{\varepsilon}}_2(\xi,\rho)\,|\,U(\xi,\rho)\,|^2\}\right]+\frac{i}{2\tilde{k}}\int_0^z \mathrm{d}\xi\,\times$$

$$\times\exp\left[i\frac{\tilde{k}}{2}\int_\xi^z \mathrm{d}\eta\,\{\tilde{\tilde{\varepsilon}}_1(\eta,\rho)+\right.$$

$$\left.+\tilde{\tilde{\varepsilon}}_2(\eta,\rho)\,|\,U(\eta,\rho)\,|^2\}\right]\Delta_\perp U(\xi,\rho), \tag{34}$$

and this equation can be solved only using the iteration method. In the first approximation we have

$$U(z,\rho)=U_0(\rho)\exp\left[i\frac{\tilde{k}}{2}\int_0^z \mathrm{d}\xi\,\{\tilde{\tilde{\varepsilon}}_1(\xi,\rho)+\right.$$

$$\left.+\tilde{\tilde{\varepsilon}}_2(\xi,\rho)\,|\,U(0,\rho)\,|^2\}\right]+\frac{i}{2\tilde{k}}\int_0^z \mathrm{d}\xi\,\times$$

$$\times\exp\left[i\frac{\tilde{k}}{2}\int_\xi^z \mathrm{d}\eta\,\{\tilde{\tilde{\varepsilon}}_1(\eta,\rho)+\right.$$

$$\left.+\tilde{\tilde{\varepsilon}}_2(\eta,\rho)\,|\,U(0,\rho)\,|^2\}\right]\Delta_\perp U(0,\rho), \tag{35}$$

and so on.

Special attention must be given to Ref. 13, that has stimulated our investigations. The randomly inhomogeneous media are not considered in the paper but the propagation of soliton wave is studied based on the generalized nonlinear Schrodinger wave equation for the homogeneous medium, Refs. 14 and 15.

It is not difficult to generalize this approach for the case of a pulsed wave propagation in conducting (randomly inhomogeneous) nonlinear media that is very important when using MHD generators for investigating the Earth's crust and in laser applications to atmospheric studies.

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