INVERSE REFRACTION PROBLEM IN THE PARTIAL IMMERSION GEOMETRY

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We present here the theory of the refraction inverse problem developed for the immersion geometry when either the receiver or a source change its position inside the atmosphere to be sounded. We consider the problem for the case when the refraction is set only within some portion of the entire altitude range studied. The inverse problem reduces to solution of integral equation of the 1st kind that is different inside and outside the altitude interval of sounding. We also present here analytical solutions obtained for the region inside the observation interval when assuming the receiving angle to be constant. Some results of numerical simulations are given in the paper as well.

The refractometric techniques of studying the atmospheres of the Earth and other planets have recently received a wide use with the development of space researches. Solving the inverse refraction problems enables one to reconstruct vertical distribution of the refractive index and the related meteorological parameters of the atmosphere as well. Peculiarities of any concrete inverse problem and the type of equation to be solved are determined by the relative position of a radiation source and a receiver, the quantity measured, and the radiation frequency range.

The refractometric measurement technique was tried, for the first time, in explorations of the Solar system planets with space vehicles (see Refs. 1–3). Then similar methods were developed for studying the Earth's atmosphere (Refs. 4–7). In these limb measurements the radiation source and the receiver are outside the atmosphere under study. In this case the problem can be reduced to solving the Abel equation, that is to the mathematically correct problem.

In the case when the receiver is on the Earth's surface the problem on reconstructing the refractive index profile is described by the Fredholm equation of the 1st kind. Its solution is already an ill-posed problem.[8–10]

In Ref. 11 we have already considered the refraction inverse problem for the case when a radiation source or the receiver change its position inside the atmosphere (the immersion geometry). This case may be rather important when investigating the planets by means of descending vehicles. We showed that in this case the measured quantities and the values to be reconstructing are related by Volterra integral equation of the 2nd kind. We have also developed there the solution algorithms and performed some numerical simulations.

In this paper we generalize the statement of the problem to the case when the refraction is known not along the whole atmospheric path, but only within a finite portion of it.

To solve the problem, we applied Tikhonov method of the generalized discrepancy.[12] We have also investigated, using numerical simulations, the accuracy that may be achieved and the optimal conditions for reconstruction of the refractive index of the Earth's atmosphere. These calculations were made for the case of solving the problem on the whole height interval considered, and when considering only the atmospheric layers above (outer region) and below (inner region) the upper measurement boundary, depending on the position of the latter. In the case when the elevation angle of refraction measurements is constant we deduced the conversion formula for the inverse problem considered, i.e., we represented the refractive index profile inside the inner region through the refraction value measured.

STATEMENT OF THE PROBLEM

The solution of the refraction inverse problem in the immersion geometry reduces to Volterra equation of the 2nd kind (see Ref. 11)

$$N(p_h) - \int\limits_{p_h}^{\infty} N(p) \frac{p p_h \cos\theta_0(p_h)}{\left[p^2 - p_h^2 \cos^2 \theta_0(p_h)\right]^{3/2}} \mathrm{d}p = 10^6 \tan(\theta_0)\, \varepsilon(p_h), \quad (1)$$

which relates the initial refractive index profile $N(p_h)$ and the observed refraction angle $\varepsilon(p_h)$. Here $p = nr$, $r = r_0 + h$, r_0 is the Earth's radius, n is the refractive index, $N = 10^6 (n - 1)$ is the refraction factor. The $N(p)$ profile may by converted into the $N(h)$ profile by the following expression:

$$h = \{p / [1 + 10^{-6} N(p)]\} - r.$$

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In our earlier paper, Ref. 11, we have considered the method of reconstruction of the refractive index profile for the elevation angle $\theta_0(p_h) = \text{const}$. In so doing, we obtained that the reconstructed profile highly accurate coincided with the initial one. We showed that the reconstruction error depends on the error of refraction angle measurement, the position of the radiation source and the receiver relative to the Earth's surface, and the elevation angle. When making the numerical experiments, we assumed that the refraction angle is measured with a given error along the entire altitude range where the refractive index profile is to be reconstructed.

If to solve a more complicated problem assuming the refraction to be set only within the interval $p_0 < p_h < p_H$ (that is up to the height H), then the interval of reconstruction splits into two physically different regions. One may assume that in the inner region the solution is close to that of the problem in its initial formulation. In the outer region the problem approaches that which is characteristic of the Fredholm equation of the 1st kind. In the latter case the solution properties are similar to those of the inverse problem solution on the astronomical refraction, Ref. 12. Under the assumption that not only the refraction angle values, but the values of the refractive index are known up to the height H either, the integral in the left-hand side of Eq. (2) will contain the known component and in the outer region, $h > H$, the problem on determining the refractive index reduces to the Fredholm equation of the 1st kind thus being close to the problem described in Ref. 12

$$\int_{p_H}^{\infty} N(p) \frac{p p_h \cos\theta_0(p_h)}{\left[p^2 - p_h^2 \cos^2 \theta_0(p_h)\right]^{3/2}} \, \mathrm{d}p =$$

$$= N(p_h) \cot\theta_0 (p_h) - 10^6 \, \varepsilon(\theta_0(p_h)) -$$

$$- \int_{p_h}^{p_H} N(p) \frac{p p_h \cos\theta_0(p_h)}{\left[p^2 - p_h^2 \cos^2 \theta_0(p_h)\right]^{3/2}} \, \mathrm{d}p. \qquad (2)$$

Moreover, at a constant elevation angle the right-hand side of the equation will not be informative, as having become a constant value. To determine the refractive index profile, it is necessary that the angle θ_0 has different values at different p_h, as, for instance, when measuring at different moments in time. It is well known that the equations of this type are ill-posed and solution of those needs for regularization methods based on *a priori* information on the properties of the exact solution.

Evidently, the Eq. (1) is, on the whole, an ill-posed problem both in the outer and the inner region, though without any certain name of its type since it combines the features of the equations of both types mentioned above, which, however, should manifest themselves in the outer and inner regions of reconstruction in different ways.

THE METHOD OF THE INVERSE PROBLEM SOLUTION

When solving ill-posed problems without use of any additional information, large errors in reconstruction can occur even at small errors in the initial data.[12] If the exact solution belongs to the class of compact functions, the problem can be solved by minimizing the discrepancy functional, as this functional is convex for compact functions. In this paper the solution is based on the Tikhonov principle of the generalized discrepancy. This principle uses some general information about the exact solution as, for example, its membership of the class of nonnegative quadratically summable functions with the quadratically summable derivative. This well agrees with the specific features of the problem being solved. This allows one to use the maximum of the solution deviation modulus from the exact one as a measure of reconstruction accuracy assuming some typical distributions of the refractive index.

Let us write Eq. (1) in the operator form

$$KN = \varepsilon^\delta, \qquad (3)$$

where K is the operator of equation (1), ε^δ is the data vector with the error $\delta\varepsilon$ that satisfies the inequality

$$\delta\varepsilon^2 = \sup\|KN - \varepsilon^\delta\|^2_{L_2} = \frac{1}{\Delta p_h} \int [\varepsilon(p_h) - \varepsilon^\delta(p_h)]^2 \, \mathrm{d}p_h, \qquad (4)$$

where N is the right-hand side of Eq. (3), that corresponds to the exact solution $N(p)$, Δp_h is the integration interval. Within the framework of the method used one can take into account the error of the kernel that includes both the digitization error δ_h in the numerical simulation, and possible inaccuracy of the quantitative description of the kernel

$$\delta_h^2 = \sup\|K_h N - KN\|^2_{L_2}, \qquad (5)$$

where K_h is the approximate kernel set when solving Eq. (1). The inconsistency measure δ_μ cannot exceed the resulting error due to measurement errors and those in the kernel

$$\delta_\mu^2 = \inf \| K_h N - \varepsilon^\delta \| \le (\delta\varepsilon + \delta_h)^2. \qquad (6)$$

The approximate solution minimizes, in the Tikhonov method, the smoothing functional

$$M^\alpha(N) = \| K_h N - \varepsilon^\delta \|^2_{L_2} + \alpha \| N \|^2_{W_2^1} . \qquad (7)$$

In the above relations L_2 is the space of quadratically summable functions, W_2^1 is the space of quadratically summable functions with the quadratically summable derivatives. The regularization parameter α that determines the degree of the approximate solution smoothing, in Tikhonov generalized discrepancy method,[12] is defined as a root of the one-dimensional equation of the generalized discrepancy

$$\rho(\alpha) = \| K_h N^\alpha - \varepsilon^\delta \|^2_{L_2} - \delta^2 = 0, \quad (8)$$

where N^α is the function that minimizes Eq. (7), $\delta^2 = (\delta\varepsilon + \delta_h)^2 + \delta^2_\mu$ is the parameter of the effective error including measurement errors, errors of discretization, and other inaccuracies of the kernel description. This parameter also includes the measure of equation inconsistency with its right-hand side that depends on these errors. Thus, the value of the regularization parameter and, therefore, the degree of smoothing the solution is related to the value of the effective error δ. When the effective error vanishes, in L_2 metrics, the approximate solution converges to the exact one, in the W^1_2 metrics. Therefore, according to Sobolev embedding theorem this solution uniformly converges to the exact one, i.e., in the metrics C where the maximum of modulus serves as norm. As a rule, the convergence rate is slower than in the correct problems where the convergence rate is proportional to a decrease in δ.

The parameters δ_h and δ_μ may be found in the process of numerical simulation when minimizing Eq. (7). Normally, the inconsistency measure limits the level of discrepancy, to which it is reasonable to minimize the functional (7). After the relevant discretization the problem of smoothing the minimizing functional reduces to its finite analog, that is to the well investigated, from the computation point of view, problem of quadratic programming.

The peculiarity of any ill-posed problem is that no certain ratio exists between the error in the right-hand side of the equation and the accuracy of reconstruction since the latter essentially depends on the view of the initial function. So, for investigating the possibility of reconstructing the refraction profiles (the accuracy of reconstruction as a function of error level) assuming some typical model profiles one can make use of a closed numerical experiment. For making the method considered practical it is necessary to establish the connection between the parameter of the effective error δ, determined by the relation (8), in the Tikhonov method, and the experimental error. The problem is to obtain the estimate of δ having in mind the circumstance, that the experimental errors have a random nature.

Let the error obey the normal distribution law with the mean value $\Delta\varepsilon$ and standard deviation $\sigma\varepsilon$. Taking also into account, that the efficiency of the method is tested numerically, and that the fact of the choice being optimal or not can be checked, we may take as the error not the maximum value of the integral in Eq. (4), but its mean value. As a result, we obtain

$$\delta\varepsilon^2 = \frac{1}{\Delta p_h} \int \langle[\varepsilon(p_h) - \varepsilon^\delta(p_h)]^2\rangle \, \mathrm{d}p_h =$$

$$= \frac{1}{\Delta p_h} \int [\sigma\varepsilon^2(p_h) + \Delta\varepsilon^2(p_h)] \, \mathrm{d}p_h. \quad (9)$$

At constant values of the parameters $\Delta\varepsilon$ and $\sigma\varepsilon$

$$\delta\varepsilon = \sqrt{\sigma\varepsilon^2 + \Delta\varepsilon^2} \, . \quad (10)$$

If the bias error equals to zero $\delta\varepsilon = \sigma\varepsilon$ and at zero-valued random error $\delta\varepsilon = \Delta\varepsilon$. Since the error in the solution of an ill-posed problem is not proportional to the error in the initial data and can be determined only numerically we have numerically made the corresponding analysis.

NUMERICAL SIMULATION

We have carried out a numerical experiment on solving the above stated problem assuming the measurement accuracy to be 1–10″, that is characteristic of measurements in the optical range (Ref. 13), and typically exponential profile of the refractive index of the atmosphere. Then, using the initial profile, we calculated the refraction angle. After that we introduced a random noise into the value of the refraction angle calculated, thus simulating the errors of the refraction angle measurements in the atmosphere. It appeared from these experiments that the reconstruction in the outer region is effective only up to the height H < 5 to 10 km.

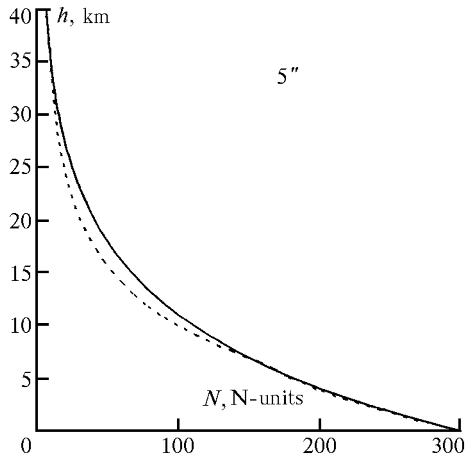

FIG. 1.

Figure 1 presents an example of reconstruction for H = 5 km with the accuracy of refraction measurements of 5″. We assumed the elevation angle to vary from 0.5°, at H = 5 km, up to 2.5° at H = 0. Figure 2 shows the reconstruction accuracy for H = 5 km versus height at different model errors of refraction measurements. In the altitude range where the refraction is set, the solution exhibit the properties characteristic of the Volterra equation, close, as a rule, to those of correct problems, that means that the reconstruction accuracy is almost proportional to the error in the initial data. Near the upper measurement boundary the solution takes the form typical for ill-posed problems. The transition region is characterized by a growth of the solution inaccuracy and occupies rather wide altitude range where the solution behavior essentially varies. Figure 3 presents similar results for $H = 20$ km (we show the case with the assumed measurement error of $0.01''$ as an example illustrating the convergence). The elevation angle of the observations varies from 0.5°, at $H = 20$ km, and up to 10° at $H = 0$. The measurements in the ranges of achievable accuracy are not informative in the outer region of reconstruction as it is seen that the error is the same at different accuracy and being close to the refractive index value at the relevant heights. There is little information about the outer layer here, and the destabilizing influence of the solution error from the outer region effects the equation solution in the inner range.

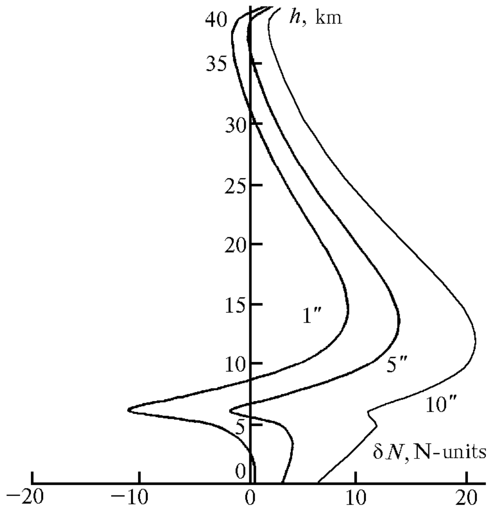

FIG. 2.

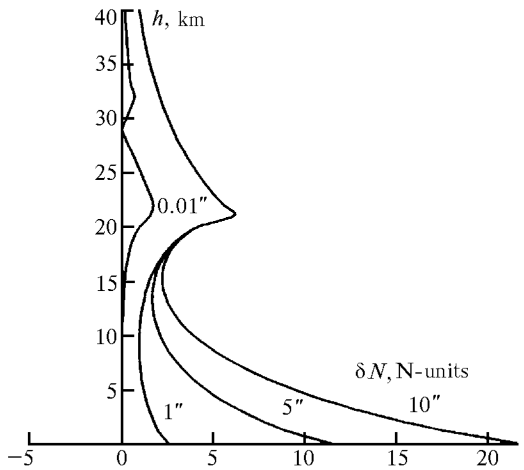

FIG. 3.

Figure 4 shows the results of the refractive index reconstruction with the $5''$ modeled error that can be considered as the limiting one for the refraction measurements at different heights H, including the case of $H = 40$ km (the refraction is set within the entire reconstruction interval). Thus, at small measurement errors of the refraction angle it is possible to obtain satisfactory results on reconstruction of the refractive index profile, while being limited in choosing heights for measurements of the refraction angle. However, it is worth noting that, on the whole, for the geometry of partial immersion, the solution has a larger errors as compared to that of a well-posed problem.[11] It is the manifestation of the fact that the problem is ill-posed that occurs not only in the outer region, but in the whole altitude range either.

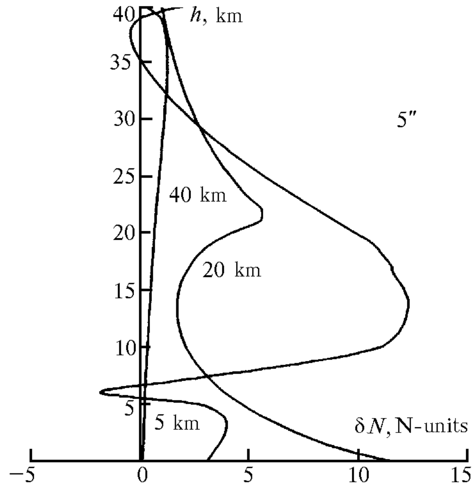

FIG. 4.

SOME EXACT SOLUTIONS

For the inner range, $0 < h < H$, one can obtain, assuming $\theta_0(p_h) = \text{const}$, the exact solution of equation (1). Let us differentiate Eq. (2) with respect to p

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}p} = 10^{-6}\,\frac{p_0 \cos\theta_0\, \mathrm{d}N(p)}{\left[p^2 - p_0^2 \cos^2\theta_0\right]^{1/2} \mathrm{d}p}\,. \tag{11}$$

By integrating the Eq. (11), using the values $\varepsilon(p_h)$ from the interval $p_0 < p_h < p_H$, we obtain

$$\varepsilon(p_h) = 10^{-6} \int\limits_{p_0}^{p_h} \frac{\mathrm{d}N(p)}{\mathrm{d}p} \times$$

$$\times \frac{p_0 \cos\theta_0}{\sqrt{p^2 - p_0^2 \cos^2\theta_0(p_h)}}\, \mathrm{d}p + \varepsilon(p_0). \tag{12}$$

Its inversion formula has the following form:

$$N(p) = 10^6 \int\limits_{p_0}^{p} \frac{\mathrm{d}\varepsilon(p_h)}{\mathrm{d}p_h}\, \frac{\sqrt{p_h^2 - p_0^2 \cos^2\theta_0(p_h)}}{p_0 \cos\theta_0} \times$$

$$\times\, \mathrm{d}p_h + N(p_0). \tag{13}$$

One can see from this that knowledge of the refractive index value on the surface is an essential requirement.

If one takes a different integration interval, then it is possible to obtain the problem solution for the case considered in Ref. 11, when the refraction is set in the entire altitude range

$$\varepsilon(p_h) = -10^{-6} \int\limits_{p_h}^{\infty} \frac{\mathrm{d}N}{\mathrm{d}p} \frac{p_0 \cos\theta_0}{\sqrt{p^2 - p_0^2 \cos^2 \theta_0(p_h)}} \mathrm{d}p \qquad (14)$$

or after the inversion

$$N(p) = -10^6 \int\limits_{p}^{\infty} \frac{\mathrm{d}\varepsilon(p_h)}{\mathrm{d}p_h} \frac{\sqrt{p_h^2 - p_0^2 \cos^2 \theta_0(p_h)}}{p_0 \cos\theta_0} \mathrm{d}p_h. \qquad (15)$$

No solution can be derived for the outer region at a constant elevation angle.

CONCLUSION

The refraction inverse problem for the immersion geometry is generalized for the case, when the refraction measurements are carried out not on the whole altitude range of reconstruction of the refractive index vertical profile, but only on its part. Analysis of results of the numerical simulation shows, that the solution accuracy essentially depends both on the value of the model data error, and on the layer height, where the refraction is considered known. The transition region is characterized by an essential increase in the solution error. Near the upper boundary of the reconstruction range the error variations are close, in magnitude, to the refractive index value. At the height $H = 20$ km the measurement error in the outer region does not depend on the refraction measurement error, in the range of achievable errors. Satisfactory error of reconstruction in the whole layer may be reached only at the error of refraction measurements less than 1″, that far exceeds the realistic accuracy of the refraction measurements.

The solution structure of the equation essentially depends on the layer height where the refraction is set. For example, at a constant model error of 5″ the reconstruction error varies for different heights over a wide range. At $H = 5$ km the solution in the inner region is close, by its properties, to that of a correct problem ($H = 40$ km), i.e., the reconstruction error also does not exceed 5N–units and gradually increases with the increasing height. The manifestation of the fact that the problem is ill-posed is most noticeable in the transition, and in the outer regions. The reconstruction error is large both in the outer, and in the inner regions at $H = 20$ km, i.e., the problem appears to be ill-posed in the whole altitude range.

In this paper we obtained some explicit relations for the direct and inverse refraction problems in the immersion geometry for the case of partially available refraction measurements at a constant elevation angle.

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