SCALING LAWS FOR PROPAGATION THROUGH TURBULENCE

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In this work it is shown that in regard to optical propagation effects associated with intensity variations and with higher order wave front distortion any pair of quite distinct cases of optical propagation through Kolmogorov turbulence may be related by simple scaling laws providing that two equivalencies apply between the cases. The first of these two equivalencies requires that from the source plane to the measurement plane the distribution of the optical strength of turbulence should follow the same form in the two cases, i.e., that there should be a simple proportionality between the strength of turbulence at the same fraction of the total distance in the total distance in the two cases. The second of the two equivalencies requires that a quantity which we will call the Rytov number, and shall denote by $\mathcal{R} = k^{7/6}Z^{5/6}\int_0^Z \mathrm{d}z(z/Z)^{5/6}(1-z/Z)^{5/6}C_N^2(z)$, *should be the same for the two cases. (The Rytov number is proportional to the log-amplitude variance as calculated by Tatarskii for spherical wave propagation over the path, using theory based on the Rytov approximation).*

1. INTRODUCTION

Scaling based (or dimensional/dimensionless) analysis is a standard technique in physics research. It has played a major role in such fields as fluid dynamics, where one has only to mention the term “Reynolds number” to recognize the importance of this type of analysis to the field. This type of analysis has been used in both theoretical and experimental work related to wave propagation through turbulence. Scaling laws have been used in experimental studies,[1] as a motivation/basis for heuristic approximations,[2] for asymptotic analysis,[3] and in numerical simulation.[4] In this work we intend to develop scaling laws for the propagation of an optical field through turbulence through manipulation of the wave equation – scaling laws applicable when the propagation effects of interest concern or are related to intensity fluctuations and/or higher order wave front distortion. These scaling laws will be of a form that applies directly to the random optical field *per se*, and not just to various moments of that field. Because extremely low spatial frequency components of the turbulence will not be treated rigorously the average, i.e., the first moment of the phase and of the tilt of the two fields will not be related by the scaling laws. This is discussed in Section 2.

We will establish in a relatively rigorous manner that the scaling laws follow directly from the form of the wave equation and from the Kolmogorov statistics of turbulence, considered in conjunction with some rather straight forward physical arguments –particularly arguments concerning the extremely low spatial frequency components of the turbulence. We shall show that providing that certain conditions are satisfied two seemingly quite different optical propagation cases (having different wavelengths, different path lengths, and different strengths of turbulence) have optical fields whose random intensities and higher order wave front distortions are related to each other by simple scaling laws. The conditions that have to be satisfied relate to the distribution of the strength of turbulence along the propagation paths and to a quantity which we call the Rytov number, whose value is defined by Eq. (1) given in Section 3.

It should be noted that we are considering propagation matters involving only a strictly monochromatic wave. Two cases that might be related through the scaling laws that we shall present can involve different wavelengths, but each of the two cases must be considered to be strictly monochromatic – since the scaling requires a well defined value for the wavelength. This restriction carries with it the implication that the scaling laws will not provide a handle on time dependent effects such as the stretching of a short pulse, since the fact that the pulse is short has inherent in it the implication that the optical field is polychromatic.

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Establishment of these scaling laws will makes it possible to justify carrying out an optical propagation experiment with parameters chosen to facilitate the implementation of the experiment and then applying the results (in scaled form) to a propagation case of interest – a case for which the characteristic parameters (for example the path length) might make the conduct of a directly relevant experiment quite difficult/expensive. Of perhaps more direct relevance to us as analysts is the fact that the existence of these scaling laws means that in considering computer simulation Monte Carlo studies of optical propagation through atmospheric turbulence, the dimensionality of the space to be explored is reduced – which can reduce the effective size of the space to be explored by orders of magnitude. This can turn a problem which otherwise might be too large to be systematically explored by Monte Carlo simulation methods into one that is small enough that such a systematic exploration would be possible.

It should be remarked that our original stimulus for the development of the scaling laws derivation that is being presented here lies in our observation that computer codes for simulation of optical propagation through turbulence have built into them what is in essence the very scaling laws that we shall be developing! Though the scaling laws *per se* were not intended (or even a consideration) when the codes were prepared, an examination of how the phase shifts associated with a turbulence phase-screen are calculated, and an examination of how the phase shifts associated with the path lengths between pairs of points on successive screens are calculated in the codes leads to these scaling laws. (Quite frankly it is from a recognition of this that we were led to seek a derivation of the scaling laws that would be more rigorous than the mere noting of the fact that the computer codes seem to have these laws inherent in them).

Interestingly, the fact that the scaling laws are inherent in the way the codes conduct propagation calculations means that we can not uses the codes in a Monte Carlo simulation to test the scaling laws. A well conducted simulation would necessarily lead to perfect confirmation. [Only if we deliberately sought unnecessary randomness (for example by deliberately not using the same random number seed in generating the turbulence phase-screens for the two cases we are comparing) would we get less than perfect confirmation – but that lack of perfect confirmation would be no more than an observation of the inability of the code to produce two exactly equal results for two statistically independent simulations of exactly the same case.] Validation of the scaling laws has to be sought in experimental tests, and in consideration of the soundness of the derivation we shall present.

The scaling laws will be derived based on the assumption that the statistics of turbulence are Kolmogorov and that the optical strength of turbulence, as measured by the refractive index structure constant, C_N^2, is similarly distributed along the two propagation paths that are to be related by the scaling laws. By similarly distributed we mean that with C_N^2 expressed as a function of the fractional position (i.e. quarter way, half way, three-quarters of the way, etc.) along the propagation path, the distributions along the two paths are directly proportional to each other. It will be shown that with the turbulence for the two paths having the properties just stated, then the requirement for the two cases to have propagation statistics that have a simple scaling law relationship to each other is that the two cases have the same value for what we call the Rytov number, $\mathscr{R}$.

A word is in order here regarding extremely low spatial frequency components of the turbulence and regarding one of the implications of the restriction of our attention to Kolmogorov turbulence. We explicitly do not introduce an outer scale of turbulence to limit the range of separations for which the Kolmogorov description of turbulence is applicable. This restriction of our attention to pure Kolmogorov turbulence carries with it the implication that the refractive index variance is *infinite*. For applications of principle interest to date (such as in the calculation of wave front distortion effects for an imaging system study or for an adaptive optics system study, or in the calculation of laser beam intensity fluctuation effects for a study of optical communications through the atmosphere) where the effects of interest concern intensity perturbations and higher order wave front distortion there appears to be no need to introduce an outer scale of turbulence. The extremely low spatial frequency components of the turbulence have no optical effect of practical interest, so it make no significant difference how we treat the matter of the outer scale of turbulence, or in fact even if we allow our treatment of such components of turbulence to be not entirely self consistent; there is no significant optical effect. Only in consideration of first moments of the optical field (a subject for which we have been unable to identify any basis for a practical interest) would the outer scale of turbulence become manifestly significant.

This matter is discussed in more detail in Section 2 where it is explained why our scaling law results should not be considered applicable when the phenomena of interest is dependent upon the variability of the phase at a point, as would be the case in calculating the first moment of the optical field of the area average phase, or when the phenomena of interest is dependent on the variability of the area average wave front tilt. While our results will not be applicable when the interest is in evaluation of such matters, for other more practically oriented interests such as those concerning imaging, adaptive optics, optical communications, etc.), which are governed by intensity variations and higher order wave front distortion effects, the scaling law results should be quite relevant. The distinction between the two classes of subjects – those for which the scaling laws are applicable and those for which it is not – is based on some rather obvious physical consideration as to whether turbulence variations whose scale sizes are extremely large will affect the propagation statistics of interest. For cases of practical interest (or at least for all cases of practical interest that we have been able to identify) this is not the case – such large scale refractive index variations will have no effect of significance, so the scaling laws should be applicable.

Before dropping the matter of moments of the optical field it seems appropriate to take note of the fact that some different scaling laws and dimensionless parameters have been derived by others by use of methods that approximated some of the statistical moments of the optical field. In particular we call attention to the work of Gurvich and Kan,[5] of Whitman and Beran,[2] and of Gozani.[6]

2. TREATMENT OF EXTREMELY LOW SPATIAL FREQUENCY COMPONENTS OF TURBULENCE

Low spatial frequencies play a rather unusual role in the description and analysis of atmospheric turbulence and of the optical effects of turbulence. In the treatment of turbulence in the inertial sub range it is convenient to be allowed to ignore the low spatial frequency limit of the inertial sub range and consider Kolmogorov statistics, i.e., the $r^{2/3}$ power law dependence of the structure function and the $\varkappa^{-11/3}$ power law dependence of the (three dimensional) power spectral density, to apply for all value of r no matter how large and for all values of $\varkappa$ no matter how small. This carries with it *inter alia* the implication that the variance of the atmospheric refractive index is infinite. Tolerance of such a clearly unphysical situation – and this matter is tolerated in many if not most studies of optical propagation through atmospheric turbulence – derives from the fact that the optical effects produced by the very low spatial frequency components of turbulence are either unobservable or so difficult to observe that there is essentially no interest in these effects. Also contributing to the fact that this unphysical situation is so often tolerated in propagation analysis is the fact that physical insight makes it so easy to evaluate the optical propagation effects of these extremely low spatial frequency components – so easy to determine whether or not the presence or absence of such components of turbulence makes any physically interesting/engineering-wise significant difference. For most applications it is concluded that the presence or absence of the low spatial frequency components makes no difference of concern. We shall utilize this lack of physically interesting consequences in optical propagation to justify our use of a not entirely self consistent approach to the modeling of these extremely low spatial frequency components of the turbulence in the analysis we shall be presenting.

It is immediately clear from physical insight that extremely low spatial frequency components of turbulence do not result in any significant effects relative to intensity fluctuations. As noted above, with no outer scale of turbulence the covariance of the refractive index is infinite. This is due to the presence of very "strong" extremely low spatial frequency components of turbulence. But because these extremely low spatial frequency components of turbulence have essential no optical propagation effect in terms of intensity perturbations the covariance of intensity fluctuations will be finite. Accordingly, if our interest is in intensity fluctuation effects then we are free to treat the very low spatial frequency components of turbulence in what every way we find most convenient – and not necessarily even in a self consistent manner. These components of the turbulence have no effect upon the intensity perturbations so how we treat these components will not effect the intensity variations; the intensity perturbation results remain valid no matter how the extremely low spatial frequency components of turbulence are treated.

It is easy to see that the dimension that is to be associated with the division between what is and what is not to be considered an extremely low spatial frequency is set by the Fresnel length, which is of the order of the square root of the optical wave length times the effective path length – where the effective path length is the distance the light travels after first encountering turbulence.

If our interest is in phase perturbation effects then the matter is not quite so simply stated. In this case we need to distinguish between three aspects of phase perturbation. These three aspects are:

1) variations with time of the phase at some point (or of the area average phase – averaging over some region of interest),

2) variations with time of the wave front tilt associated with some region (the region presumably corresponding to some optics aperture), and

3) within some region the point-to-point phase difference (at some instant of time) excluding from the phase difference the portion due to the average wave front tilt associated with the region (at that instant of time) – what is often called higher-order wave front distortion.

It should be fairly obvious from physical insight that the very low spatial frequency components of turbulence will have no significant effect relative to the third of three items – higher-order wave front distortion. The extremely low spatial frequency components of turbulence can be seen to produce only correspondingly low spatial frequency components of phase variation with essentially no contribution to the higher order wave front distortion. (To the extent that there is any significant phase perturbation it is almost entirely incorporated in the wave front tilt – the second of the three aspects listed above.) Accordingly, if our interest is in higher order wave front distortion effects, for the study of something like the short exposure resolution of an imaging system, like the performance of an adaptive optics system for correction of higher order wave front distortion, or like the instantaneous antenna gain of a laser transmitter, then we are free to treat the very low spatial frequency components of turbulence in what every way we find most convenient – and not necessarily even in a self consistent manner.

In this case the dividing line between just what is not to be considered to be an extremely low spatial frequency is set by the size of the region being considered, presumably the aperture size (or by the Fresnel length, which ever is larger).

If our interest is related to the wave front tilt associated with some aperture, as would be the case if we were considering the servo bandwidth of the tip-tilt tracker of an adaptive optics system, then we note that even with no outer scale of turbulence – so that there are significant contributions to the tilt from some extremely low spatial frequency components of turbulence – the tilt variance will be finite. The value of that variance will, however, depend on some extremely low spatial frequency components of the turbulence. In this case the dividing line between significant spatial frequencies and those that are so low that the tilt statistics are virtually uninfluenced by whether or not such low spatial frequency components are include, is some very large multiple of the aperture diameter. Let us consider just where this dividing line is. To do this, it is convenient to consider the contributions by the different spatial frequency components of turbulence to the wave front tilt variance.

The tilt variance's value is set almost entirely by spatial frequency components lower than the inverse of the aperture diameter, D. If we set a dividing line at some spatial frequency $\varkappa_{\rm Low}$ and say that for spatial frequencies smaller than $\varkappa_{\rm Low}$ the spatial frequency is so extremely low that the spatial frequency component makes no significant contribution to the area average tilt and so can be ignored in calculating the tilt variance, then how much of the true tilt variance have we left out? For Kolmogorov turbulence the tilt variance, which is proportional to the integral of $\varkappa^2$ times the turbulence PSD, integrated from some low spatial frequency end, $\varkappa_{\rm Low}$ to an effective upper limit of $1/D$, has a value of $D^{-1/3} - \varkappa_{\rm Low}^{1/3}$. To get a variance equal to 75% (or 90%) of the true value, the value we get with no cut off of the very low spatial frequency components of turbulence, which true value is proportional to $D^{-1/3}$, we have to set the dividing line, $\varkappa_{\rm Low}$, between just what is considered a significant contributor to the tilt and what is to be considered to be a such an extremely low spatial frequency that it makes no significant contribution to the tilt – with significance taken to be defined in terms of the 75% (or 90%) of the full amount – at $\varkappa_{\rm Low} = (1/4)D^{-1/3} = (64D)^{-1/3}$ (or at $\varkappa_{\rm Low} = (1/10)D^{-1/3} = (1{,}000D)^{-1/3}$) corresponding to a spatial frequency period of $64D$ (or $1{,}000D$).

This dividing line will, in general, correspond to such a large length that we probably will not be able to justify the treatment of the lower spatial frequencies of turbulence as being so extremely low that they have no significant effect upon optical wave front tilt for moderate to large size regions/apertures. Accordingly we conclude that the scaling law results we will develop here will be of only limited/questionable applicability if the optical propagation effect of interest significantly involves wave front tilt over moderate or large aperture diameters.

If our interest lies in the variation of the phase at a point, as it might be if we were thinking of using a Mach Zender type of interferometer with one leg of the interferometer passing through an evacuated pipe – so as to monitor ground stability (seismology effects), or if we were interested in determining the first moment of an optical field then it should be clear that we must properly include all spatial frequency components down to the lowest physically present. If the turbulence is Kolmogorov then the phase variance is infinite (and the first moment of the optical field is equal to zero) and any incorrect treatment of the lowest spatial frequencies can have a major effect on the quantity of interest. For such a case we must be punctilious in our treatment of the lowest spatial frequencies. Accordingly we must conclude that the scaling laws we shall develop here will be inapplicable for study of optical effects involving the variations of the phase at a point (or involving the variations of an area average phase). Accordingly in this work we shall restrict our attention to optical effects for which the treatment of the lowest spatial frequency components of the turbulence can be adjusted at our convenience to facilitate the analysis without there being any concern for the effect of such "adjustments" upon the optical effects of interest. Basically we are restricting our attention to turbulence induced intensity fluctuations and to turbulence induced higher order wave front distortion. (It should be noted that the matters we have excluded – related to wave front tilt, to long term variations of phase at a point, and to area average phase, which matters are dominated by extremely low spatial frequency components of turbulence – can be treated reasonably well with physical insight and with simple ray-optics type analysis.) The optical effects for which the scaling laws are applicable, related to intensity variations and to higher order wave front distortion, are the effects which constitute the real challenge in the development of optical propagation theory.

3. BASIC FORMULAS

The Rytov number, $\mathscr{R}$, which will prove to be central element in our scaling law results, is defined by the equation

$$\mathscr{R} = k^{7/6} Z^{5/6} \int_0^Z {\rm d}z (z/Z)^{5/6} (1 - z/Z)^{5/6} C_N^2(z) . \quad (1)$$

Here Z denotes the total length of the propagation path and $k = 2\pi/\lambda$ is the optical wave number. We have assigned the quantity $\mathscr{R}$ the name "Rytov number" in acknowledgment of the fact that $\mathscr{R}$ is proportional to the log-amplitude variance for infinite plane wave propagation as it was calculated by Tatarskii using the Rytov approximation.

For normalization of position in the plane transverse to the nominal propagation direction we use the length L, which is what we may call the Fresnel-length for the propagation path. Its value is given by the equation

$$L = \sqrt{Z/k} \ . \tag{2}$$

The scaling laws we shall derive will relate the random optical field, expressed as a function of the transverse dimension, $\mathbf{r}$, (i.e. the dimension associated with the plane perpendicular to the nominal direction of propagation) for one case to the corresponding random optical field for the other case – with the relationship established by expressing the random optical fields for the two cases as functions of $\mathbf{r}/L$.

The scaling laws will be developed from the paraxial version of the wave equation approximated in a phase-screen form, and will rely on replacement of results for one ensemble of random source functions by results developed for an equivalent ensemble of random source functions. The phase-screen formulation with statistically independent phase-screens, which we shall be using in deriving the scaling laws, is based on the physical argument set forth in the first paragraph of this paper concerning the inconsequential nature of the effect of extremely low spatial frequency components of the turbulence. Strictly speaking the phase screens should be considered to be correlated. It is to be noted that this argument, leading to the introduction of statistically independent phase-screens leads with no farther insight (other than that double back scattering is negligible) to the so called "Markov approximation" for the optical field[7,8] – the approximation "that the field is statistically independent of the medium inhomogeneities that the wave has not yet passed through."

We shall be working with the paraxial wave equation, solving for the perturbation induced by turbulence on an optical field $U(\mathbf{r}, z, t)$, where $U(\mathbf{r}, z, t)$ represents a nominally plane wave traveling in the z-axis direction. The field is expressible as

$$U(\mathbf{r}, z, t) = u(\mathbf{r}, z) \exp[ik(\pm z + ct)] \ , \tag{3}$$

where $u(\mathbf{r}, z)$ is a function which (nominally) varies slowly, particularly in its z-coordinate dependence. The paraxial equation, governing the value of the perturbation-carrying function, $u(\mathbf{r}, z)$, is

$$[\nabla^2_{\mathbf{r}} \pm 2ik\partial_z + 2k^2 n(\mathbf{r}, z)]\ u(\mathbf{r}, z) = 0 \ . \tag{4}$$

Here the notation ∂_z denotes the partial derivative with respect to z (corresponding to $\partial/\partial z$) and the notation $\nabla^2_{\mathbf{r}}$ denotes the sum of the two second derivative components of the Laplacian, the derivatives being taken with respect to the two variables perpendicular to the z-axis. The notation $n(\mathbf{r}, z)$ denotes the turbulence induced perturbation of the atmosphere's refractive index, and is presumably very small.

The paraxial equation is obtained from Maxwell's wave equation on the basis of the two approximations that the refractive index perturbation, $n(\mathbf{r}, z)$, is so small that its square can be neglected, and that the variation of the perturbation-carrying function, $u(\mathbf{r}, z)$, varies so slowly along the z-axis that we can neglect $\partial^2_z u(\mathbf{r}, z)$, the second derivative with respect to z of $u(\mathbf{r}, z)$. Equation (4) represents the starting point for the analysis to follow.

4. PHASE-SCREEN FORMULATION

Following the arguments and practices used by Uscinski,[9] Prokhorov et al.,[10] Taylor,[11] Ishimaru,[12] and many others, and in accordance with the practice generally utilized in wave optics propagation simulation, we shall consider the propagation path, whose total length we denote by Z, to be divided in to P equal length segments with boundaries at $z = \{z_0, z_1, z_2,\ldots,z_p,\ldots,z_P\}$, where $z_0 = 0$ and $z_P = Z$ – a total of $P + 1$ values of z_p, and will consider each such segment's refractive index pattern, $n(\mathbf{r}, z)$, to be collapsed into a zero thickness "phase screen" located at the mid-point of the segment. We shall use the notation $\bar{z}_p$ to denote the mid point of the pth segment, so that

$$\bar{z}_p = \frac{1}{2}(z_p + z_{p-1}) \ . \tag{5}$$

Our rule for selecting the value of P is that it should be as small as possible compatible with the requirement that

$$(Z/P)\lambda/r_0 \ll r_0^{\rm seg}(p) \tag{6}$$

for all values of p, where r_0 denotes the effective coherence diameter for infinite plane wave propagation, with a value given by the equation

$$r_0 = \left[\frac{2.91}{6.88}\, k^2 \int_0^Z {\rm d}z\ C_N^2(z)\right]^{-3/5} , \tag{7}$$

and where $r_0^{\rm seg}(p)$ denotes the effective coherence diameter to be associated with infinite plane wave propagation through the pth segment of the propagation path – with a value given by the equation

$$r_0^{\rm seg}(p) = \left[\frac{2.91}{6.88}\, k^2 \int_{z_{p-1}}^{z_p} {\rm d}z\ C_N^2(z)\right]^{-3/5} . \tag{8}$$

The physical reasoning underlying the imposition of Eq. (6) to govern the selection of P is as follows. In replacing the refractive index pattern, distributed over the finite length of the segment, with a zero thickness phase screen surrogate, positioned at the center of the segment, we are assuming that the random wave front distortion induced spread in propagation directions – a spread that at worst (at the far end of the path) is of the order of λ/r_0 – has no *significant* impact on the additional phase perturbation introduced by propagating through the segment. This will be so if the random amount of lateral displacement – which is at worst of the order of $(Z/P)\lambda/r_0$ – is small enough. The quantity defining just what is a "small enough" amount of lateral displacement is given by $r_0^{\rm seg}(p)$. Thus if Eq. (6) is satisfied then the effect of the lateral spread is "small enough" and we can replace the refractive index pattern extended over the segment by the phase screen located at the mid-point of the segment.

We also require that P be large enough that

$$(Z/P)\lambda \ll r_0^2 \, , \tag{9}$$

in order to justify the use of ray optics in considering the propagation through the thickness, Z/P, of the segment in the way just have. But since this requirement is satisfied, if Eq. (6) is satisfied, we need not take Eq. (9) as presenting an additional constraint on the allowed value of P.

We shall also assume that the selected value of P is small enough that the segment thickness, Z/P, is great enough to justify our treatment of the phase screens as being statistically independent. As discussed latter statistical independence will be justified as an acceptable approximation if Z/P is significantly larger than the period associated with the dividing line between spatial frequency components of the turbulence whose period (spatial frequency) is so extremely large (small) that they do not have a significant impact on the optical propagation effects of interest and those components whose period (spatial frequency) is sufficiently small (high) that they do have significant optical propagation effects of interest. We assume that we are not dealing with a propagation case for which the turbulence effects are too severe, that a value of P can be found which satisfies both this phase screen statistical independence requirement and also the requirement posed by Eq. (6).

With P having a value that satisfies Eq. (6) [and Eq. (9)] we can, on the basis of physical considerations, regard all of the refractive index variations in the pth segment to be collapsed into a thin phase-screen which we shall denote by the notation $N(\mathbf{r}, p)$, positioned at the midpoint of the pth segment, i.e. located at $\bar{z}_p$. We have use the term "collapsed" to imply that the phase-screen, $N(\mathbf{r}, p)$, can be written as

$$N(\mathbf{r}, p) = \int\limits_{z_{p-1}}^{z_p} \mathrm{d}z \; n(\mathbf{r}, z) \tag{10}$$

and accordingly replace $n(\mathbf{r}, z)$ in Eq. (4) as

$$n(\mathbf{r}, z) \Rightarrow \sum_{p=1}^{P} N(\mathbf{r}, p)\, \delta(z - \bar{z}_p) \, . \tag{11}$$

so that in place of Eq. (4) we have

$$\{\nabla_{\mathbf{r}}^2 \pm 2ik\partial_z + 2k^2[\textstyle\sum N(\mathbf{r}, p)\, \delta(z - \bar{z})_p]\}\, u(\mathbf{r}, z) = 0 \, . \tag{12}$$

This, it should be noted, is the form of the wave equation that is assumed in the formulation of most (if not all) computer wave optics propagation simulation codes.

A key part of the physics insight that goes into the scaling law derivation lies in the recognition that for the optical effects of interest to us Eqs. (6) and (9) justify our writing Eq. (11) [and from that Eq. (12)]. That physics insight lies in our recognition that whether the actual refractive index variation are distributed in a continuous form as specified by $n(\mathbf{r}, z)$, or are distributed in a spatially quantized form (quantized along the z-axis) as specified by $N(\mathbf{r}, z)$ the way the refractive index variations affect the optical propagation will be very nearly the same. As noted in the discussion following Eqs. (8) and (9) the basis for this physics insight lies in the fact that if Eqs. (6) and (9) are satisfied then the propagation of a distorted optical field from one screen to the next screen with refractive index induced phase shifts applied only at the screens (and with no refractive index variation *between* the screens) will result in essentially the same field incident on the second screen as would arrive at the second screen when propagating through the continuous distribution of refractive index variation between the screens (with no phase shifts being applied by the screens themselves).

We now introduce the farther assumption (also inherent in the use of computer wave optics propagation simulation codes) that

$$Z/P \gg |\,\mathbf{r} - \mathbf{r}'| \, , \tag{13}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are any two transverse position vectors that we may want to consider. The validity of Eq. (13), which is certainly true for all cases we are aware of and probably for all that we could conceive of, allows us to infer from the two-thirds power law of Kolmogorov statistics that applies for the refractive index variations, namely that

$$\langle[n(\mathbf{r}_1, z_1) - n(\mathbf{r}_2, z_2)]^2\rangle = C_N^2\left(\tfrac{1}{2}(z_1 + z_2)\right) \times$$
$$\times \left[|\mathbf{r}_1 - \mathbf{r}_2|^2 + (z_1 - z_2)^2\right]^{1/3}, \tag{14}$$

that the corresponding statistics of the phase-screen $N(\mathbf{r}, p)$ are given by the equation

$$\langle[N(\mathbf{r}_1, z_1) - N(\mathbf{r}_2, z_2)]^2\rangle \approx$$
$$\approx \left(2.91 \int\limits_{z_{p-1}}^{z_p} \mathrm{d}z\, C_N^2(z)\right)[|\mathbf{r}_1 - \mathbf{r}_2|^{5/3}] \, . \tag{15}$$

Consideration of Eq. (13) suggests that we introduce the quantity $R = |\mathbf{r} - \mathbf{r}'|$ where $\mathbf{r}$ and $\mathbf{r}'$ represent a pair of transverse position vectors that are as widely separated as any two that we may wish to consider. (For system performance studies this might correspond to a length some what larger than the aperture diameter or the beam diameter if the beam diverges, or it might correspond to several times the Fresnel length, L, for point source propagation studies. Exactly what will govern the determination of the particular value of R need not be specified here.) The inverse of this dimension R or of some moderate multiple of R can be considered to define the concept referred to earlier of an exceptionally low spatial frequency. So long as Eq. (13), or as we shall write it here

$$Z/P \gg R , \tag{16}$$

is satisfied then we can conclude that for all of the spatial frequency components of the turbulence which do significantly impact the optical effects of interest – effects concerning intensity variations and concerning higher order wave front distortion – the period of the component's spatial frequency is very much less than the thickness of a segment. For these components there should consequently be only a very small correlation between the contribution to two separate segments and their associated phase screens, even if the phase screens are adjacent. For the extremely low spatial frequency components, since they have no optical effect of interest to us here, we can make the incorrect (but inconsequential) assumption that their contribution to any two phase screens are also uncorrelated. This allows us to write

$$<[N(\mathbf{r}_1,p) - N(\mathbf{r}_2,p)][N(\mathbf{r}'_1,p') - N(\mathbf{r}'_2,p')]> \approx 0$$

$$\text{for } p \neq p' . \tag{17}$$

(It may be noted that this approximation is customarily incorporated into the running of most computer wave optics propagation simulation codes.) Equation (17) carries the implication that the individual phase-screens can be treated as being statistically independent from phase-screen to phase-screen.

We shall consider the phase-screens to be defined in terms of an ensemble of "unit strength" random functions; $\nu(\boldsymbol{\rho}, *)$, governed by the same sort of five-thirds power law as in Eq. (15). Here $\boldsymbol{\rho}$ denotes a dimensionless two-component ("position") vector. (That we can use such a function to present the statistics of turbulence is due to the essentially scaleless nature of Kolmogorov turbulence.) The asterisk, *, appears as an "argument" of the ν-functions to indicate that we are interested in a randomly selected realization of the function ν, and that the appearance of this function two or more times in an equation does not imply that the same realization is intended. *Only* if the asterisk appears subscripted, with the *same* value of the subscript in each occurrence, is to be understood that the same realization is intended. For example if we write $\sum_m \nu(\boldsymbol{\rho}, *_m)$, since in each occurrence of $\nu(\boldsymbol{\rho}, *_m)$ in the summation the value of m is different, the individual terms, i.e. $\nu(\boldsymbol{\rho}, *_1)$, $\nu(\boldsymbol{\rho}, *_2)$, $\nu(\boldsymbol{\rho}, *_3)$, ..., are to be understood as each representing different, randomly selected realizations of ν.

The function $\nu(\boldsymbol{\rho}, *)$ is to be understood, by definition, to have the statistical property that

$$<[\nu(\boldsymbol{\rho}_1, *_a) - \nu(\boldsymbol{\rho}_2, *_a)]\ [\nu(\boldsymbol{\rho}_1, *_b) - \nu(\boldsymbol{\rho}_2, *_b)]> =$$

$$= \begin{cases} 2.91|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^{5/3}, & \text{if } *_a = *_b, \\ 0, & \text{if } *_a \neq *_b. \end{cases} \tag{18}$$

Considering Eq. (15), we can see that this allows us to consider the ensemble of $N(\mathbf{r}, p)$-functions to be replaceable by the ensemble of $\nu(\boldsymbol{\rho}, *)$-functions with the replacement equation being

$$N(\mathbf{r}_1, p) \Rightarrow L^{5/6}\, C_p \nu(\mathbf{r}/L, *_p) , \tag{19}$$

where

$$C_p = \sqrt{\int\limits_{z_{p-1}}^{z_p} \mathrm{d}z\ C_N^2(z)} . \tag{20}$$

It will prove convenient if we introduce here the notations C and c_p, defined by the equations

$$C = \sqrt{\int\limits_0^Z \mathrm{d}z\ (z/Z)^{5/6}\ (1 - z/Z)^{5/6}\ C_N^2(z)} \tag{21}$$

and

$$c_p = C_p/C . \tag{22}$$

This allows Eq. (19) to be rewritten as

$$N(\mathbf{r}_1, p) \Rightarrow L^{5/6}\, C\, c_p\, \nu(\mathbf{r}/L, *_p) . \tag{23}$$

Making use of Eq. (23) we can recast Eq. (12) as

$$\Big(\nabla^2_{\mathbf{r}} \pm 2ik\partial_z + 2k^2 C\Big(\sum_{p=1}^{P} L^{5/6}\, c_p \nu(\mathbf{r}/L, *_p)\ \delta(z - \bar{z}_p)\Big)\Big) \times$$

$$\times\ u(\mathbf{r}, z) = 0 . \tag{24}$$

With this result in hand we are ready to turn our attention to the scaling law formulation. This is treated in the next section.

5. SCALING THE FORMULATION

We shall reformulate Eq. (24) by making the change of variables from $\mathbf{r}$ to $\boldsymbol{\rho}$ and from z to ζ, where

$$\boldsymbol{\rho} = \mathbf{r}/L \tag{25}$$

and

$$\zeta = z/Z . \tag{26}$$

We note that accordingly we have

$$\nabla^2_{\mathbf{r}} = L^{-2}\ \nabla^2_{\boldsymbol{\rho}} \tag{27}$$

and

$$\partial_z = Z^{-1}\,\partial_\zeta\ . \qquad (28)$$

Defining $\bar{\zeta}_p$ by the equation

$$\bar{\zeta}_p = \bar{z}_p/Z \qquad (29)$$

and taking note of the fact that the property of the Dirac delta-function is such that

$$\delta(z - z_p) \Rightarrow Z^{-1}\,\delta(\zeta - \zeta_p)\ , \qquad (30)$$

we can reformulate Eq. (24) as

$$\Big\{L^{-2}\,\nabla^2_\rho \pm 2ikZ^{-1}\partial_\zeta + 2k^2 C\Big[\sum_{p=1}^{P} L^{5/6}\,c_p \nu(\rho,*_p)\,Z^{-1}\,\delta(\zeta - \bar{\zeta}_p)\Big]\Big\} \times u(L\rho, Z\zeta) = 0\ . \qquad (31)$$

If we introduce the function $v(\rho, \zeta)$ defined by the equation

$$v(\rho, \zeta) = u(L\rho, Z\zeta)\ , \qquad (32)$$

with the inverse relationship

$$u(\mathbf{r}, z) = v(\mathbf{r}/L, z/Z)\ . \qquad (33)$$

and make use of Eq. (2) we can recast Eq. (31) as

$$\Big\{(k/Z)\,\nabla^2_\rho \pm 2i(k/Z)\partial_\zeta + 2(k/Z)k^{7/12}\,Z^{5/12}C\Big[\sum_{p=1}^{P} c_p \nu(\rho,*_p)\,\delta(\zeta - \bar{\zeta}_p)\Big]\Big\} \times v(\rho, \zeta) = 0\ . \qquad (34)$$

If we divide through by the quantity k/Z and recognize from consideration of Eqs. (1) and (21) that

$$k^{7/12}\,Z^{5/12}\,C = \mathscr{R}^{1/2}, \qquad (35)$$

then we see that Eq. (34) can be rewritten as

$$\Big(\nabla^2_\rho \pm 2i\partial_\zeta + 2\mathscr{R}^{1/2}\Big(\sum_{p=1}^{P} c_p \nu(\rho,*_p)\,\delta(\zeta - \bar{\zeta}_p)\Big]\Big\} \times v(\rho, \zeta) = 0\ . \qquad (36)$$

This non dimensional form of the propagation equation is our basic analytic result. Any solution for $v(\rho, \zeta)$ can be scaled, using Eq. (33), to apply to a variety of different cases – cases with the same $\mathscr{R}$ and the same distribution of c_p values.

Equation (36) should be interpreted as follows. For any two propagation cases, if the distribution of turbulence along the two paths follows the same pattern (for example that both are uniform, or that both fall off exponentially each decreasing by the same factor from one end of the path to the other end of the path) as would be indicated by having the same set of values for c_p and if the absolute strength of turbulence, the path length, and the wavelength are such that the two have the same Rytov number, $\mathscr{R}$, then both will have the same form of the non dimensional propagation equation, i.e. of Eq. (36), and will have identical forms for the non dimensional wave function, $v(\rho, \zeta)$.

This result concerning the equivalence of the propagation results, i.e. the random optical fields for two distinct cases implies that if we can run an experiment (or conduct a computer simulation) for one case, measuring the various random realizations of $u(\mathbf{r}_1, Z_1)$ for that case, we can calculate the associated values of $v(\rho, \zeta)$ using Eq. (32), and then can consider these v-values to apply with corresponding likelihood to the other case and use Eq. (33) to calculate a statistically equivalent set of values of $u(\mathbf{r}_2, Z_2)$ for the second case. [Here and in what follows the subscripts 1 and 2 are being used to denote the variables (or parameters) of the two cases.] Any result calculated from the intensity and the higher order wave front distortion will be properly evaluated for case 2 if the results are obtained from the thus scaled results obtained for case 1.

Considering Eq. (25), we can see that if the scaling requirements are satisfied, then what we measure for $\mathbf{r}_1$ will apply for $\mathbf{r}_2$ with

$$\mathbf{r}_2 = \mathbf{r}_1(L_2/L_1)\ . \qquad (37)$$

Interestingly, since the pattern of the distribution of the strength of turbulence is the same for the two cases and the Rytov number, $\mathscr{R}$, is the same for the two, then Eq. (37) also implies that what we measure for $\mathbf{r}_1$ will apply for $\mathbf{r}_2$ with

$$\mathbf{r}_2 = \mathbf{r}_1[(r_0)_2/(r_0)_1]\ , \qquad (38)$$

It can be inferred from this that in two experiments that are dependent only upon intensity (variations) and on higher order wave front distortion, if both experiments have the same Rytov number, $\mathscr{R}$, and if there are two apertures whose diameters, D_1 and D_2, are in the same ratio as $(r_0)_1$ is to $(r_0)_2$, or as L_1 is to L_2, then the two optical systems will be equally affected by turbulence. Similarly it can be argued that for angles scaled in accordance with λ/D anisoplanatism effect will be identical. In all of its dependences adaptive optics antenna gain (which is dependent only on intensity variations and upon higher order wave front distortion) will be equivalent for two cases that have appropriately scaled parameter values.

6. COMMENTS

It is in a way quite instructive to consider what these scaling law results mean for the investigation of optical propagation effects related to intensity variations and to higher order wave front distortion when, for example, the strength of turbulence, C_N^2, is uniform along the entire length of the propagation path, and the inner scale of turbulence, l_0, is small enough to be considered to be inconsequential. There is only a one dimensional array of cases to be considered. This makes the idea of a fairly comprehensive numerical study of the entire span of the problem appear plausible – certainly an intriguingly possibility.

It is perhaps worth remarking here that our exclusion of area average wave front tilt from the phenomena that were covered by the scaling laws (and possibly even the exclusion of phase at a point and area average phase) may have been overly severe. If the variation over only moderately short time periods is being considered then the extremely low spatial frequency components of the turbulence are relatively inconsequential. In such a case the scaling laws would apply to a study of tilt and phase effects, as well as to intensity variations and to higher order wave front distortion.

REFERENCES

1. M.E. Gracheva, A.S. Gurvich, S.S. Kashkarov, and Vl.V. Pokasov, in: *Laser Beam Propagation in the Atmosphere*, J.W. Strohbehn, ed. (Springer-Verlag, Berlin, 1978), pp. 107–128.

2. A.M. Whitman and M.J. Beran, J. Opt. Soc. Am. **A–2**, 2133–2143 (1985).

3. J.M. Martin and S.M. Flatte, J. Opt. Soc. Am. **A–7**, 838–847 (1990).

4. S.M. Rytov, Yu.A. Karavtsov, and V.I. Tatarskii, *Principles of Statistical Radiophysics*. Vol. 4. *Wave Propagation through Random Media* (Springer-Verlag, Berlin, 1988).

5. A.S. Gurvich and V. Kan, Radiophys. and Quant. Electron. **22**, 131–134 (1979).

6. J. Gozani, Waves Random Media **3**, 279–306 (1993).

7. V.I. Klyatskin and V.I. Tatarskii, Radiophysics and Quantum Electronics **14**, 1100–1111 (1971).

8. V.U.Zavorotnyi, Soviet Physics JETP **48**, 27–31 (1978).

9. B.J. Uscinski, in: *Wave Propagation and Scattering*, B.J. Uscinski, ed. (Clarendom Press, Oxford, 1986).

10. A.M. Prokhorov, F.V. Bunkin, K.S. Gochelashvily, and V.I. Shishov, Proc. IEEE **63**, 790–811 (1975).

11. L.S. Taylor, J. Math. Phys. **13**, 590–595 (1972).

12. A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic, New York, 1978); Sect. 20–14 through 20–16.