APPLICATION OF PHASE MODULATION FOR WAVE PHASE RESTORATION FROM THE AMPLITUDE DATA

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The efficiency of introduction of the controlled phase modulation for an image-forming wave is justified for solution of a phase problem. As applied to the adaptive optics, the method is proposed, which allows separation of the restoration problem for even and odd components of the distorted wave phase by an image.

1. INTRODUCTION

As known, atmospheric instability has a negative effect upon an object's image. The methods of compensation for atmospheric instability can be divided into two types.

The first type covers the methods which can be referred to as interferometric. They are based on the specific processing method. A series of short-exposition images of an object is processed. This allows obtaining of information about object's spatial frequencies. The final stages of these methods is restoration of intensity distribution on an object by spectrum modulus and, conceivably, by some data on spectrum phase.

Methods of the second type (they can be joined under the name "adaptive") assume that an optical system (OS) operates by the new principle.[1] An optical system includes the adaptive element. This element in real time makes the controlled change of a phase of the image-forming wave. The diffraction image of an object can be obtained, if the controlled change is equal to the wave phase distortion introduced by the medium of wave propagation in the absolute value, but opposite in sign. The principles of adaptive optics can be used not only for compensation for atmospheric instability, but also for solution of the problems of large-size optical elements. The control over the adaptive element depends on the measurement method used for phase distortions: in one method, measurement of phase distortion and its compensation by the adaptive element are made at once (with regard for dynamics); in the second method, phase distortion measurement (conceivably, inaccurate) and compensation are stepwise. It is as if OS self-tunes to the diffraction image. This tuning can follow some criteria or condition, or both of them.

The second method is most simple from the viewpoint of phase measurement. It attracts many researchers.[2] That is why just the second method is considered in this paper. To implement it, the methods and algorithms are required to extract the information about phase from the object's image distorted by this phase.

Let us use $G(\xi, \eta) = A(\xi, \eta) \times \exp(i\Phi(\xi, \eta))$ for the complex function (pupil function), which describes amplitude and phase wave distortions at the OS exit pupil Ω. The amplitude $A = 0$ outside Ω. Intensity in the image of a point object is[3]:

$$h(x, y) = |\, g(x, y)|^2 = |\, F(G)|^2, \qquad (1)$$

where F is used for two-dimensional Fourier transform.

Phase restoration by an image is reduced to the following so-called phase problem (PhP): determination of Φ from Eq. (1) from the known A on Ω and h on ω. In particular, the problem of image restoration by the interferometric method is, in essence, the problem given by Eq. (1) for the unknown function $G = A \geq 0$. Problems associated with the phase problem are rather numerous,[4] and they are not restricted by astronomic observations. Difficulties in the phase problem solution are well known. It is necessary to obtain the reliable data on the phase (function G) by h with regard for measurement errors.

Two algorithms for optimization of the numerical solution for PhP can be separated: the Gershberg-Zakstone (GZ) algorithm and the gradient one.[2] In both methods, a solution of the phase problem is the global extreme point of the defining functional, which characterizes the discrepancy between the measured and calculated values of intensity.

The adaptive algorithm for PhP solution, which will be considered below in Section 2, was first proposed in Ref. 5 and then generalized in Ref. 6. Starting from the essence of an adaptive OS, it should solve the following equation:

$$h(x, y; \Phi) - h(x, y; 0) = 0\ , \quad (x, y) \in \omega\ , \qquad (2)$$

where Φ is the difference between the unknown phase and the controlled phase introduced by the adaptive element; $h(x, y, 0)$ is the known intensity corresponding to the diffraction image of a point object. To solve Eq. (2), we use the modified Newton method:

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$$h(x, y; \Phi_n) - h(x, y; 0) =$$
$$= \delta h(x, y; 0) = h'(x, y; 0)\delta\Phi_n\,, \quad (x, y) \in \omega\,, \tag{3}$$

where δh is a variation of h at the point $\Phi = 0$ due to $\delta\Phi$. A variation $\delta\Phi_n$ is just this controlled phase, by which Φ_n should be changed with the help of the adaptive element. The algorithm (3) can be realized, if the derivative $h'(x, y; 0)$ is the reversible operator. A narrower version of the algorithm (3) can be considered. Let us present the phase distortion as a truncated series by some system of basis functions:

$$\Phi = \sum_{s=1}^{n} \zeta_s\, \Phi_s\,. \tag{4}$$

Then PhP is reduced to determination of the coefficients (modes) ζ_s. The algorithm (3) can be realized if the derivatives $\partial h/\partial\zeta_s$ are linearly independent on ω. Just the problem of mode restoration by the algorithm (3) was considered in Refs. 5 and 6.

Sections 2 and 3 of this paper prove the efficiency of the use of the fixed controlled wave phase modulation for solution of the phase problem. In addition, it is shown in Section 3 that if the PhP defining functional is taken as a discrepancy of wave functions (rather than amplitudes) and the method of "dimensionality increase" is applied, then the optimization methods for PhP solution become more numerous.

2. RESTORATION OF THE EVEN AND ODD COMPONENTS OF THE WAVE FRONT

Let us present the phase of the pupil function as

$$\Phi = \Phi^1 + \Phi^2,$$

where $\Phi^1(-\xi, -\eta) = -\Phi^1(\xi, \eta)$ is the odd component of Φ, and $\Phi^2(-\xi, -\eta) = \Phi^2(\xi, \eta)$ is the even one. The components can be expressed in terms of Φ as follows:

$$\Phi^1(\xi, \eta) = \frac{\Phi(\xi, \eta) - \Phi(-\xi, -\eta)}{2}\,,$$

$$\Phi^2(\xi, \eta) = \frac{\Phi(\xi, \eta) + \Phi(-\xi, -\eta)}{2}\,.$$

As to the known amplitude A of the pupil function, let us assume that it is an even function on the exit pupil. Let us also assume that there is a possibility to introduce a fixed controlled action upon the pupil function. This action is reduced to multiplication of the pupil function by the phase factor $G_\theta = e^{-i\alpha\varphi(\xi,\,\eta)}$, where the real function $\varphi(\xi, \eta)$ defines the character of the phase action, and the real coefficient α defines the magnitude of this action. For example, measurement of the intensity $h(x, y)$ in the plane with the coordinate $\alpha = z$, parallel to the focal plane, rather than in the focal plane itself, is tantamount to introduction of the phase modulation defined by the function

$$\varphi(\xi, \eta) = (\xi^2 + \eta^2)/2 = \rho^2/2\,.$$

References 5 and 6, if considered from this viewpoint, apply the modulation of the type of defocusing, and it is shown that at $z = 0$ the modes of particular components do not affect δh. Consequently, they cannot be restored from solution of the problems given by Eqs. (3) and (4).

Let us introduce the following shorthand symbols:

$$C = F(A \cos \alpha\varphi)\,, \quad S = F(A \sin \alpha\varphi)$$
$$F_n(\Phi) = F(\Phi A \cos \alpha\varphi)\,,$$
$$F_S(\Phi) = F(\Phi A \sin \alpha\varphi)\,.$$

With regard for the designations, the function $g(x, y)$ in the OS focal plane with the pupil function $G(\xi, \eta) = A(\xi, \eta)\exp(i\Phi(\xi, \eta))$ and at phase modulation can be written in the following form:

$$g(\Phi) = F(G_0\, A\, e^{i\Phi}) = F_n(e^{i\Phi}) - iF_S(e^{i\Phi})\,.$$

At $\Phi = 0$, the function $g(0) = C - iS$, and the variation $\delta g(0)$ at the point $\Phi = 0$ as a response to the variation $\delta\Phi$ is equal to

$$\delta g(0) = F_n(i\delta\Phi) - iF_S(i\delta\Phi)\,.$$

Let us consider the expression for the variation δh depending on whether φ is even or odd.

1. The function φ is even. The C and S are also even functions, while $F_C(\Phi)$ and $F_S(\Phi)$ are real even or imaginary odd functions depending on whether Φ is even or odd. The intensity variation δh as a response to the wave front variation $\delta\Phi$ at the point $\Phi = 0$ is defined by the following expression:

$$\delta h = 2\mathrm{Re}\; g^*(0)\; \delta g(0) =$$
$$= 2\mathrm{Re}\;(C + iS)\;[F_n(i\delta\Phi) - iF_S(i\delta\Phi)] =$$
$$= 2C\;[F_n(i\delta\Phi^1) + F_S(\delta\Phi^2)] +$$
$$+ 2S\;[-F_n(\delta\Phi^2) + F_S(i\delta\Phi^1)] =$$
$$= 2\begin{pmatrix} C \\ -S \end{pmatrix}^T \begin{pmatrix} F_n(i\delta\Phi^1) + F_S(\delta\Phi^2) \\ -F_S(\delta\Phi^2) + F_n(i\delta\Phi^1) \end{pmatrix} =$$
$$= 2\begin{pmatrix} C \\ -S \end{pmatrix}^T \begin{pmatrix} F_n & F_S \\ -F_S & F_n \end{pmatrix} \begin{pmatrix} i\delta\Phi^1 \\ \delta\Phi^2 \end{pmatrix}, \tag{5}$$

where the symbol T is used for transposition, $\begin{pmatrix} F_n & F_S \\ -F_S & F_n \end{pmatrix}$ is the matrix of linear operators of transformation over the vector of functions $\begin{pmatrix} i\delta\Phi^1 \\ \delta\Phi^2 \end{pmatrix}$. At $\alpha = 0$, the function $S = 0$, F_S is zero operator, and

$$\delta h = 2CF_n(i\delta\Phi^1)\,.$$

This equality can be reversed with good accuracy:

$$\delta\Phi^1 = (2i)^{-1} F^{-1}[C\delta h^1/(\gamma + C^2)] ,$$

where $\gamma \cong 0$ is the positive regulating parameter which excludes division by zero.

Thus, at $\alpha = 0$ the algorithm (3) allows restoration of only odd wave front components. In other words, without application of modulation, the algorithm (3) does not restore even components of wave front.

At $\alpha \neq 0$ $\delta h = \delta h^1 + \delta h^2$, where

$$\delta h^1 = 2 \begin{pmatrix} C \\ -S \end{pmatrix}^T \begin{pmatrix} F_n \\ -F_S \end{pmatrix} (i\delta\Phi^1);$$

$$\delta h^2 = 2 \begin{pmatrix} C \\ -S \end{pmatrix}^T \begin{pmatrix} F_S \\ F_n \end{pmatrix} (\delta\Phi^2) . \qquad (6)$$

The equality (3) breaks into two equalities for respectively odd and even components of the wave front:

$$h^1(\Phi) - h^1(0) = \delta h^1(0) \text{ and } h^2(\Phi) - h^2(0) = \delta h^2(0) . \qquad (7)$$

Realization of the algorithm (7) depends on the properties of the transformations $\delta\Phi^1 \to \delta h^1$ and $\delta\Phi^2 \to \delta h^2$. The condition $\delta h^1 = 0$ is tantamount to stating that vectors $\begin{pmatrix} C \\ -S \end{pmatrix}$ and $\begin{pmatrix} F_n \\ -F_S \end{pmatrix} (i\delta\Phi^1)$ are normal at any $(x, y) \in \omega$. At the points where $CS \neq 0$, this is tantamount to the equalities

$$F_n i\delta\Phi^1 = k^1 S \quad \text{and} \quad -F_S i\delta\Phi^1 = k^1 C , \qquad (8)$$

where k^1 is the odd function determined through $\delta\Phi^1$. If the set, on which $CS \neq 0$, has zero measure, then it follows from Eq. (8) that the set of zeros of the transformation $\delta\Phi^1 \to \delta h^1$ is not very wide.

Similar reasoning shows that the set of zeros of the transformation $\delta\Phi^2 \to \delta h^2$ is determined by the equalities

$$F_n \, \delta\Phi^2 = k^2 C \quad \text{and} \quad -F_S \, \delta\Phi^2 = k^2 S , \qquad (9)$$

where k^2 is the even coefficient of proportionality. It is different for every function $\delta\Phi^2$, for which $\delta h = 0$. In particular, $\delta\Phi^2 = k = \text{const}$ satisfies Eq. (9). But this case is of no interest, because it is sufficient when the even component of phase is known accurate to a constant term. And again we discover that the set of zeros for the second transformation given by Eq. (7) is not very wide.

It follows here from that the algorithm (7) at $\alpha \neq 0$ can be realized for restoration of wave front modes, and the proper choice of α and the function φ can provide for linear independence of partial derivatives $\partial h/\partial \xi_s$. It is also possible to choose α and φ for every mode in such a way that derivatives will be not only linearly independent, but also their Gram matrix will be well-posed.

2. In spite of the function $\alpha\varphi$ defining the phase modulation, let us consider the sum $\psi + \alpha\varphi$, where $\psi = 0$ at the polar angle in the pupil plane having the value within the range $[0, \pi)$ and $\psi = \pi$ with the polar angle in the range $[\pi, 2\pi)$, while φ is the odd function. For example, a controlled total inclination of the wave front gives the odd function $\varphi = \beta x + \gamma y$, it can be created by parallel transfer of the coordinate system in the plane of intensity recording h by the vector $(-\beta, -\gamma)$.

At such modulation, the product $Ae^{i(\psi + \alpha\varphi)} = (Ae^{i\psi})\, e^{i\alpha\varphi}$ has the odd amplitude $A_1 = Ae^{i\psi}$ and the even phase φ. The functions C, S, F_n, and F_S have the following properties: S, $F_n(\delta\Phi^1)$ and $F_S(\delta\Phi^2)$ are even real functions, while C, $F_S(\delta\Phi^1)$ and $F_n(\delta\Phi^2)$ are odd imaginary functions. With regard for the above-said,

$$\delta h = 2 \,\mathrm{Re}\, (C^* + iS)\, [F_n(i\delta\Phi) - i\, F_S(i\delta\Phi)] =$$
$$= 2 \,\mathrm{Re}\, (iC^* - S)\, [F_n(\delta\Phi) - i\, F_S(\delta\Phi)] =$$
$$= 2\, (iC^* - S)\, [F_n(\delta\Phi^1) - i\, F_S(\delta\Phi^2)] ;$$

$$A_1\, \delta\Phi^1 \cos\alpha\varphi - A_1\, \delta\Phi^2 \sin\alpha\varphi = F^{-1}\left[\frac{(iC^* - S)\delta h/2}{\gamma + (iC^* - S)^2}\right] . \qquad (10)$$

If measurements are done at two different modulations, such that the determinant $\begin{vmatrix} \cos(\alpha_1\varphi_1) & \sin(\alpha_1\varphi_1) \\ \cos(\alpha_2\varphi_2) & \sin(\alpha_2\varphi_2) \end{vmatrix} = \sin(\alpha_2\varphi_2 - \alpha_1\varphi_1) \neq 0$, then these measurements allow unambiguous determination of $\delta\Phi^1$ and $\delta\Phi^2$ from Eq. (10). Consequently, the Newton method can be fully used in the problem of subsequent compensation of the wave front.

As we found, restoration of even and odd modes by the algorithm (7) can be done separately. It may be used to decrease the problem dimensionality by choosing the basis from even and odd functions. Thus, the Zernike basis initially consists of even and odd functions. Piecewise linear functions form very convenient basis. Let the exit pupil be divided into $2n$ equal subapertures Ω_s, and the apertures Ω_s and Ω_{s+n}, $s = 1, \ldots, n$, are centrally symmetric. Let the wave front is given by the linear law $\Phi_s = \alpha_s + \beta_s\xi + \gamma_s\eta$ on Ω_s. If χ_s is the characteristic function on Ω_s, equal to unity on Ω_s and zero outside Ω_s, then the following functions form the basis:

$$\chi_s, \chi_s\, \xi, \chi_s\, \eta, \quad s = 1, \ldots, 2n .$$

Let us construct the new basis of them:

$$(\chi_s + \chi_{s+n}),\ (\chi_s + \chi_{s+n})\, \xi ,\ (\chi_s + \chi_{s+n})\, \eta ,$$
$$(\chi_s - \chi_{s+n}),\ (\chi_s - \chi_{s+n})\, \xi ,\ (\chi_s - \chi_{s+n})\, \eta ,$$
$$s = 1, \ldots, 2n ,$$

which consists of even and odd functions. Due to transition to the new basis, the problem dimensionality is halved. Other combinations of subapertures could also be formed, in order to construct the basis consisting of even and odd functions. This reasoning is also applicable to the odd number of subapertures.

3. OPTIMIZING METHOD FOR SOLUTION OF THE PHASE PROBLEM

Let us consider the pupil function G, the product GG_0, and the Fourier transform $F(GG_0)$ as elements of the Hilbert space H of complex-valued functions, square integrable on the pupil plane and the image plane, respectively. Let us form two sets of the initial data in the phase problem:

$$V_1 = \{ g: |g(x, y)| \le h^{1/2}(x, y), \ (x, y) \in \omega \}$$

and

$$V_2 = \{ g: g = F(GG_0), |G(\xi, \eta)| \le A(\xi, \eta) \text{ for } \Omega, \ G(\xi, \eta) = 0 \text{ outside } \Omega \} .$$

Thus, to solve the phase problem, means to find the point of boundary intersection of these two sets $\partial V_1 \cap \partial V_2$. Let G' be the solution of the phase problem, and $g'_2 = F(G_0G') \in \partial V_2$. The method of seeking the point from $\partial V_1 \cap \partial V_2$ depends on the selected strategy. One strategy may consist in construction of the sequence of points $g_n \in H$, approaching $\partial V_1 \cap \partial V_2$. Another strategy may consist in construction of the sequence of two points $g_{1n} \in \partial V_1$ and $g_{2n} \in \partial V_2$, which become closer with increasing n. Let us consider the second strategy named the method of problem dimensionality extension.

Let us give the following functional at the set of pairs $(g_1, g_2) \in \partial V_1 \times \partial V_2$:

$$J(g_1, g_2) = \| g_1 - g_2 \|^2_\omega = \iint_\omega |(g_1 - g_2)|^2 \, dxdy. \quad (11)$$

The pair $(g'_1 = g'_2, g'_2)$ gives the absolute minimum to the functional (11) at $\partial V_1 \times \partial V_2$, which is equal to zero. Herefrom the choice of the algorithm for construction of the minimizing sequence (g_{1n}, g_{2n}) of the functional (11) gives the iteration method for PhP solution.

The following definition will be needed in our consideration. Let g_0 is the point of H, and V is the closed set of H. Let us call the point $\tilde{g} \in V$ a projection of g_0 on V, if

$$\| \tilde{g} - g_0 \| = \min_{g \in V} \| g - g_0 \| .$$

The $\tilde{g}$ projection will be designated as $\tilde{g} = P_V \, g$.

It follows from the equality

$$\min_{(g_1, g_2) \in \partial V_1 \times \partial V_2} J(g_1, g_2) = \min_{g_2 \in \partial V_2} \left(\min_{g_1 \in \partial V_1} J(g_1, g_2) \right)$$

that the minimization problem of the functional (11) at the set $\partial V_1 \times \partial V_2$ can be reduced to the minimization problem of the functional

$$J_2(g_2) = \min_{g_1 \in \partial V_1} J(g_1, g_2) = \| \tilde{g}_1 - g_2 \|^2_\omega = \| \tilde{g}_1 - g_2 \|^2_{\omega + \omega'} ,$$

at ∂V_2, where

$$\tilde{g}_1 = P_{\partial V_1} \, g_2 = \begin{cases} h^{1/2} \, e^{i \arg g_2} & \text{for } \omega, \\ g_2 & \text{for } \omega'; \end{cases}$$

ω' is the space complementing ω to the whole plane OXY.

The functional $J_2(g_2)$ can be also written as follows:

$$J_2(g_2) = \| h^{1/2} - |g_2| \|^2_\omega .$$

The solution of PhP based on minimization of the functional $J_2(g_2)$ was studied in Ref. 2.

Let us construct the algorithm of the coordinate-wise descent to the minimum of the functional $J(g_1, g_2)$. Let $g_{2,n}$ be the nth approximation to g'_2. The approximation of $g_{1,n}$ to g'_1 will be found from the condition $J(g_{1,n}, g_{2,n}) = \min_{g_1 \in \partial V_1} J(g_1, g_{2,n})$, i.e. $g_{1,n} = P_{\partial V_1} \, g_{2,n}$. At fixed $g_{1,n}$ let us find $g_{2,n+1}$ from the condition

$$J(g_{1,n}, g_{2,n+1}) = \min_{g_2 \in \partial V_2} J(g_{1,n}, g_2) \le J(g_{1,n}, g_{2,n}) .$$

From the definition of the functional $J(g_1, g_2)$ and the projection onto a set, it follows that

$$g_{2,n+1} = P_{\partial V_2} \, g_{1,n} = P_{\partial V_2} \, P_{\partial V_1} \, g_2 .$$

However, it is just the GZ functional, which is one of the main methods for PhP solution mentioned in Section 1. Thus, application of the second strategy allows increase in the number of methods for PhP solution. For example, the gradient descent to the minimum of the functional $J(g_1, g_2)$ by both coordinates may prove preferable, at least at some iterations.

Let us made the following transformations in the functional (11):

$$J(g_1, g_2) = \iint_\omega (|g_1|^2 - 2\text{Re} \, g_1^* g_2 + |g_2|^2) \, dxdy =$$

$$= \iint_\omega |g_1|^2 \, dxdy - 2\text{Re} \iint_\omega g_1^* g_2 \, dxdy -$$

$$- \iint_{\omega'} |g_2|^2 \, dxdy + \iint_{\omega + \omega'} |g_2|^2 \, dxdy .$$

With the help of the Parseval equality we establish that the sum

$$\iint_\omega |g_1|^2 \, dxdy + \iint_{\omega + \omega'} |g_2|^2 \, dxdy =$$

$$= \iint_{\omega} h\, dx dy + \iint_{\Omega} A^2\, d\xi d\eta$$

is constant at $\partial V_1 \times \partial V_2$, therefore the minimization problem of the functional (11) is tantamount to the maximization problem of the functional

$$J(g_1, g_2) = 2\mathrm{Re} \iint_{\omega} g_1^*\, g_2\, dx dy +$$

$$+ \iint_{\omega'} |g_2|^2\, dx dy \text{ for } V_1 \times V_2 . \qquad (12)$$

It is shown in Ref. 7 that the intensity distribution in the image space contains the complete information about phase of the wave Φ even in the case, when an object is unknown. Many authors[2] use the intensity distributions in several planes (parallel to the focal one) for obtaining the reliable estimate of the phase from PhP solution. In Section 2 the efficiency of application of defocusing and, in more general form, controlled phase modulation of the wave for PhP solution is demonstrated. Therefore, with regard for phase modulation $G_0(\alpha) = e^{i\alpha\varphi(\xi,\eta)}$, we can consider the following functional, more general than that given by Eq. (11):

$$J_1(g_1, g_2) = \int J_1(g_1, g_2, \alpha)\, d\mu(\alpha) \qquad (13)$$

at the set of pairs $(g_1, g_2) \in \partial V_1 \times \partial V_2$,
where

$V_1 = \{g_1: |g_1(x, y, \alpha)| \le h^{1/2}(x,y,\alpha), (x,y) \in \omega\}$,

and $J_1(g_1, g_2, \alpha)$ is the functional (12),

$V_2 = \{g_2: g_2(x, y, \alpha) = F(G_0(\alpha)G), \; |G| \le A \text{ at } \Omega;$

$g_2 = 0 \text{ outside } \Omega\}$

at different α, $d\mu(\alpha)$ is the measure at the set of α values.

Let us find the variation of the functional (13) at the point (g_1, g_2):

$$\delta J_1(g_1, g_2) = \int \delta J_1(g_1, g_2, \alpha)\, d\mu(\alpha) ,$$

where

$$\delta J_1(g_1, g_2, \alpha) = 2\mathrm{Re} \iint_{\omega} (g_2^* \delta g_1 + g_1^* \delta g_2)\, dx dy +$$

$$+ 2\mathrm{Re} \iint_{\omega'} g_2^* \delta g_2\, dx dy .$$

Introduction of the function $\tilde{g}_1 = \begin{cases} g_1 \text{ at } \omega \\ g_2 \text{ at } \omega' \end{cases}$ allows us to write down the variation

$$\delta J_1(g_1, g_2, \alpha) = 2\mathrm{Re} \iint_{\omega} g_2^* \delta g_1\, dx dy +$$

$$+ 2\mathrm{Re} \iint_{\omega+\omega'} \tilde{g}_1^* \delta g_2\, dx dy .$$

In the polar coordinates $g_1 = h^{1/2} e^{i\theta}$ and $G_2 = A e^{i\Phi}$, hence

$$\delta J_1(g_1, g_2, \alpha) = 2\mathrm{Re} \iint_{\omega} i g_2^*\, g_1 \delta\theta\, dx dy +$$

$$+ 2\mathrm{Re} \iint_{\omega+\omega'} i G_2^* G_2 \delta\Phi\, dx dy .$$

Here, it is taken into account that the Parseval holds true

$$\iint_{\omega+\omega'} \tilde{g}_1^* \delta g_2\, dx dy = \iint_{\omega+\omega'} \left(F^{-1}(\tilde{g}_1)^{-1}\right)^* \delta F^{-1}(g_2)\, d\xi d\eta ;$$

$G_0(\alpha)\tilde{G}_1 = F^{-1}(\tilde{g}_1)$, $\quad G_0(\alpha)\tilde{G}_2 = F^{-1}(\tilde{g}_2)$.

Finally, we obtain that

$$\delta J_1(g_1, g_2) = -2\mathrm{Im} \int d\mu(\alpha) \iint_{\omega} g_2^* g_1 \delta\theta(x,y,\alpha)\, dx dy -$$

$$- 2\mathrm{Im} \iint_{\omega+\omega'} \left(\int \tilde{G}_1^* d\mu(\alpha) \right) G_2 \delta\Phi\, d\xi d\eta$$

and the functional derivative J_1' with respect to the phase components of the functions g_1 and G is equal to

$$J_1' = -2\mathrm{Im} \left(i g_2^*\, g_1, \left(\int \tilde{G}_1^* d\mu(\alpha) \right) G_2 \right) .$$

It can be directly checked that the GZ algorithm for the functional (13) is constructed by the scheme

$$g_{2,n}(\alpha) ; \; g_{1,n}(\alpha) = \left\{ \begin{array}{ll} h^{1/2}(\alpha) e^{i \arg g_{2,n}(\alpha)} & \text{at } \omega, \\ g_{2,n}(\alpha) & \text{at } \omega'; \end{array} \right\}$$

$$G_{2,n+1} = A \arg \int \tilde{G}_1\, d\mu(\alpha);$$

$$g_{2,n+1}(\alpha) = F(G_0\, G_{2,n+1}) .$$

The method of dimensionality extension can be also applied for determination of the point of intersection of the finite number of sets.

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