OPTIMAL ESTIMATION OF THE PHASE FRONT AND IMAGE RECONSTRUCTION AT THE DISTORTIONS OF PHASE

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We present here an optimal recursion procedure for reconstructing images of remote objects. The approach proposed has been developed, using the statistical method of maximum likelihood, and provides for reconstructing at the presence of phase distortions of a signal in the turbulent atmosphere and of the additive background noise at the input of an optical receiver. It is shown that an optimal algorithm can be realized using an adaptive optical system (AOS) based on the wave front generator of a heterodyne type. It is also shown that in contrast to traditional AOSs, the AOS optical arrangement should necessarily include an amplitude corrector, in addition to phase corrector and wave front generator. The amplitude corrector provides for improvement of the accuracy of estimation of the phase distortions in the presence of noises. The potentialities of the optical procedure are shown to be such that no principle need in using an additional reference source occurs.

INTRODUCTION

As known,[1] distortions in the refractive index of a propagation medium result in distortions of the wave front of received signal that may significantly worsen performance characteristics of the traditional optical systems. Such a derating of the performance parameters is especially pronounced when the size of receiving aperture far exceeds the coherence radius of a signal distorted at propagation through the turbulent atmosphere. If there is a non-resolvable reference source within the area of an object isoplanatism, the efficient methods to control signal distortions (and, consequently, distortions of an object image to be reconstructed) in the turbulent medium are the adaptive procedures for processing received signals (and, in particular, the algorithm of phase conjugation).[2] In such cases, the efficient way of separating out the information about signal distortions in the atmosphere from the information about the characteristics of the observed object could be the way of measuring distortions of a signal wave front (WF) and compensating for those when reconstructing the object image. If the reference source has high enough power and parameters of the measuring and correcting devices have been chosen properly, the end characteristics of the adaptive optical systems (AOSs) may be close to the diffraction limited ones.[3] However, the angular misalignment between the observed object and the reference source used to measure the WF distortions must not exceed 2–4 seconds of arc in the visible range. The reference source brightness should be no less than 7–8 star brightness units, in this case.[4] The estimates from Ref. 4 show that the probability of finding a suitable natural reference star within the field of view, when working in an arbitrary area of the sky, is rather low ($\approx 10^{-6}$).

It seems so that at present there are two ways that most likely could enable one to overcome, to a certain extent, the above restrictions when working in the visible wavelength range. One of the ways is connected with creation of artificial laser "stars".[5] It is possible that such a "star" with a relevant brightness can, in principle, be formed in a direction needed. Another one way is related to the development of an optimal procedure for processing the distorted signal from an extended object observed. The procedure must provide for reconstruction of an object image without the use of any additional reference source. The development of such a procedure is the subject of the study presented in this paper.

1. MODEL OF RECEIVED SIGNAL

Let us consider the problem of observing an extended, well resolved in angular coordinates, object that emits spatially incoherent radiation. The radiation scattered by such an object propagates through a random medium. Let this medium be the turbulent atmosphere. For simplicity reasons, let us assume the following conditions (that usually hold in practice):

– the exposure time T is short as compared to the characteristic time of variation of the signal complex phase $\psi(\mathbf{r}, \boldsymbol{\rho}, t) \approx \psi(\mathbf{r}, \boldsymbol{\rho})$, describing the distortions of light waves propagated through a turbulent medium from the point $\boldsymbol{\rho}$ on an object surface to the point $(z, \mathbf{r})$ of the receiving aperture Ω;

– the size of the observed object is small as compared to that of the area of the object isoplanatism

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relative to the receiving aperture; that means that the random phase increment ψ is the same for any point $\boldsymbol{\rho}$ on the object surface, $\psi(\mathbf{r}, \boldsymbol{\rho}) \approx \psi(\mathbf{r})$.

Besides, we assume that the observation system involves a preset, sufficiently narrow-band optical filter (OF).

Taking the above assumptions into account, a model of the received signal can be proposed to have the form

$$y(\mathbf{r}, t) = \mathrm{Re} \int\limits_{-\infty}^{\infty} E(\rho, t)\, G(z, \mathbf{r}, \rho) \mathrm{d}^2\rho \times$$

$$\times \exp\{\psi(\mathbf{r}) - i\omega_0 t\} + n(\mathbf{r}, t),\ \mathbf{r} \in \Omega,\ 0 < t \le T. \tag{1.1}$$

In Eq. (1.1), the function $n(\mathbf{r}, t)$ describes the noise (background) radiation, that is usually considered as the white Gauss noise with the spectral density N_{noi}:

$$<n(\mathbf{r}, t)> = 0,\ <n(\mathbf{r}_1, t_1)\, n(\mathbf{r}_2, t_2)> =$$

$$= N_{\mathrm{noi}}\, \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \delta(t_1 - t_2). \tag{1.2}$$

The angular brackets are used here to denote averaging over an ensemble of background samples, and $\delta(.)$ is the Dirac delta function.

The function $E(\rho, t)$ is the complex amplitude describing the spatial distribution of the field in the plane of the object image. If objects emit spatially incoherent radiation, the function $E(\rho, t)$ can be considered as a sample of a random Gaussian process delta-correlated over the spatial coordinate:

$$<E(\rho, t)> = 0;$$

$$<E(\rho_1, t_1)\, E(\rho_2, t_2)> = 0; \tag{1.3}$$

$$<E(\rho_1, t_1) E^*(\rho_2, t_2)> = O(\rho) \delta(\rho_1 - \rho_2) b(t_1 - t_2),$$

where the function $O(\rho)$ describes the spatial distribution of the radiation power density over the object surface; $b(t)$ is the time correlation function of the signal; the asterisk denotes complex conjugation.

Finally, the Green's function $G(z, \mathbf{r}, \rho)$, entering into Eq. (1.1), describes the propagation of radiation from the object plane to the plane of the receiving aperture. In the Fresnel approximation it has the form

$$G(z, \mathbf{r}, \rho) = \frac{k}{2\pi i z} \exp\left\{ ik \left[z + \frac{|\mathbf{r} - \rho|^2}{2z} \right] \right\}, \tag{1.4}$$

where z is the distance to the object; $k = \omega_0 / c$ is the wave number, and ω_0 is the central frequency of the OF transmission band.

The model given by Eqs. (1.1)–(1.4) sufficiently accurately describes the process of formation of the signal by a rough surface of the object observed as well as the propagation of this signal through the turbulent atmosphere. Now the problem is to extract the information about the object's image $O(\rho)$ from the signal with the above-indicated characteristics, that is, the signal distorted by the turbulent atmosphere. As known, such problems can successfully be solved when using the statistical decision-making theory.[6]

2. OPTIMAL PROCEDURE FOR RECONSTRUCTING DISTORTED IMAGES

The synthesis of optimal systems for processing the information signals, in the statistical decision-making theory, is based on the formation and technical implementation of the so-called functional of the likelihood ratio (FLR).[7] The technique of seeking the FLR for a Gaussian signal observed is well known.[7] The application of this technique to signals described by Eqs. (1.1)–(1.4) allows one to write the conditional logarithm of the FLR in the following form:

$$\Lambda[y(\mathbf{r}, t) / O(\rho), \varphi(\mathbf{r})] = -\frac{m_t\, S}{\lambda^2\, z^2} \int\limits_{-\infty}^{\infty} \ln \left[1 + \frac{O(\rho)}{4N_{\mathrm{noi}}} \right] \mathrm{d}^2\rho +$$

$$+ \frac{1}{N_{\mathrm{noi}}} \int\limits_{-\infty}^{\infty} \frac{O(\rho) / 4N_{\mathrm{noi}}}{1 + O(\rho)/4N_{\mathrm{noi}}}\, P(\rho)\, \mathrm{d}^2\rho, \tag{2.1}$$

where $m_t = \Delta\omega T$ is the number of temporal cells of the signal coherence within the observation interval; $\Delta\omega$ is the OF transmission bandwidth; λ is the average wavelength of the recorded signal; S is the area of the receiving aperture, and the function $P(\rho)$ can be treated as the spatial energy density of the received signal:

$$P(\rho) = \int\limits_0^T |U(\rho, t)|^2\, \mathrm{d}^2\rho; \tag{2.2}$$

$$U(\rho, t) = \int\limits_{\Omega} Y^*(\mathbf{r}, t)\, G(z, \mathbf{r}, \rho)\, \exp\{i\varphi(\mathbf{r})\}\, \mathrm{d}^2 r;$$

$$Y(\rho, t) = \int\limits_0^T y(\mathbf{r}, \tau)\, h(t - \tau)\, \exp\{i\omega_0\tau\}\, \mathrm{d}\tau$$

is the complex amplitude of the field at the OF output; $h(t)$ is the envelope of the OF transfer function.

The functional (2.1) has been obtained under the following additional assumptions:

– signal fluctuations, within the observation interval, are fast ($m_t \gg 1$);

– the observed object is well resolved in angular coordinates ($m_r = SS_0/\lambda^2 z^2 \gg l$, S_0 is the area of the object's projection upon the image plane);

– amplitude fluctuations arising in the signal at propagation through a randomly inhomogeneous medium are ignored ($\psi(\mathbf{r}) \cong i\varphi(\mathbf{r})$, $\varphi(\mathbf{r})$ is the function describing the fluctuations of the real phase);

– the time power spectrum of the signal is approximated by an equivalent "rectangle".

The further step according to the theory used, that is, averaging of Eq. (2.1) over the noise factors, like, for example, phase distortions $\varphi(\mathbf{r})$, does not lead to a success, while yielding cumbersome and hard-to-interpret operations. Therefore, let us do the following: simultaneously with estimating the object's image, estimate the signal distortions described by the function $\varphi(\mathbf{r})$. As shown below the application of such a procedure leads to synthesis of a tracking adaptive optical system, that can easily be realized.

Let us first find, with the help of the functional (2.1), the estimate of the maximum likelihood of the image $\hat{O}(\rho)$ of the object $O(\rho)$ at a fixed sample of the distortions $\varphi(\mathbf{r})$. As seen, this estimate has the following form:

$$\hat{O}(\rho) = \begin{cases} \dfrac{4\lambda^2 z^2}{m_t S}\,[P(\rho/\varphi(\mathbf{r})) - P_{\text{noi}}], & P(\rho) > P_{\text{noi}}, \\ 0, & P(\rho) < P_{\text{noi}}, \end{cases} \quad (2.3)$$

where $P_{\text{noi}} = E_{\text{noi}}/S_0$ is the energy density of the background radiation in one spatial cell of the signal coherence, $E_{\text{noi}} = N_{\text{noi}} m_t m_r$ is the average energy of the noise background in the bulk of the observational data recorded.

Thus obtained procedure of the object image estimation is very similar to the traditional algorithm of optical image formation with the use of ordinary optical systems. The process of image formation under noisy conditions and with the phase distortions present differs from the traditional procedure, as seen from Eq. (2.3), by the requirement to introduce the noise threshold in the image plane and the phase mask in the entrance aperture plane. The noise threshold serves to exclude areas of unreliable data, with low SNR, from the traditionally formed image, while the phase mask compensates for phase distortions, $\varphi(\mathbf{r})$, of the signal.

To find an optimal procedure to compensate for these distortions, at a completely unknown object under study, the estimate of the object image presented by the Eq. (2.3) is substituted into the FLR logarithm, Eq. (2.1). Thus we obtain the estimate of the FLR logarithm, which is independent of the object image. Consequently, this estimate can be used for estimating the wave front distortions:

$$\Lambda[y(\mathbf{r}, t)/\hat{O}(\rho), \varphi(\mathbf{r})] = \\ = \frac{m_t S}{\lambda^2 z^2} \int\limits_{P(\rho) > P_{\text{noi}}} \left[\frac{P(\rho)}{P_{\text{noi}}} - 1 - \ln \frac{P(\rho)}{P_{\text{noi}}}\right] \mathrm{d}^2\rho. \quad (2.4)$$

It should be noted that the logarithmic criterion of the optimal compensation for the phase distortions of a signal has been earlier derived in Ref. 8.

The functional (2.4) allows one to derive an explicit expression describing the optimal procedure for the WF estimation. However, one ought to keep in mind that it is impossible to estimate a continuous function under noisy conditions using the maximum likelihood method, that is, without the account of *a priori* information on the function. It is obvious that only the information about the finite number of parameters of this function can be extracted with a sufficiently high accuracy. In the case under consideration, it is convenient to pass to a step-wise approximation of the function $\varphi(\mathbf{r})$:

$$\varphi(\mathbf{r}) = \sum_{j=1}^{N} \varphi_j\, \chi_j(\mathbf{r}), \quad (2.5)$$

where N is the number of the parameters φ_j, estimated and $\chi_j(\mathbf{r})$ is the indicator function of the WF area within the receiving aperture analyzed: $\chi_j(\mathbf{r}) = 1$ for $\mathbf{r} \in \Omega_j$ and $\chi_j = 0$ in other cases, and $\bigcup_{j=1}^{N} \Omega_j = \Omega$.

Now, to estimate the unknown values of the "steps" in Eq. (2.5), we can apply the maximum likelihood method. At the same time, the available *a priori* information about the function $\varphi(\mathbf{r})$ (and about the intensity of the noise, in the bulk of observation data) should be used for selecting optimal number N of the parameters to be estimated.

Assuming that we have already gotten an estimate of the object image $\hat{O}^{(n)}(\rho)$, we can easily obtain, from Eq. (2.4) with the account of the approximation (2.5), the following recursion procedure for recovering the observed object image from the distortions while simultaneously estimating the distortions and compensating for them:

$$\begin{cases} \hat{O}^{(n)}(\rho) = \dfrac{4\lambda^2 z^2}{m_t S}[P(\rho/\hat{\varphi}_j^{(n)}, j=1,\ldots,N) - P_{\text{noi}}], \; P(\rho) > P_{\text{noi}}; \\ \exp\{i\hat{\varphi}_j^{(n+1)}\} = \dfrac{\int_0^T \mathrm{d}t \int_{\Omega_j} Y(\mathbf{r},t) X^{(n)}(\mathbf{r},t)\,\mathrm{d}^2r}{\left|\int_0^T \mathrm{d}t \int_{\Omega_j} Y(\mathbf{r},t) X^{(n)}(\mathbf{r},t)\,\mathrm{d}^2r\right|}, \; n = 1, 2, \ldots, \end{cases} \quad (2.6)$$

where $\hat{\varphi}_j^{(1)} = 0$; $j = 1, \ldots, N$; n is the running iteration number. Besides, the following designations have been used in Eqs. (2.6): $\hat{O}^{(n)}(\rho)$ and $\hat{\varphi}_j^{(n)}$ are the estimate of the object image and of the phase value that provides for compensation within the area Ω_j of the aperture at the nth step of the recursion procedure; $X^{(n)}(\mathbf{r}, t)$ is the complex amplitude of the field in the plane conjugate to the plane of the entrance aperture at the nth step:

$$X^{(n)}(\mathbf{r}, t) = \int\limits_{\hat{O}^{(n)}(\rho) > 0} A^{(n)}(\rho)\, U^{(n)}(\rho, t)\, G^*(z, \mathbf{r}, \rho)\, \mathrm{d}^2\rho, \quad (2.7)$$

$$U^{(n)}(\rho, t) = \sum_{j=1}^{N} \exp\left\{i\hat{\varphi}_j^{(n)}\right\} \int\limits_{\Omega_j} Y^*(\mathbf{r}, t)\, G(z, \mathbf{r}, \rho)\, \mathrm{d}^2r, \quad (2.8)$$

$$A^{(n)}(\rho) = 1 - \frac{P_{\text{noi}}}{P(\rho/\hat{\varphi}_j^{(n)}, j = 1, \ldots, N)}. \quad (2.9)$$

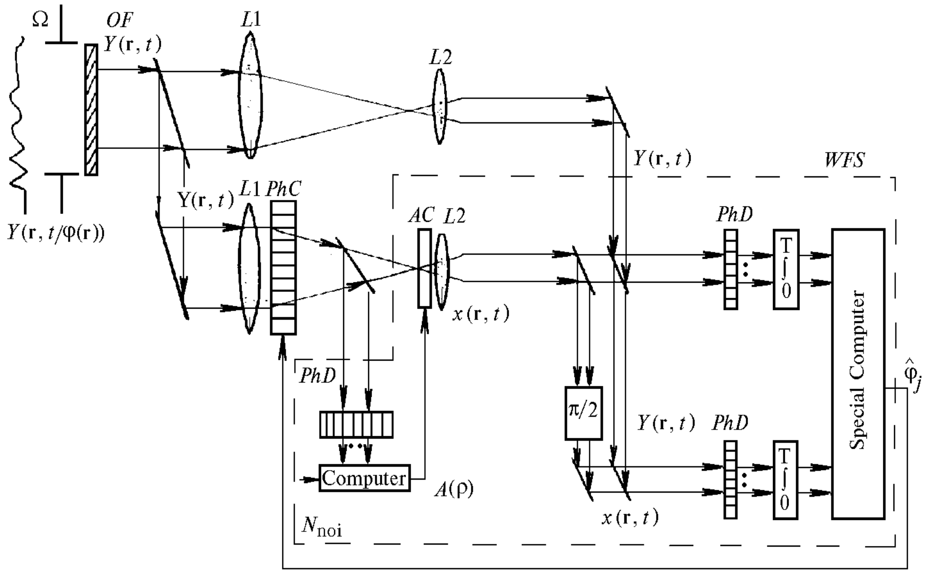

FIG. 1. Block diagram of the AOS for reconstruction of a distorted image of an object.

It is easy to see that the above procedure (2.6)–(2.9) can be implemented when using a device with the optical arrangement shown in the Figure 1. According to Eqs. (2.6)–(2.9) the optical lay-out involves two main elements: a multichannel phase corrector (PhC) and a multichannel WF sensor (WFS). The wave front sensor involves an amplitude correcting mask whose action is described by the function (2.9). This mask lowers the weights of the image points processed (in order to extract information about the phase distortions) where the signal-to-noise ratio is high enough. In situations, where an additive noise is negligible (the signal-to-noise ratio tends to infinity), the amplitude corrector may be excluded, because in this case its transfer coefficient approaches unity for all points of the image processed. In such cases, an optimal scheme becomes close to that of a traditional AOS with a single PhC. In the general case, the optimal scheme of an AOS involves two controlled correctors, that is one for phase and the other one for amplitude corrections. Let us describe its operation using Eqs. (2.6)–(2.9).

At the first step ($n = 1$) of the iterative process, no compensation is being introduced for a distortion of phase front, that means that the processed phase front coincides with the initial one, and the correcting phases $\hat{\varphi}_j^{(1)} = 0$, $j = 1, ..., N$. In this case the first expression in Eq. (2.6) corresponds, at the first step, to the distorted image $\hat{O}^{(1)}(\rho)$, that has not been corrected (except for sections with low signal-to-noise ratio). The "reference" signal $X^{(1)}(\mathbf{r}, t)$ for the WFS is formed based on this image. The complex amplitude of this, already partially corrected, signal is determined by Eq. (2.7). At the first stage, the "reference" signal is corrected only in the amplitude corrector (AC) channel situated in the WFS image plane. The "reference" signal comes to a multichannel photodetector of the WFS being placed in the plane conjugate to the plane of the entrance lens. Simultaneously the initial signal $Y(\mathbf{r}, t)$ enters the photodetector. The described sensor is apparently a multichannel WFS of the heterodyne type. At the first step it forms correcting phases $\hat{\varphi}_j^{(2)}$ according to the second expression of the Eq. (2.6), thus changing the state of the PhC. The change in the PhC state results in a correction of the object image based on the first expression of the Eq. (2.6). The image takes, at this stage, the form $\hat{O}^{(2)}(\rho)$. Then the AC changes its state in accordance with the image obtained. Then, the "reference", for the WFS, signal $X^{(2)}(\mathbf{r}, t)$ is repeatedly corrected. Now, according to Eqs. (2.7)–(2.9), the correction is being done simultaneously in two places, that is, in the PhC that is in the aperture plane and in the AC in the image plane. The PhC in this case is used to improve image reconstruction and to improve the quality of estimating the phase distortions at the next iteration, while the AC is only used to improve the accuracy of estimating the phase distortions.

The process is reiterated until the stationary state is reached, at which certain improvement of the image quality is achieved. This is the way to realize the process of adaptive correction of a distorted image in an optimal AOS.

CONCLUSIONS

1. With the use of the statistical approach, the optimal recursion procedure has been developed to reconstruct the image of a remote object, the optical signal from which is distorted by the turbulent atmosphere and whose *a priori* characteristics are completely unknown.

2. The procedure can be implemented in an adaptive tracking optical system. However, in

contrast to traditional AOSs, the main elements of an optimal arrangement of such a system are not only a multichannel phase corrector and a WF sensor of the heterodyne type, but also a controlled amplitude corrector placed in the image plane. The amplitude corrector serves to increase the accuracy of estimating the phase distortions in the presence of noise. In the case, when the additive noise is negligibly low, there is no need in an amplitude corrector.

3. The potentialities of the procedure sought eliminate, in principle, the necessity of using any additional reference source of light. This, to a significant degree, enables one to overcome the difficulty connected with the low probability of finding a natural star within the zone of isoplanatism of an object observed with astronomical systems in the visible spectral range.

4. One particular, but very interesting result should certainly be noted in the conclusion. It is the determination of an objective criterion of optimum in the problem of estimation of the wave front distortions in a signal from an extended object. It turns out that the maximum of integral of some nonlinear function of the energy density of a received signal over the area occupied by the object image estimate serves as the criterion of optimum. Generally speaking, a random realization of the spatial energy density of a received signal in the image plane during the exposure time forms the statistics that is sufficient for solving the problems of reconstructing the images of extended objects that are distorted, as well as for estimating the phase fronts of signals from such objects.

5. To estimate the convergence rate of the optimal recursion reconstruction algorithm and, consequently, to reveal the necessary characteristics of the operation rate of the executive elements of the phase and amplitude correctors, some additional investigations into the statistical modeling of the algorithm operation are needed.

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