GREEN'S FUNCTIONS OF A LENS–LIKE MEDIUM

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The Green's functions are derived for the parabolic equation of "quasioptics" describing the propagation of optical radiation through a lens–like defocusing medium with a variable focal length. It is shown that the reciprocity does not hold for the Green's functions of a lens–like medium with a variable focal length. It is fulfilled for a lens–like medium with a constant focal length.

The propagation of optical radiation in a lens–like defocusing medium, whose optical axis coincides with the $0x$ axis is described by the parabolic equation of "quasioptics"[1–3]

$$\left\{2ik\frac{\partial}{\partial x}+\Delta_\perp+\frac{k^2\rho^2}{F^2(x)}\right\}U(x,\ \rho)=0,$$

$$U(0,\ \rho)=U_0(\rho), \tag{1}$$

where $U(x,\ \rho)$ is the parabolic amplitude of the optical field, x is the longitudinal coordinate, $\rho=\{y,\ z\}$ are the transverse coordinates, $k=2\pi/\lambda$, λ is the radiation wavelength, $F(x)$ is the local focal length of the lens–like medium (refraction channel), and $\Delta_\perp=\delta^2/\delta y^2+\delta^2/\delta z^2$ is the transverse Laplacian operator. The corresponding Green's function satisfies the equation[4–6]

$$\left\{2ik\frac{\partial}{\partial x}+\Delta_\perp+\frac{k^2\rho^2}{F^2(x)}\right\}G(x,\ \rho;\ x',\ \rho')=0 \tag{2}$$

with the boundary condition

$$G(x,\ \rho;\ x',\ \rho')\big|_{x=x'}=\delta(\rho-\rho').$$

The Green's function $G(x,\ \rho;\ x,\ \rho')$ describes the field of the spherical wave which propagates from the point $(x',\ \rho')$ in the positive direction along the $0x$ axis. It is possible to show that the solution of Eq. (2) for $0\le x'<x$ has the form

$$G(x,\ \rho;\ x',\ \rho')=$$

$$=\frac{k}{2\pi iF_0U_2\left(\frac{x-x'}{F_0}\right)}\exp\left\{\frac{ikU_2'\left(\frac{x-x'}{F_0}\right)}{2F_0U_2\left(\frac{x-x'}{F_0}\right)}\rho^2-\right.$$

$$\left.-\frac{ik}{F_0U_2\left(\frac{x-x'}{F_0}\right)}\rho\rho'+\frac{ikU_1\left(\frac{x-x'}{F_0}\right)}{2F_0U_2\left(\frac{x-x'}{F_0}\right)}\rho'^{\,2}\right\}, \tag{3}$$

where the functions $U_1\left(\frac{x-x'}{F_0}\right)$ and $U_2\left(\frac{x-x'}{F_0}\right)$ are the particular solutions of the equation

$$U''\left(\frac{x-x'}{F_0}\right)-\frac{F_0^2}{F^2(x)}U\left(\frac{x-x'}{F_0}\right)=0$$

with the boundary conditions

$$U_1(0)=U_2'(0)=1,\quad U_1'(0)=U_2(0)=0,$$

while $F_0=F(\mathrm{x}=\mathrm{x}')$ is the "initial" value of the focal length of the lens–like medium.

The Green's function $G(x,\ \rho;\ x',\ \rho')$ of Eq. (3) satisfies the normalization conditions

$$\int_{-\infty}^{\infty}\!\!\int d\rho G(x,\ \rho;\ x',\ \rho'))\big|_{x=x'}=\int_{-\infty}^{\infty}\!\!\int d\rho' G(x,\rho;\ x',\rho'))\big|_{x=x'}=1$$

and the orthogonality relations

$$\int_{-\infty}^{\infty}\!\!\int d\rho G(x,\ \rho;\ x',\ \rho')G^*(x,\ \rho;\ x',\ \rho'')=\delta(\rho'-\rho'')\ ;$$

$$\int_{-\infty}^{\infty}\!\!\int d\rho' G(x,\ \rho_1;\ x',\ \rho')G^*(x,\ \rho_2;\ x',\ \rho')=\delta(\rho_1-\rho_2)\ .$$

In solving the problems of the reflection of the optical waves from a mirror, it is necessary to know the Green's function $\tilde G(x',\ \rho';\ x,\ \rho)$ which describes a spherical wave propagating in the negative direction of the 0x axis from the point (x, ρ). For $x\gg x'$,

$$\tilde G(x',\ \rho';\ x,\ \rho)=$$

$$=\frac{k}{2\pi iF_0\tilde U_2\left(\frac{x-x'}{F_0}\right)}\exp\left\{\frac{ik\tilde U_2'\left(\frac{x-x'}{F_0}\right)}{2F_0\tilde U_2\left(\frac{x-x'}{F_0}\right)}\rho'^2-\right.$$

$$\left.-\frac{ik}{F_0\tilde U_2}\left(\frac{x-x'}{F_0}\right)\rho\rho'+\frac{ik\tilde U_1\left(\frac{x-x'}{F_0}\right)}{2F_0\tilde U_2\left(\frac{x-x'}{F_0}\right)\rho^2}\right\}, \tag{4}$$

where the functions $\tilde U_1\left(\frac{x-x'}{F_0}\right)$ and $\tilde U_2\left(\frac{x-x'}{F_0}\right)$ are the particular solutions of the equation

$$\tilde{U}''\left(\frac{x-x'}{F_0}\right) - \frac{F_0^2}{\tilde{F}^2(x')}\tilde{U}\left(\frac{x-x'}{F_0}\right) = 0$$

with the boundary conditions

$$\tilde{U}_1(0) = \tilde{U}_2'(0) = 1\ , \quad \tilde{U}_1'(0) = \tilde{U}_2(0) = 0\ ,$$

while $\tilde{F}(x')$ is the mirror image of the function $F(x)$ (see Ref. 2).

It follows from formulas (3) and (4) that the reciprocity relation for the Green's function of a lens–like medium

$$G(x,\ \rho;\ x',\ \rho') = \tilde{G}(x',\ \rho';\ x,\ \rho)\ (x-x') \tag{5}$$

holds only under the following conditions:

$$\begin{cases} U_2\left(\frac{x-x'}{F_0}\right) = \tilde{U}_2\left(\frac{x-x'}{F_0}\right), \\ U_1\left(\frac{x-x'}{F_0}\right) = \tilde{U}_2'\left(\frac{x-x'}{F_0}\right), \\ U_2'\left(\frac{x-x'}{F_0}\right) = \tilde{U}_1\left(\frac{x-x'}{F_0}\right). \end{cases} \tag{6}$$

In a lens–like medium (refraction channel) with a constant focal length $F(x) = \tilde{F}(x') = F_0$ (see Ref. 1–3) these conditions are satisfied as follows:

$$\begin{cases} U_2\left(\frac{x-x'}{F_0}\right) = \tilde{U}_2\left(\frac{x-x'}{F_0}\right) = \mathrm{sh}\left(\frac{x-x'}{F_0}\right), \\ U_1\left(\frac{x-x'}{F_0}\right) = \tilde{U}_2'\left(\frac{x-x''}{F_0}\right) = \mathrm{ch}\left(\frac{x-x'}{F_0}\right), \\ U_2'\left(\frac{x-x'}{F_0}\right) = \tilde{U}_1\left(\frac{x-x'}{F_0}\right) = \mathrm{ch}\left(\frac{x-x'}{F_0}\right), \end{cases}$$

and, consequently, the reciprocity relation holds for the Green's functions (5) is valid. An analogous situation is observed for lens–like media with a symmetric distribution of the local focal length $F(x)$ with respect to the point $(x-x')/2$. In this case $F(x) = \tilde{F}(x')$ and, consequently,

$$U_1\left(\frac{x-x'}{F_0}\right) = \tilde{U}_1\left(\frac{x-x'}{F_0}\right), U_2\left(\frac{x-x'}{F_0}\right) = \tilde{U}_2\left(\frac{x-x'}{F_0}\right),$$

and

$$U_1'\left(\frac{x-x'}{F_0}\right) = \tilde{U}_1\left(\frac{x-x'}{F_0}\right), U_2'\left(\frac{x-x'}{F_0}\right) = \tilde{U}_2\left(\frac{x-x'}{F_0}\right),$$

i.e., conditions (6) and the reciprocity relation (5) are satisfied. For lens–like media with variable focal length for which $F(x) \ne \tilde{F}(x')$, conditions (6) are not satisfied, since

$$U_1\left(\frac{x-x'}{F_0}\right) \ne \tilde{U}_1\left(\frac{x-x'}{F_0}\right)$$

and

$$U_2\left(\frac{x-x'}{F_0}\right) \ne \tilde{U}_2\left(\frac{x-x'}{F_0}\right).$$

For example, for $\alpha\,\frac{x-x'}{F_0} \ll 1$ and

$$F^2(x) = F_0^2\left(1 + \alpha\,\frac{x-x'}{F_0}\right)$$

the following expressions:

$$U_1\left(\frac{x-x'}{F_0}\right) \simeq \mathrm{ch}\left(\frac{x-x'}{F_0}\right) + \frac{\alpha}{4}\left(\frac{x-x'}{F_0}\right) \times$$
$$\times \left[\mathrm{ch}\left(\frac{x-x'}{F_0}\right) - \left(\frac{x-x'}{F_0}\right)\mathrm{sh}\left(\frac{x-x'}{F_0}\right)\right] -$$
$$- \frac{\alpha}{4}\,\mathrm{sh}\left(\frac{x-x'}{F_0}\right),$$

$$U_2\left(\frac{x-x'}{F_0}\right) \simeq \mathrm{sh}\left(\frac{x-x'}{F_0}\right) + \frac{\alpha}{4}\left(\frac{x-x'}{F_0}\right) \times$$
$$\times \left[\mathrm{sh}\left(\frac{x-x'}{F_0}\right) - \left(\frac{x-x'}{F_0}\right)\mathrm{ch}\left(\frac{x-x'}{F_0}\right)\right],$$

and

$$\tilde{U}_1\left(\frac{x-x'}{F_0}\right) \simeq \mathrm{ch}\left(\frac{x-x'}{F_0}\right) - \frac{\alpha}{4}\left(\frac{x-x'}{F_0}\right) \times$$
$$\times \left[\mathrm{ch}\left(\frac{x-x'}{F_0}\right) - \left(\frac{x-x'}{F_0}\right)\mathrm{sh}\left(\frac{x-x'}{F_0}\right)\right] +$$
$$+ \frac{\alpha}{4}\,\mathrm{sh}\left(\frac{x-x'}{F_0}\right),$$

$$\tilde{U}_2\left(\frac{x-x'}{F_0}\right) \simeq \mathrm{sh}\left(\frac{x-x'}{F_0}\right) - \frac{\alpha}{4}\left(\frac{x-x'}{F_0}\right) \times$$
$$\times \left[\mathrm{sh}\left(\frac{x-x'}{F_0}\right) - \left(\frac{x-x'}{F_0}\right)\mathrm{ch}\left(\frac{x-x'}{F_0}\right)\right],$$

i.e.,

$$U_2\left(\frac{x-x'}{F_0}\right) - \tilde{U}_2\left(\frac{x-x'}{F_0}\right) \simeq \frac{\alpha}{2}\left(\frac{x-x'}{F_0}\right) \times$$
$$\times \left[\mathrm{sh}\left(\frac{x-x'}{F_0}\right) - \left(\frac{x-x'}{F_0}\right)\mathrm{ch}\left(\frac{x-x'}{F_0}\right)\right] \ne 0,$$

$$U_1\left(\frac{x-x'}{F_0}\right) - \tilde{U}_2'\left(\frac{x-x'}{F_0}\right) \simeq -\frac{\alpha}{2}\left(\frac{x-x'}{F_0}\right)^2 \mathrm{sh}\left(\frac{x-\boldsymbol{x}'}{F_0}\right) \ne 0\ ,$$

$$U_2'\left(\frac{x-x'}{F_0}\right) - \tilde{U}_1'\left(\frac{x-x'}{F_0}\right) \simeq -\frac{\alpha}{2}\left(\frac{x-x'}{F_0}\right)^2 \mathrm{sh}\left(\frac{x-x'}{F_0}\right) \ne 0.$$

Thus, the reciprocity relation for the Green's function in this case is not satisfied. Because of this, the use of the reciprocity relation for the Green's function of a lens–like medium with variable focal length in Refs. 7 and 8 is erroneous.

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