APPLICABILITY LIMITS OF THE SMOOTH PERTURBATION METHOD

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The smooth perturbation method (SPM) or Rytov's method is considered in this paper in the context of more rigorous theory of multiple wave scattering. It is shown that the SPM adequately describes the multiple scattering of waves only by those inhomogeneities of a scattering medium for which an observation point is in their near zone.

The smooth perturbation method (SPM) or Rytov's method is one of the most well–known approximate methods for solving the problems on optical and acoustic wave propagation through randomly inhomogeneous media like the turbulent atmosphere.[1-3]

The applicability limits of the smooth perturbation method are usually determined by the condition of small magnitudes of the field amplitude fluctuations A

$$\frac{<A^2> - <A>^2}{<A>^2} \ll 1 . \quad (1)$$

The condition given by Eq. (1) is formal mathematical in character, and the physical meaning of restrictions inherent in this method remains unresolved.

In this paper we consider the SPM in the context of the more vivid theory of multiple wave scattering and estimate the applicability limits of the SPM in terms of this theory.

1. In the beginning we outline the basic principles of the theory of multiple wave scattering.[4] We describe the wave propagation in an arbitrary medium by the generalized operator equation

$$(L - V)\Psi = 0 , \quad (2)$$

where Ψ is the arbitrary wave field, L is the operator describing the wave propagation in free space, and V is the operator describing the medium. Let us break up the operator V into the sum in an arbitrary way

$$V = \sum V_j , \quad (3)$$

where every summand is referred to as the j th scatterer.

Solution to the problem on wave propagation through the medium V in the theory of multiple wave scattering is expressed in terms of the solution to the problem of scattering by each individual scatterer

$$(L - V_j)\,\Psi_j = 0 . \quad (4)$$

As is well–known, solution of Eq. (4) is the superposition of incident and scattered fields

$$\Psi_j = \Psi_0 + \Psi_{sj} . \quad (5)$$

Although the calculation of the scattered field in each concrete case can be a cumbersome mathematical problems it can be easy written down in terms of formal operators. Usually it is convenient to write down the scattered field in terms of the so–called T–matrix of a given scatterer

$$\Psi_{sj} = L^{-1}\, T_j\, \Psi_0 , \quad (6)$$

where the T–matrix is defined by the Born expansion

$$T_j = V_j + V_j L^{-1} V_j + V_j L^{-1} V_j L^{-1} V_j + \ldots \quad (7)$$

or by the corresponding operator equation. There is no need to write it out here.

Based on the above–made definitions given by Eqs. (4)–(7), we can easy find the solution to the problem on propagation or multiple scattering of an incident wave in a randomly inhomogeneous medium [Eq. (3)] on the level of operator relations. Namely, it is determined by the expansion in the multiplicity of scattering

$$\Psi = \Psi_0 + \sum L^{-1} T_j \Psi_0 + \sum_{j|l} L^{-1} T_l L^{-1} T_j \Psi_0 + \ldots . \quad (8)$$

Physical meaning of expansion (8) is obviously clear. Here the second summand means singly scattered field or superposition of waves produced due to scattering by each individual scatterer, the third summand is the superposition of waves scattered sequentially by two scatterers, and so on.

It should be stressed that relations (4)–(8) are quite general and independent of both the nature of a wave field and way of division of the medium into the individual scatterers. In particular, the scatterers can be not only one–dimensional for layered inhomogeneous media but also three–dimensional. They can either not to overlap each other, like aerosols in the atmosphere, or be inserted one inside the other, like turbulent eddies of the refractive index, and so on. The division of the medium into the scatterers is only a matter of convenience for physical interpretation or mathematical description.

2. Now we discuss the application of the theory of multiple scattering to such a randomly inhomogeneous medium as the turbulent atmosphere. It is clear that concrete realization of the turbulent atmosphere distorts, i.e., scatters the propagating wave by some inhomogeneities in the refractive index existing at the moment and being bounded in space. These inhomogeneities are referred to as

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scatterers in the theory of wave multiple scattering. If we divide the turbulent medium into the system of discrete scatterers applying a certain rule chosen by us, we can interpret the process of wave propagation through this medium as the multiple scattering and describe it by series (8).

To create the quantitative theory of wave propagation through the turbulent atmosphere in this way, rather cumbersome model of scatterers (3) is necessary. This model should take into account the wide range of variation of their dimensions from the internal to the outer turbulent scale, the insertion of the refractive index eddies one inside the other, and so on.

In our case to make qualitative estimation of the applicability limits of the SPM, we restrict ourselves to an examination of the simplest model of the randomly inhomogeneous medium. We assume that the medium consists of randomly located inhomogeneities of the close size a, which can overlap in space (see Fig. 1). In addition, we assume longitudinal and transverse dimensions being of the same order and equal to a without loss of generality of our model. Very large range of variation of the inhomogeneities in size, inherent in the turbulent atmosphere, will be taken into account on the qualitative level.

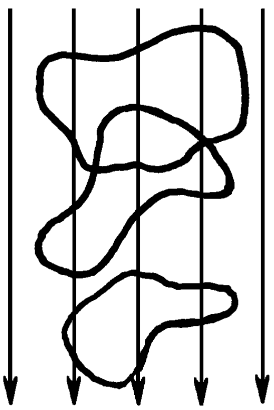

FIG. 1 Multiple wave scattering in the near zone of scatterers is equivalent to shading of some scatterers by others.

As mentioned above, expansion in the multiplicity of scattering (8) is quite general. Thus we can interpret any approximation of the problem on propagation or multiple wave scattering in the inhomogeneous media as some approximations of the terms of expansion (8), that is, of the field of different multiplity of scattering. It is precisely such interpretation that is used in the SPM below.

In the SPM approximation the field is usually written down on the basis of the parabolic equation

$$\left[2ik \frac{\partial}{\partial z} + \Delta_{\perp} - \nu(\;\; , z) \right] u(\rho, z) = 0 , \qquad (9)$$

where z is the longitudinal coordinate, $\rho = x, y$ specifies the transverse coordinates, $\Delta_{\cdot}$ is the Laplacian with respect to ρ, $v(\rho, z) = k^2 [n^2(\rho, z) - 1]$, k is the wave number, and n is the refractive index of a medium. If we compare Eq. (9) with generalized Eq. (1), we see that

$$\Psi = u(\rho, z) , L = 2ik \frac{\partial}{\partial z} + \Delta_{\perp}, V(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \nu(\rho, z). \qquad (10)$$

In the simplest case of the plane incident wave $u_0 = 0$, the field in the SPM approximation on account of Eq. (10) has the following simple form:

$$u = \exp (L^{-1} \nu) . \qquad (11)$$

The division of the medium into a sum of discrete scatterers (see Eq. (3)) reduces initial Eq. (11) to the form

$$u = \exp \left(\sum L^{-1} \nu_j \right) \qquad (12)$$

which can be represented by the product of multipliers

$$u_j = \exp (L^{-1} \nu_j) . \qquad (13)$$

Let us consider the physical meaning of such multiplier. For this purpose we first expand the exponent in Eq. (13) in the Taylor series

$$u_j = 1 + L^{-1} \nu_j + (L^{-1} \nu_j)^2 / 2! + \dots . \qquad (14)$$

The first summand in Eq. (14) is the incident field. The second summand

$$\omega_{jB} = L^{-1}\nu_j \qquad (15)$$

is the scattered field in the so–called Born approximation when the T–matrix is substituted by the first summand of expansion (7).

It is well known that the Born approximation holds when the run–on of the phase within the scatterer is small

$$k(n_j - 1) a_j \ll 1 . \qquad (16)$$

We usually have $n - 1 \sim 10^6$ in the visible range for the turbulent atmosphere. In this case, for example, for He–Ne laser radiation with the parameter $k \sim 10^7$ m^{-1}, the Born approximation holds only for inhomogeneities of size $a \ll 0.15$ m. Thus for most real inhomogeneities in the turbulent atmosphere the Born approximation is inapplicable, and the next terms of expansion (7) in scattered field must be taken into account.

Let us show that the terms of series (14) correspond to the terms of series (7), but only in the definite spatial region. Each inhomogeneity in the turbulent atmosphere is a large and optically soft scatterer, that is

$$k a_j \gg 1 , \quad |n_j - 1| \ll 1 . \qquad (17)$$

When a plane wave is incident on such a scatterer, the resulting field within it differs from the plane wave only in the additional run–on of the phase accumulated along the direct rays ρ = const

$$u_j^0 = \exp\left[(2\,i\,k)^{-1} \int \nu_j(\rho,\,z')\,\mathrm{d}z'\right]. \tag{18}$$

Expressions like Eq. (18) are often used in solving the problems on wave propagation in the turbulent atmosphere, where they are associated with the geometric optics approximation. At the same time ray refraction is inherent in the geometric optics approximation, but it is ignored here. Thus it is more natural to refer to approximation (18) as the direct ray approximation (DRA). On the basis of parabolic equation (9) the DRA is obtained, if we drop the operator $\Delta_\perp$ describing the field diffraction, i.e., substitute the operator L in the parabolic equation by the operator L_0 in the direct ray approximation

$$L_0 = 2\,i\,k\,\frac{\partial}{\partial z}\,. \tag{19}$$

As we can see, physically justified expression (18) coincides with Eq. (13) obtained from the SPM in the context of the direct ray approximation

$$u_j^0 = \exp\,(L_0^{-1}\,\nu_j)\,. \tag{20}$$

In this case the scattered field, according to definitions (5), is equal to

$$\omega_j^0 = \exp\,(L_0^{-1}\,\nu_j)\;L_1\,. \tag{21}$$

Equations (18), (20), and (21) hold not only within the region occupied by the scatterer, but also in its near zone at distances

$$z - z_j \ll k\,a^2 \tag{22}$$

from the scattering center z_j located on the z axis, where diffraction is still negligible. These expressions become incorrect starting from distances $z - z_j \sim ka^2$, where scattered field (21) is distorted by Fresnel diffraction. In the wave zone of the scatterer

$$z - z_j \gg k\,a^2 \tag{23}$$

the Fraunhofer diffraction transforms the scattered field into a diverging spherical wave

$$\omega_j = f_j\,(\mathbf{n},\,\mathbf{n}_0)\,\frac{\exp\left[i\,k\,|\mathbf{r}-\mathbf{r}_j| - i\,k(z-z_j)\right]}{|\mathbf{r}-\mathbf{r}_j|}\,, \tag{24}$$

where $\mathbf{r}_j = (\rho_j\,,\,z_j)$ specifies the position of the scattering center and f_j is the scattering amplitude in the direction $\mathrm{n} = (\mathbf{r}-\mathbf{r}_j)/|\mathbf{r}-\mathbf{r}_j| \approx (\rho-\rho_j)(z-z_j)$. Here $|\mathbf{n}| < 1$ and $\mathbf{n}_0$ is the unit vector along the z axis. In addition, the scattering amplitude is the two–dimensional Fourier transform of the scattered field in the near zone

$$f_j(\mathbf{n},\,\mathbf{n}_0) = \frac{k}{2\pi i}\int \exp\,(-\,i\,k\mathbf{n}\rho)\;\omega_j^0(\rho)\;\mathrm{d}\rho\,. \tag{25}$$

Now we compare exact expressions for scattered field (21) and (24) with series (14), obtained with the use of the SPM. In expansion (14) we can substitute the operator L by L_0 for the near zone of scatterer (22). As a result, all the powers of the Born approximation of scattered field $(L_0^{-1}\,\nu_j)$ given by Eq. (14) are summed in exact scattered field given by Eqs. (6), (7), and (21). This follows from the relations of the type

$$(L_0^{-1}\,\nu)^2\,/\,2! = L_0^{-1}\,V\,L_0^{-1}\,V\,\Psi_0\,, \tag{26}$$

which hold only in the direct ray approximation. In the wave zone of scatterer (23) powers of the Born scattered field $L^{-1}\,\nu_j$ have no physical sense. But in this case we can use the estimate $|\omega_j| \ll 1$, which is valid for scattered fields in the wave zone, and drop all the powers of the field $L^{-1}\,\nu_j$ starting from quadratic power.

As a result, the term given by Eq. (13) and obtained by the SPM has physical meaning of superposition of incident and scattered waves

$$u_j \approx 1 + \omega_j\,. \tag{27}$$

In the near zone expression (27) transforms into exact relation (21). It becomes approximate when diffraction appears. As we have satisfied ourselves, the scattered field ω_j in approximate relation (27) within the wave zone is spherical wave (24) having scattering amplitude written down in the Born approximation.

Let us consider the multiple scattering. The substitution of Eq. (27) into initial Eq. (12) gives the desired expansion in terms of the multiplicity of scattering for the SPM

$$u = \exp\left(\sum L^{-1}\,\nu_j\right) \approx \Pi(1+\omega_j) = 1 + \sum \omega_j + \sum_{j\;l} \omega_l\,\omega_j + \dots\,. \tag{28}$$

As in general expansion (8), here the second summand is the singly scattered field, the third summand is the doubly scattered field, and so on.

As we have seen, the most distinctive feature of the SPM approximation is the representation of field multiscattered by several scatterers by the product of fields scattered by each scatterer

$$\omega_{jlk\,\dots} = \omega_j\,\omega_l\,\omega_k \cdots\,. \tag{29}$$

Let us discuss the applicability limits of expressions (28) and (29). On account of Eq. (21), Eq. (28) is exact when the observation point is in the near zone of all the scatterers intersecting the ray $\rho = \mathrm{const}$ (see Fig. 1). In this case the wave propagation in the medium is described by the direct ray approximation and the multiple scattering is equivalent to multiple shading of some scatterers by others (see Refs. 5 and 6). The SPM adequately describes this process. When the scatterers and the observation point are in the wave zone of each other (see Fig. 2), first, the SPM substitutes exact Eq. (25) for the scattered amplitude by that in the Born approximation; second, the product of fields given by Eq. (29) becomes incorrect.

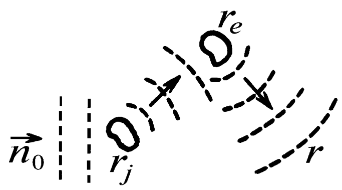

FIG. 2 Multiple scattering by scatterers located in the wave zone of each other corresponds to re–scattering of spherical waves.

Actually, the exact expression, for example, of the field scattered by two scatterers can be written down by analogy with Eq. (24)

$$\omega_{jl} = f_l\left(\frac{\mathbf{r}-\mathbf{r}_l}{|\mathbf{r}-\mathbf{r}_l|}, \frac{\mathbf{r}_l-\mathbf{r}_j}{|\mathbf{r}_l-\mathbf{r}_j|}\right)\frac{\exp\left[ik|\mathbf{r}-\mathbf{r}_l| - ik(z-z_l)\right]}{|\mathbf{r}-\mathbf{r}_l|}\times$$

$$\times f_j\left(\frac{\mathbf{r}_l-\mathbf{r}_j}{|\mathbf{r}_l-\mathbf{r}_j|}, \mathbf{n}_0\right)\frac{\exp\left[ik|\mathbf{r}_l-\mathbf{r}_j| - ik(z_l-z_j)\right]}{|\mathbf{r}_l-\mathbf{r}_j|}. \quad (30)$$

The product of scattered waves (29), according to the SPM, yields another value

$$\omega'_{jl} = f_l\left(\frac{\mathbf{r}-\mathbf{r}_l}{|\mathbf{r}-\mathbf{r}_l|}, \mathbf{n}_0\right)\frac{\exp\left[ik|\mathbf{r}-\mathbf{r}_l| - ik(z-z_l)\right]}{|\mathbf{r}-\mathbf{r}_l|}\times$$

$$\times f_j\left(\frac{\mathbf{r}-\mathbf{r}_j}{|\mathbf{r}-\mathbf{r}_j|}, \mathbf{n}_0\right)\frac{\exp\left[ik|\mathbf{r}-\mathbf{r}_j| - ik(z-z_j)\right]}{|\mathbf{r}-\mathbf{r}_j|}. \quad (31)$$

The most substantial difference in field amplitudes (30) and (31) is contributed by the following terms:

$$\omega_{jl} \sim \frac{1}{|\mathbf{r}-\mathbf{r}_l|}\frac{1}{|\mathbf{r}_l-\mathbf{r}_j|}, \qquad \omega'_{jl} \sim \frac{1}{|\mathbf{r}-\mathbf{r}_l|}\frac{1}{|\mathbf{r}-\mathbf{r}_j|}. \quad (32)$$

Of practical interest are the moments of the field averaged over configurations rather than the fields at fixed configuration of scatterers. Let us estimate the most important average intensity of fields (30) and (31). The average intensity of singly scattered field is equal to the optical thickness of the medium τ

$$I_1 = |\sum \omega_j|^2 = c\int\frac{|f|^2}{|\mathbf{r}-\mathbf{r}_j|^2}\,\mathrm{d}\mathbf{r}_j =$$

$$= c\int \mathrm{d}|\mathbf{r}-\mathbf{r}_j| \int |f|^2\,\mathrm{d}\left(\frac{\mathbf{r}-\mathbf{r}_j}{|\mathbf{r}-\mathbf{r}_j|}\right) = c\sigma\, d = \tau\,, \quad (33)$$

where c is the number density of scatterers, σ is the cross section of scattering by an individual scatterer, and d is the linear dimensions of the medium in the direction to the observation point. Analogous calculation of the intensity of doubly scattered field on account of Eq. (30) yields $\tau^2/2$, while relation of the SPM (Eq. (31)) yields a double amount τ^2. Consequencely, product of fields (29) inherent in the SPM introduces significant error (starting from double scattering).

Thus the foregoing allows us to formulate the applicability limits of the SPM in terms of the theory of multiple wave scattering in the following way.

The SPM holds within the whole spatial region in the special case in which it coincides with the small perturbation method[1-3]

$$\exp(L^{-1}\nu) \approx 1 + L^{-1}\nu\,, \quad (34)$$

that is, under condition

$$|L^{-1}\nu| < 1\,. \quad (35)$$

Here the scattered field $L^{-1}\nu$ has physical sense of a singly scattered field, i.e., superposition of fields scattered by each individual scatterer

$$L^{-1}\nu = \sum L^{-1}\nu_j\,, \quad (36)$$

In so doing, the field scattered by an individual scatterer is taken in the Born approximation.

When condition (35) is violated, which is characteristic of optical and acoustic wave propagation in the turbulent atmosphere, formula (11), obtained by the SPM, takes into account multiple wave scattering by spatial inhomogeneities of a medium only in the case, in which waves come to the observation point primarily from the near zone of scattering inhomogeneities. The SPM physically adequately describes both the scattering by an individual inhomogeneity and multiple re–scattering of waves by these inhomogeneities. In the opposite case, when diffraction essentially contributes to the wave propagated from inhomogeneities to the observation point, both scattering by an individual inhomogeneity and multiple re–scattering are described physically incorrectly by the smooth perturbation method.

This conclusion is in a qualitative agreement with the other well–known estimations of the SPM applicability limits. For example, it is well known that the SPM describes fairly well the phase fluctuations of wave in the turbulent atmosphere even though condition (1) is violated. It is also well known that the phase fluctuations are primarily determined by the large–scale inhomogeneities. If we assume that the condition of near zone is satisfied for these inhomogeneities, it becomes apparent the successful use of the SPM in this case. The amplitude fluctuations of wave propagating in the turbulent atmosphere, on the contrary, are engendered by the small–scale inhomogeneities. The field scattered by these inhomogeneities has time to essentially diffract during its propagation to the observation point. Thus the applicability limits of the SPM given by Eq. (1) are analogous to that of the small perturbation method in this case.

It is interesting to discuss in passing the approximation similar to the SPM which was proposed by N.P. Kalashnikov and M.I. Ryazanov[7] for the medium consisting of discrete scatterers. In this approximation the field is written down in the form of exponent (12) in which the scattered fields are written down exactly instead of scattered field in the Born approximation

$$u = \exp\left(\sum \omega_j\right) = 1 + \sum \omega_j + \sum_{j,\,l} \omega_j \omega_l /2! + \ldots \,. \qquad (37)$$

A series given by Eq. (37) differs from that of the SPM (see Eq. (28)) only by terms with identical indeces $j = l$ and so on. We may neglect such terms for the large number of scatterers. Otherwise all the foregoing about the applicability limits of the SPM is also true for this method.

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