VARIANCE OF MEASUREMENT NOISE AND SIGNAL–TO–NOISE RATIO IN THE OPTICAL BEAM PHASE RECONSTRUCTION FROM THE INTENSITY DISTRIBUTION

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Received July 29, 1993

The stability of an earlier obtained solution to a phase problem in optics is investigated with respect to the small random errors in measured intensity distributions. Based on calculations of the signal–to–noise ratio, the feasibility of calculational formulas for creation of the phase reconstruction algorithms implemented in wave front sensors and systems of optical control is analyzed.

In Ref. 1 the author derived an analytic solution to the problem on reconstruction of the real optical wave phase $S(x, z)$ from the measured intensity $I(x, z)$ distributions. The solution was based on the parabolic approximation of wave equation[2] for the field $U(x, z)$ depending solely on the transverse coordinate.

Advantages of analytic solution of any inverse problem over numerical solution are obvious for theoretical analysis. However, the feasibility of the derived exact relations is frequently limited due to the measurement noise. The aim of the present paper is to study the stability of the obtained solution to the phase problem against the small random errors in the measured intensities. The analysis is based on calculation of the phase fluctuation variance. In accordance with Ref. 1, the phase can be written as

$$S(x, z) = S(0, z) + \int_0^x dx' \frac{\partial S(x', z)}{\partial x'} , \quad (1)$$

where

$$\frac{\partial S(x', z)}{\partial x'} = \frac{k}{2} \{I(x', z)\}^{-1} \int_{-\infty}^{+\infty} \mathrm{sgn}(x'' - x') \frac{\partial}{\partial z} I(x'', z)\, dx'' \quad (2)$$

is the local tilt of the phase front, k is the wave number, $\mathrm{sgn}(x)$ is the signum function. Further we are interested in the correlation function of the random phase field

$$K_S(x_1, x_2; z_1, z_2) =$$

$$= \overline{[S_r(x_1, z_1) - \overline{S_r(x_1, z_1)}\,]\,[S_r(x_2, z_2) - \overline{S_r(x_2, z_2)}\,]} \,. \quad (3)$$

The horizontal bar denotes averaging over random realizations. The phase is random owing to the fact that the intensity $I(x, z)$ is measured with a certain error and instead of $I(x, z)$ the following function is known:

$$I_r(x, z) = I(x, z) + \delta_I(x, z) , \quad (4)$$

where $\delta_I(x, z)$ is the random error caused by an imperfection of recording facilities. Let us refer to this additive term as the measurement noise. The error $\delta_I(x, z)$ is assumed to be uniform random variable with the mean value

$$\overline{\mathrm{d}_I(x, z)} = 0 , \quad (5)$$

and the correlation function

$$K_I(x_1 - x_2; z_1 - z_2) = \overline{\delta_I(x_1, z_1)\, \delta_I(x_2, z_2)} \,. \quad (6)$$

Below we assume that the intensity fluctuations are small as compared with the mean value

$$\overline{\delta_I(x, z)} \ll I(x, z) . \quad (7)$$

Moreover, the intensity $I(x', z)$ is considered to meet the condition

$$I(x', z) > 0 , \quad x' \in [0, x] ,$$

with the exception of phase front dislocations.[3]

On the basis of the above–made assumptions the phase correlation function has the form

$$K_S(x_1, x_2; z_1, z_2) = \int_0^{x_1} dx_1' \int_0^{x_2} dx_2'\, K_{YS}(x_1', x_2'; z_1, z_2) , \quad (8)$$

where

$$K_{YS}(x_1', x_2'; z_1, z_2) = \left(\frac{k}{2}\right)^2 \frac{\partial^2}{\partial z_1 \partial z_2} \int_{-\infty}^{+\infty} dx_1'' \int_{-\infty}^{+\infty} dx_2'' \times$$

$$\times\, \mathrm{sgn}(x_1'' - x_1')\, \mathrm{sgn}(x_2'' - x_2') \frac{K_I(x_1'' - x_2'', z_1 - z_2)}{I_I(x_1', z_1)\, I(x_2', z_2)} \,. \quad (9)$$

By transforming to new coordinates $z_1 - z_2 = z$ and $z_1 + z_2 = \tilde{z}$ in Eq. (9), we obtain

$$K_{YS}(x_1', x_2'; \tilde{z}, \tilde{z}) = -\frac{k^2}{4} \int_{-\infty}^{+\infty}\!\! \int dx_1''\, dx_2'' \times$$

$$\times\, \mathrm{sgn}(x_1'' - x_1')\, \mathrm{sgn}(x_2'' - x_2') \times$$

$$\times \frac{\partial^2}{\partial z^2} K_I(x_1'' - x_2'', z) \Big|_{z=0} \{I(x_1' \tilde{z}), I(x_2' \tilde{z})\}^{-1} . \quad (10)$$

0235-6880/93/11 775-05 $02.00

Let us define the one–dimensional spectral power density $V(x, \kappa)$ by the relation

$$K_I(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V(x, \kappa)\, e^{-ijz}\, d\kappa\,. \tag{11}$$

In conformity with the properties of the Fourier transform, we can then write

$$\frac{\partial^2}{\partial z^2} K_I(x, z) \Big|_{z=0} = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \kappa^2\, V(x, \kappa)\, d\kappa\,. \tag{12}$$

The longitudinal derivative in Eq. (2) is assumed to be calculated from the measured intensity distributions in two planes, which are Δz apart, with subsequent formation of the finite difference. The linear interpolation of the function $I(x, z)$ over the variable z corresponds to this operation. The frequency characteristic of such a transform is well known[4]

$$H(\kappa) = \{\sin(\kappa\, \Delta z/2) \,/\, (\kappa\, \Delta z/2)\}^2\,, \qquad -\infty < \kappa < \infty\,. \tag{13}$$

On account of signal filtering

$$K_I(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} V(x, \kappa)\, |H(\kappa)|^2\, e^{-ijz}\, d\kappa\,,$$

and we have instead of Eq. (10)

$$K_{YS}(x_1', x_2'; \tilde{z}, \tilde{z}) = \frac{1}{2\pi} \frac{k^2}{4} \{I(x_1'\, \tilde{z}),\, I(x_2'\, \tilde{z})\}^{-1} \times$$

$$\times \int_{-\infty}^{+\infty}\int \mathrm{sgn}(x_1'' - x_1')\, \mathrm{sgn}(x_2'' - x_2') \times$$

$$\times \int_{-\infty}^{+\infty} \kappa^2\, \{\sin(\kappa\, \Delta z/2)/(\kappa\, \Delta z/2)\}^4\, V(x_1'' - x_2'', \kappa)\, d\kappa\, dx_1''\, dx_2''. \tag{14}$$

In the specific calculations we use the function

$$K_I(x_1 - x_2, z) = \sigma_I^2 \exp\left\{-\frac{(x_1 - x_2)^2}{2\, l_\perp^2} - \frac{z^2}{2\, l_{||}^2}\right\}. \tag{15}$$

The measurements in different planes are considered to be independent. At first, the case of the white noise is considered when $l_{é}, l_{\cdot} \to 0$, $\sigma_I^2 = N_0/(2\pi l_{||} l_\perp)$, and N_0 is the spectral power density. After integrating Eq. (14) over the variable k we have

$$K_{YS}(x_1', x_2'; \tilde{z}, \tilde{z}) = k^2 \frac{N_0}{2} (\Delta z)^{-3} \left\{I(x_1'\, \tilde{z}),\, I(x_2'\, \tilde{z})\right\}^{-1} \times$$

$$\times \int dx_1''\, \mathrm{sgn}(x_1'' - x_1')\, \mathrm{sgn}(x_1'' - x_2')\,. \tag{16}$$

It is obvious that the infinite integration in Eq. (16) gives the infinite variance of the random phase front tilts due to the measurement noise. Really the integration limits are finite in the measurements. Let us denote the lower and upper limits by $-\beta_x$ and β_x, respectively. The most natural limits are dictated by the optical beam radius. The integration between the finite limits of Eq. (16) yields

$$K_{YS}(x_1', x_2'; \tilde{z}, \tilde{z}) =$$

$$= k^2\, N_0\, (\Delta z)^{-3} \left\{I(x_1'\, \tilde{z}),\, I(x_2'\, \tilde{z})\right\}^{-1} \left[\beta_x - |x_1' - x_2'|\right], \tag{17}$$

$$|x_{1,2}'| \le \beta\,.$$

Assuming that

$$I(x', z) = I(0, z)\,, \quad 0 \le x' \le x\,,$$

we obtain in conformity with Eq. (8) for the correlation function of phase fluctuations

$$K_S(x_1, x_2; \tilde{z}, \tilde{z}) = k^2\, N_0\, (\Delta z)^{-3} \left\{I(0, z)\right\}^{-2} \times$$

$$\times \left\{ b_x\, x_1\, x_2 - \int_0^{x_1} dx_1' \int_0^{x_2} dx_2'\, |x_1' - x_2'| \right\}. \tag{18}$$

By substituting the coordinates and changing the order of integration in Eq. (18), we have

$$K_S(x_1, x_2; \tilde{z}, \tilde{z}) = k^2\, N_0 \left\{I(0, z)\right\}^{-2} (\Delta z)^{-3} \left\{ \beta_x\, x_1\, x_2 - \right.$$

$$\left. - \left[\int_0^{x_1} d\eta |\eta| (x_1 - \eta) + x_1 \int_0^{x_2 - x_1} d\eta |\eta| + \int_{x_2 - x_1}^{x_2} d\eta\, (x_2 - \eta) \right] \right\}. \tag{19}$$

For the phase fluctuation variance we derive

$$\sigma_S^2(x, z) = K_S(x, x; \tilde{z}, \tilde{z}) =$$

$$= k^2\, N_0 \left\{I(0, z)\right\}^{-2} (\Delta z)^{-3} \left\{ \beta_x\, x^2 - \frac{|x|^3}{3} \right\}, \quad |x| \le \beta_x\,. \tag{20}$$

Consequently, the phase fluctuation variance reconstructed from the measured intensity with additive noise increases linearly with the increase of the measurement base and reaches the maximum

$$\sigma_S^2(\beta_x, z) = \frac{2}{3}\, k^2\, N_0 \left\{I(0, z)\right\}^{-2} (\Delta z)^{-3}\, \beta_x^3\,.$$

It is natural to assume that if the intensity fluctuation field includes the areas of anticorrelation, where the correlation function is negative, the increase of the transverse base causes the spatial averaging of the phase fluctuations.

Equations (17)–(20) indicate that the reconstruction problem is ill–posed (unstable), which is connected with differentiation of $I(x, z)$. It manifests in an infinite increase of σ_S^2 when the recording planes draw close together.

To calculate $\sigma_S^2(x, z)$ for the arbitrary spectral power density of noise we rewrite formula (14) taking into account the finite transverse base

$$K_{YS}(x_1', x_2'; z, z) = (2\pi)^{-1}\, k^2\, (\Delta z)^{-2} \left\{I(x_1'\, z),\, I(x_2', z)\right\}^{-1} \times$$

$$\times \int_{-\beta_x}^{\beta_x} dx_1'' \int_{-\beta_x}^{\beta_x} dx_2''\, \mathrm{sgn}\,(x_1'' - x_1')\, \mathrm{sgn}\,(x_2'' - x_2') \times$$

$$\times \int\limits_{-\infty}^{+\infty} d\kappa \sin^4(\kappa\,\Delta z/2)\,/\,(\kappa\,\Delta z/2)^2\; V(x_1'' - x_2'', \kappa)\ . \tag{21}$$

Now we separate out the convolution transform in Eq. (21)

$$J_1(x_1'') = \operatorname{sgn} x_1''\; \Theta(\beta_x - x_1'')\; \Theta(\beta_x + x_1'') * V_I(x_1'', \kappa) =$$

$$= \int\limits_{-\infty}^{+\infty} \Theta(\beta_x - x_2'')\,\Theta(\beta_x + x_2'')\operatorname{sgn}(x_2'' - x_2')\, V_I(x_1'' - x_2'', \kappa)\; dx_2'', \tag{22}$$

where $\Theta(x)$ is the Heaviside function, act on this convolution by the direct $\hat{F}$ and inverse $\hat{F}^{-1}$ Fourier operators, and make use of the direct convolution theorem[5]

$$\hat{F}\{f(x) * V_I(x, \kappa)\} = \hat{F}\{f(x)\}\; \hat{F}\{V_I(x, \kappa)\}\ . \tag{23}$$

Here

$$\hat{F}\{f(x)\} = \int\limits_{-\infty}^{+\infty} f(x)\, e^{ix\varkappa}\, dx$$

is the direct Fourier transform over the transverse coordinate. The inverse Fourier transform is written in the form

$$f(x) = \hat{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} F(\omega)\, e^{-ix\varkappa}\, dx\ .$$

Let us denote the two–dimensional spectrum of the noise fluctuations by

$$\hat{F}\{V_I(x, \kappa)\} = \int\limits_{-\infty}^{+\infty} F_I(x, \kappa)\, e^{ix\varkappa}\, dx = \Phi_I(\omega, x)\ .$$

Since the equality

$$\hat{F}\Big\{\Theta(\beta_x - x_2'')\;\Theta(\beta_x + x_2'')\;\operatorname{sgn}(x_2'' - x_1')\Big\} =$$

$$= \frac{2}{i\mathrm{w}}\Big\{\cos\omega\,\beta_x - \exp(i\omega\, x_2')\Big\}$$

holds, after integration of Eq. (21) over the variable x_1' we obtain

$$K_{YS}(x_1', x_2'; z, z) = (2\pi)^{-2}\, 4k^2\,(\Delta z)^{-2}\Big\{I(x_1'\, z),\, I(x_2', z)\Big\}^{-1} \times$$

$$\times \int\limits_{-\infty}^{+\infty} d\kappa \sin^4(\kappa\,\Delta z/2)\,/\,(\kappa\,\Delta z/2)\}^2 \int\limits_{-\infty}^{+\infty} d\omega\; \Phi_I(\omega, x)\;\omega^{-2} \times$$

$$\times \Big\{\cos\omega\beta_x - \exp(i\omega x_2')\Big\}\Big\{\cos\omega\beta_x - \exp(-\,i\omega x_1')\Big\}\ . \tag{24}$$

Integrating Eq. (24) over the variables x_1' and x_2' and taking the optical beam intensity in the interval of phase reconstruction as being constant, we have

$$\sigma_S^2(x, z) = \pi^{-2}\, k^2\,(\Delta z)^{-2}\,\{I(0, z)\}^{-2} \times$$

$$\times \int\limits_{-\infty}^{+\infty} d\kappa \sin^4(\kappa\,\Delta z/2)\,/\,(\kappa\,\Delta z/2)^2 \int\limits_{-\infty}^{+\infty} d\omega\; \Phi_I(\omega, x)\;\omega^{-2} \times$$

$$\times \left\{x^2\cos^2\omega\beta_x - 2\,\frac{x}{\omega}\sin\omega x\cos\omega\beta_x + \frac{4}{\omega^2}\sin^2\frac{\omega x}{2}\right\}\ . \tag{25}$$

As an example let us choose the correlation function in the form of Eq. (15) and assume that $l_{||} \to 0$. The spectral power density in this case is written in the form

$$\Phi_I(\omega, x) = 2\pi\,\sigma_I^2\, l_\perp\, l_{||}\,\exp\left\{-\,\frac{\omega^2 l_\perp^2}{2}\right\}.$$

Calculating the integral in Eq. (25), we obtain

$$\sigma_S^2(x, z) = \sqrt{2\pi}\;\Omega_{\mathrm{b}}^2\left(\frac{l_\perp}{\beta_x}\right)^2\left(\frac{l_{||}}{\Delta z}\right)\frac{\sigma_I^2}{\{I(0, z)\}^2} \times$$

$$\times\left\{-\left(\frac{x}{\beta_x}\right)^2\left[1 + {}_1F_1\left(-\frac{1}{2}, \frac{1}{2}; -\frac{2\beta_x^2}{l_\perp^2}\right)\right] + \right.$$

$$+\,\frac{4}{3}\left(\frac{l_\perp}{\beta_x}\right)^2\left[1 - {}_1F_1\left(-\frac{3}{2}, \frac{1}{2}; -\frac{x^2}{2l_\perp^2}\right)\right] +$$

$$+\,2\left(\frac{x}{\beta_x}\right)\left(1 + \frac{x}{\beta_x}\right){}_1F_1\left(-\frac{1}{2}, \frac{3}{2}; -\frac{(\beta_x + x)^2}{2l_\perp^2}\right) -$$

$$\left.-\,2\left(\frac{x}{\beta_x}\right)\left(1 - \frac{x}{\beta_x}\right){}_1F_1\left(-\frac{1}{2}, \frac{3}{2}; -\frac{(\mathrm{b}_x - x)^2}{2l_\perp^2}\right)\right\},\ x \le \beta_x, \tag{26}$$

where $\Omega_{\mathrm{b}} = \dfrac{k\beta_x^2}{\Delta z}$ and ${}_1F_1(\alpha, \gamma; x)$ is the confluent hypergeometric function.

The asymptotic estimation of Eq. (20) follows from Eq. (26) as $l_\cdot \to 0$. It is of interest to calculate the phase distribution of an optical beam for the given values of Δz and β_x and to estimate the influence of these parameters on the reconstruction quality. The choice of the finite values of Δz and β_x permit us to restrict the increase in the fluctuation variance caused by the ill–posed inverse problem. However, the limitation on the transverse base of measurements β_x and the choice of the finite value of Δz in the calculation of the derivative $\mathrm{d}I(x, z)/\mathrm{d}z$ influence the quality of reconstruction of the phase itself.

Now we calculate $S(x, 0)$ for the Gaussian beam

$$U(x, 0) = U(0, 0)\exp\left\{-\,\frac{1}{2a_t^2}\,x^2 - i\,S(x, 0)\right\} \tag{27}$$

with the phase distribution

$$S(x, 0) = -\,\frac{k}{2\mathrm{j}(0)}\,x^2\ . \tag{28}$$

By substituting Eq. (27) into Eq. (1), taking into account the finite limits of integration in Eq. (2), and replacing the derivative by the finite difference $\{I(z + \Delta z) - I(z)\}/\Delta z$, we obtain the regularized inversion formula

$$S_R(x,\,0) = \frac{1}{2}\sqrt{\pi}\,\Omega_a \int\limits_0^{x/a_t} d\xi\,\exp(\xi^2) \times$$

$$\times \left\{ \mathrm{erf}\left(\gamma\Omega_a \left\{1 + \Omega_a^2\left(1 - \frac{\Delta z}{\varphi(0)}\right)^2\right\}^{-1/2}\right) - \right.$$

$$\left. - \,\mathrm{erf}\left(\xi\Omega_a \left\{1 + \Omega_a^2\left(1 - \frac{\Delta z}{\varphi(0)}\right)^2\right\}^{-1/2}\right) - \mathrm{erf}(\gamma) + \mathrm{erf}(\xi) \right\}, \quad (29)$$

$$\Omega_a = \frac{k a_t^2}{\Delta z}\,, \quad \gamma = \frac{\beta_x}{a_t}\,.$$

For the optical wavelengths and laser beams the inequality $\Omega \gg 1$ is fulfilled, and Eq. (29) is inverted in the following way:

$$S_R(x,\,0) = \frac{1}{2}\sqrt{\pi}\,\Omega_a \left\{ \mathrm{erf}\left(\frac{\gamma}{1 - \dfrac{\Delta z}{\varphi(0)}}\right) - \mathrm{erf}(\gamma)\right\} \times$$

$$\times \int\limits_0^{x/a_t} d\xi\, e^{\xi^2} + \int\limits_0^{x/a_t} d\xi\, e^{\xi^2} \left[\mathrm{erf}(\xi) - \mathrm{erf}\left(\frac{\xi}{1 - \dfrac{\Delta z}{\varphi(0)}}\right)\right]. \quad (30)$$

For the phase fluctuation variance in the framework of the model of white noise the following formula is obtained:

$$\sigma_S^2(x,\,0) = \eta_I^2\,\Omega_a^2 \left\{ \frac{\pi}{4}\,\gamma \left[\mathrm{erf}\, i\left(\frac{x}{a_t}\right)\right]^2 - \right.$$

$$\left. - \int\limits_0^{x/a_t} d\xi_1 \int\limits_0^{x/a_t} d\xi_2\, \exp\{\xi_1^2 + \xi_2^2\}\,|\,\xi_1 - \xi_1| \right\}, \quad (31)$$

$$\left|\,\frac{x}{a_t}\,\right| \le \gamma\,, \quad \gamma = \frac{\beta_x}{a_t}\,.$$

Here

$$\eta_I = \frac{I_0}{\sqrt{N_0}\,(\Delta z)^{-1/2}\,a_t^{-1/2}} \quad (32)$$

is the SNR on the beam axis, $I_0 = I(0,\,0)$,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-t^2}\,dt\,,$$

$$\mathrm{erf}\, i(x) = \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{t^2}\,dt$$

are the error functions of the real and imaginary arguments. To estimate the feasibility and efficiency of phase distribution reconstruction against the noise background, we make use of the generalized criterion of the SNR

$$\eta_S = \left|\ \overline{S_R(x,\,z)\ /\ \sigma_S(x,\,z)}\ \right|\,.$$

In its turn, for the given η_S the maximum admissible value of N_0 can be calculated for the specific values Δz and β_x, or the optimal values of Δz and β_x can be found using the model phase distribution for the given level of fluctuations as well as the beam intensity starting from which the phase reconstruction is possible.

On the basis of Eqs. (29)–(31) the signal–to–noise ratio variance was numerically analyzed when reconstructing $S(x,\,0)$. The Rayleigh criterion was taken as a reconstruction performance criterion. By this criterion

$$\delta S = |\,S_R(x,\,0) - S(x,\,0)\,| \le \frac{\pi}{2}\,. \quad (33)$$

Results of calculation for the optical beam with the parameters $\varphi(0) = 1$ m, $a_t = 0.05$ m, $\lambda_1 = 0.63\cdot10^{-6}$m, $\lambda_2 = 10.6\cdot10^{-6}$m, and $\beta_x = 0.25$ m are shown in Figs. 1–4.

Figures 1 and 2 illustrate the quality of $S_R(x,\,0)$ reconstruction. The general view of the reconstructed wavefront is shown in Fig. 1. Figure 2 shows in more detail the aberration pattern, obtained by substitution of exact solution (1) by regularized solution (29). All the curves for $\Delta z = 10^{-4}$, $3\cdot10^{-4}$, $5\cdot10^{-4}$, and 10^{-3} m, which can be clearly distinguished in Fig. 2 on the wavelength scale, are indistinguishable in the general view (Fig. 1). The dashed curve corresponds to the level $\lambda/4$. All the lines of Fig. 2 are broken due to the errors in numerical integration and differentiation. The behavior of these lines is characteristic of the ill–posed problems and reflects their contradiction. On the one hand, stringent criterion (32) dictates shortening of the transverse base up to $\Delta z = 0.1$ mm; on the other hand, this leads to an increase in σ_S^2 and, as a result, to a decrease of the signal–to–noise ratio. The corresponding trends in the behavior of σ_S^2 and η_S are shown in Figs. 3 and 4. The value of η_S normalized by $I_0/\sqrt{N_0}$, in m^{-1}, is laid off as ordinate. Such a normalization allows us to identify the diffraction (geometric) and energy parameters entering into η_I given by Eq. (32) and to obtain the "universal" dependences.

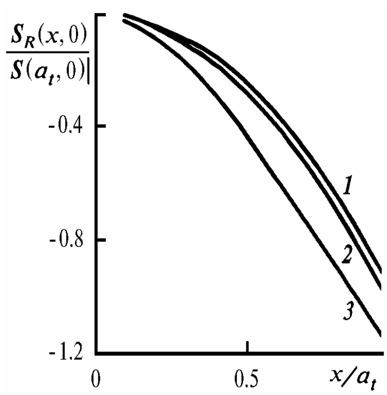

FIG. 1. The general view of the reconstructed mean phase of a single–mode Gaussian beam for different longitudinal bases of measurements Δz: 1) exact values of the phase for $\Delta z = 10^{-4}$, $3\cdot10^{-4}$, $5\cdot10^{-4}$, 10^{-3}, 10^{-2} m; *2)* $\Delta z = 10^{-1}$ m; and, *3)* $\Delta z = 5\cdot10^{-1}$ m, $\gamma = 5$.

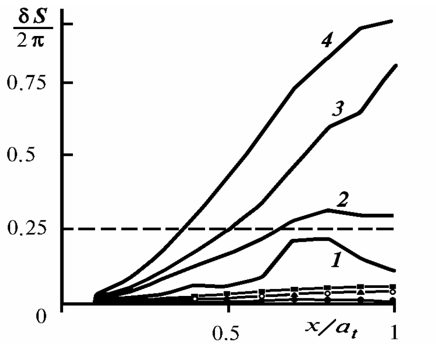

FIG. 2. Deviation of the reconstructed values of the phase from the model ones. Solid line is for $\lambda_1 = 0.63\cdot10^{-6}$ m, *the symbols* ●, ▲, ○, *and* ■ *denote the phase deviation for* $\lambda_2 = 10.6\cdot10^{-6}$ m *and* $\Delta z = 10^{-4}$, $3\cdot10^{-4}$, $5\cdot10^{-4}$, *and* 10^{-3} m, *respectively.* $\Delta z = 10^{-4}$ *(1),* $3\cdot10^{-4}$ *(2),* $5\cdot10^{-4}$ *(3), and* 10^{-3} m *(4).*

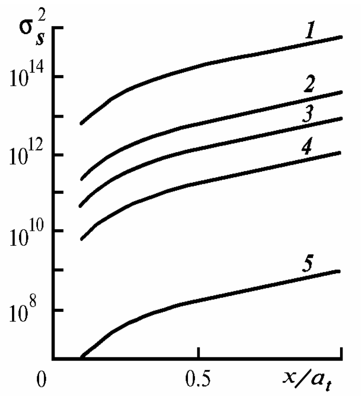

FIG. 3. The variance of the phase fluctuations reconstructed over the beam cross section. $\Delta z = 10^{-4}$ *(1),* $3\cdot10^{-4}$ *(2),* $5\cdot10^{-4}$ *(3),* 10^{-3} *(1), and* 10^{-2} m *(5).*

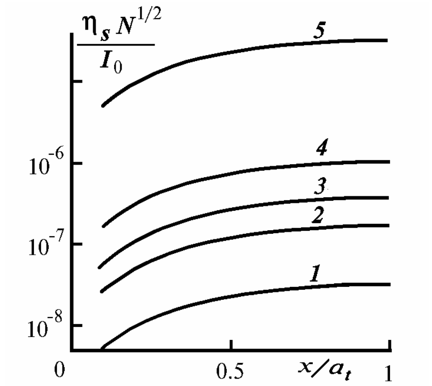

FIG. 4. SNR as a function of the observation point coordinates. Curve numbering is the same as in Fig. 3.

Although $S_R(x, 0)$, σ_S^2, and η_S have been calculated for the specific values of a_t and $\varphi(0)$, the results can be used to estimate the corresponding quantities for the other values of these parameters. To evaluate η_S, one can start from the following asymptotic relation:

$$\eta_S(x) = \frac{\frac{1}{2}\{\varphi(0)\}^{-1}\left(\frac{x}{a_t}\right)^2 (\Delta z)^{3/2} a_t^{1/2}}{\left\{\gamma\left(\frac{x}{a_t}\right)^2 - \frac{1}{3}\left(\frac{x}{a_t}\right)^3\right\}^{1/2}} \frac{I_0}{\sqrt{N_0}}, \quad x \le a_t .$$

Apparently, the accuracy of the wavefront reconstruction for $\lambda_1 = 0.63\cdot10^{-6}$ m satisfying criterion (32) is difficult to attain since it requires to perform measurements with practically unachievable ratio of the intensities of optical wave to noise in order to meet the condition $\eta_S > 1$. For a He–Ne laser, for example, with the output power $P_0 = 50$ kW the threshold $\eta_S = 1$ is reached when $I_0/\sqrt{N_0} \sim 10^8$ m^{-1}. This means that the spectral power density of noise must meet the requirement $N_0 < 4.1\cdot10^{-15}$ W^2/m^2. For $\lambda_2 = 10.6\cdot10^{-6}$ m the Rayleigh criterion has been attained already for $\Delta z = 1$ mm. In this case the threshold $\eta_S = 1$ is reached for $I_0/\sqrt{N_0} = 10^{-6}$ m^{-1}. For example, for $N_0 = 10^{-4}$ W^2/m^2 this means that $I_0 = 10^4$ W/m^2. Such intensities are not unique for the radiation of modern CO_2–lasers.

Thus to measure wavefront aberrations which are smaller in magnitude than the wavelength in the visible range, it is unlikely to use measurement techniques based on solving the inverse problems. Here the techniques for direct measurements are more effective. For less stringent requirements for the accuracy (all depend on the specific problem) as well as in the near and especially far IR ranges the use of the above–considered techniques for phase measurement is justified especially if the object under study is the wavefront topology and transformation of phase surfaces in the process of radiation propagation, investigation of which with the use of interferometric techniques[6] is apparently insuperable problem.

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