RECURSIVE FILTRATION IN APPLICATION TO CALCULATION OF COORDINATES OF A SOURCE OF OPTICAL RADIATION BY DIFFERENTIAL RANGING METHOD

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In this paper we analyze the algorithm for recursive estimation of the coordinates of a point isotropically emitting pulsed source from data of remote observations from space. We present computational programs which are used for calculating source coordinates based on the Kalman and Potter techniques.

In Ref. 1 one finds the formulated problem and the proposed differential ranging technique to determine coordinates of a point pulsed source of optical radiation from data of remote observations from space by a network "Navstar" satellite system.[2] An algorithm is also proposed in Ref. 1 to solve this problem directly, by inverting the product matrix $G^{\mathrm{T}}(\Theta_0)\, G(\Theta_0)$ that appears in the linearized equation relating the source coordinates to observational data

$$\tilde{\upsilon} = G(\Theta_0)\, \tilde{\Theta} + \boldsymbol{N} \tag{1}$$

after the latter is multiplied on the left by the transposed matrix $G^{\mathrm{T}}(\Theta_0)$. We assumed the following notations for Eq. (1):

$\tilde{\Theta} = \Theta - \Theta_0$; $\tilde{\upsilon} = \mathrm{t} - \boldsymbol{L}(\Theta_0)$; $\Theta = [x, y, z, T_0]^{\mathrm{T}}$; $\upsilon = [\upsilon_{12}, \upsilon_{13}, ..., \upsilon_{1n}]^{\mathrm{T}}$; x, y, z are the coordinates of the source; T_0 is the systematic error of measured signal delays; t_{ij} are differences between times at which the signal reaches different space vehicles (SVs) or delays; Θ_0 is the nominal value of the Θ vector; $\boldsymbol{N}$ is the column-vector of random measurement errors with a Gaussian distribution with zero average and a variance σ^2 ; $\boldsymbol{L}(\Theta_0)$ is the column-vector with elements $r_j - r_1 - cT_0$ $(j = 2.3, ..., n)$, calculated for nominal values x, y, z, T_0; r_j is the distance from the source to the jth SV; c is the speed of light. The matrix $\mathrm{G}(\Theta_0)$ has the form

$$G(\Theta_0) = -\,[G_1(\Theta_0), G_2(\Theta_0), ..., G_{n-1}(\Theta_0)]^{\mathrm{T}} , \tag{2}$$

where

$$\left.\begin{aligned} &G_j(\Theta_0) = [\alpha_{j+1}, \beta_{j+1}, \gamma_{j+1}, 1] ,\\ &\alpha_j = (x_j - x_0)/r_{j0} - (x_1 - x_0)/r_{10} ,\\ &\beta_j = (y_j - y_0)/r_{j0} - (y_1 - y_0)/r_{10} ,\\ &\gamma_j = (z_j - z_0)/r_{j0} - (z_1 - z_0)/r_{10} ; \end{aligned}\right\} \tag{3}$$

r_{i0} $(i = 1, 2, ..., n)$ are the values of r_i, calculated for the nominal coordinates of the source x, y, z. Following the direct technique the vector $\tilde{\Theta}$ and covariation matrices of the estimation errors P are estimated by the following formulas:

$$\left.\begin{aligned} &\hat{\tilde{\Theta}} = (G^{\mathrm{T}}(\Theta_0)\, G(\Theta_0))^{-1}\, G^{\mathrm{T}}(\Theta_0)\, \upsilon ,\\ &\hat{P}(\tilde{\Theta}_0) = (G^{\mathrm{T}}(\Theta_0)\, G(\Theta_0))^{-1}\, \sigma^2 . \end{aligned}\right\} \tag{4}$$

However, the direct technique becomes cumbersome when one tries to use expressions (4) for large n. It is more convenient to employ one of the recursive techniques, in which the estimation is a step–by–step procedure which follows the access of data from different SVs, so that the new improved estimate is presented as a linear combination of the preceding estimate and a new one. Below we consider two approaches to the task of recursive estimation of the coordinates of a point pulsed source of optical radiation from satellite measurement data. Recursive techniques are then compared to a direct one. The techniques considered are based on the Kalman algorithm and on its modification – the algorithm of square root of the matrix of covariation of estimation errors, which is also called the Potter algorithm.[3]

If the vector of data $\tilde{\upsilon}$ is related to the vector of parameters $\tilde{\Theta}$ by equation (1), the Kalman technique[3-4] follows the procedure according to equations

$$\hat{\tilde{\Theta}} = \bar{\tilde{\Theta}} + K\,(\tilde{\upsilon} - G\,\bar{\tilde{\Theta}}) \; ; \quad \hat{P} = \bar{P} - K\,G\,\bar{P},$$

where $\tilde{\Theta}$ and $\bar{P}$ are the *a priori* estimates and the *a priori* matrix of covariation; K and D are defined by the equations

$$K = \bar{P}\, G^{\mathrm{T}}\, D^{-1} , \quad D = G\,\bar{P}\, G^{\mathrm{T}} + I\sigma^2 .$$

Input parameters to the recursive filter are the initial values $\tilde{\Theta}_0 = 0$ and $P_0 = (G^{\mathrm{T}}(\Theta_0)\, G(\Theta_0))^{-1}\, \sigma^2$, the data z_j and the set of coefficients $\boldsymbol{A}_j$, where z_j is the jth element of the column-vector $\tilde{\upsilon}$, i.e. is a scalar, and $\boldsymbol{A}_j$ is the jth row of the matrix $G(\Theta_0)$ (the row-vector). Computations follow the scheme:

$\boldsymbol{l}_j = P_j\, \boldsymbol{A}_j^{\mathrm{T}}$, $r_j = \boldsymbol{A}_j\, \boldsymbol{l}_j + 1$ – covariation of the forecasted residual;

$\boldsymbol{K}_j = \boldsymbol{l}_j / r_j$ – vector of gain factors;

$\tilde{\upsilon}_j = z_j - \boldsymbol{A}_j\, \tilde{\Theta}_j$ – the forecasted residual;

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$\tilde{\Theta}_{j+1} = \tilde{\Theta}_j + \boldsymbol{K}_j\, \tilde{\upsilon}_j$ – new estimate of vector $\tilde{\Theta}$;

$\overline{P}_{j+1} = \overline{P}_j - \boldsymbol{K}_j\, \boldsymbol{l}_j^{\mathrm{T}}$ – new covariation;

$\overline{\boldsymbol{l}}_j = \overline{P}_{j+1}\, \boldsymbol{A}_j^{\mathrm{T}}$, $P_{j+1} = (\overline{P}_{j+1} - \overline{\boldsymbol{l}}_j\, \boldsymbol{K}_j^{\mathrm{T}}) + \boldsymbol{K}_j\, \boldsymbol{K}_j^{\mathrm{T}}$ – the stabilized new covariation, where $\tilde{\Theta}_j$ and P_j are the estimates of the vector Θ and the matrix of covariation of estimates retrieved after processing j observations.

Table I. Source coordinates: $x = 2879.592$; $y = 2249.784$; $z = 5218.817$.

No.SV	SV coordinates: x_i , y_i , z_i			Time of signal arrival, t_i
	Experiment *1*			
	Nominal values			
	x_0=2882.544	y_0=2252.107	z_0=5224.175	T_0 = 0.300
1	15338.253	20331.950	348.959	0.07502580
2	17241.558	9276.542	16292.524	0.06487520
3	9044.992	–7212.935	22692.147	0.06940000
4	–12974.502	1986.421	21828.613	0.07659680
5	–4876.855	11856.692	22009.251	0.06952020
	Experiment *2*			
	Nominal values			
	x_0=2882.580	y_0=2252.156	z_0=5224.273	T_0 = 0.300
1	15341.989	20331.475	162.199	0.07516880
2	17313.940	9369.671	16161.949	0.06492040
3	9143.620	–7080.761	22694.248	0.06930500
4	–12910.977	1840.856	21878.980	0.07657960
5	–5035.433	11876.259	21962.940	0.06962480
	Experiment *3*			
	Nominal values			
	x_0= 2882.416	y_0= 2252.008	z_0= 5223.957	T_0 = 0.300
1	15347.692	20325.722	–295.894	0.07552140
2	17490.615	9593.661	15837.113	0.06503740
3	9387.755	–6758.228	22692.955	0.06907640
4	–12757.058	1481.394	21996.293	0.07654380
5	–5422.602	11926.737	21843.135	0.06988440

As to the square-root algorithm, the value used for its input filter is the *a priori* value of the square root of P_0 , instead of the *a priori* value P_0 of the matrix of covariation P. In our case this a square root is equal to $S_0 = G^{-\mathrm{T}}(\Theta_0)\,\sigma$, where $G(\Theta_0)$ is calculated by formulas (2) and (3) for $n = 5$. Calculations follow the scheme:

$\boldsymbol{l}_j^{\mathrm{T}} = \boldsymbol{A}_j\, S_j$, $r_j = 1/(\boldsymbol{l}_j^{\mathrm{T}}\, \boldsymbol{l}_j + 1)$ – the inverse value of the covariance of the forecasted residual;

$\boldsymbol{K}_j = S_j\, \boldsymbol{l}_j$ – vector of gain factors;

$\tilde{t}_j = z_j - \boldsymbol{A}_j\, \tilde{\Theta}_j$ – the forecasted residual;

$\tilde{\Theta}_{j+1} = \tilde{\Theta}_j + \boldsymbol{K}_j\, (\tilde{\upsilon}_j\, r_j)$ – new estimate of vector $\tilde{\Theta}$;

$\gamma_j = r_j \,/\, (1 + \sqrt{r_j})$, $S_{j+1} = S_j - (\gamma_j\, \boldsymbol{K}_j\,)\boldsymbol{l}_j^{\mathrm{T}}$ – new square root of covariation;

$P_{j+1} = S_{j+1}\, S_{j+1}^{\mathrm{T}}$ – covariation.

To verify the efficiency of the described algorithms a numerical experiment was performed. The initial data and the observational data for it are presented in Table I. The direct technique was employed for n=5, and the recursive techniques were used for n=14. Computational results are presented in Table II.

Table II.

Technique	Calculated (estimated) parameters x, y, z, cT_0 and diagonal elements of the matrix of covariation (in brackets)
	Experiment 1
Direct	2881.296 (0.000) 2251.138 (0.001) 5222.083 (0.007) –0.166 (0.003)
Kalman	2880.445 (0.000) 2250.237 (0.001) 5221.257 (0.007) 0.300 (0.003)
Potter	2879.842 (0.015) 2249.976 (0.032) 5220.137 (0.084) 0.598 (0.053)
	Experiment 2
Direct	2881.288 (0.000) 2251.155 (0.001) 5222.106 (0.007) –0.169 (0.003)
Kalman	2880.336 (0.000) 2250.276 (0.001) 5221.483 (0.007) 0.300 (0.003)
Potter	2879.889 (0.016) 2249.897 (0.032) 5220.439 (0.084) 0.598 (0.052)
	Experiment 3
Direct	2881.286 (0.000) 2251.137 (0.001) 5222.056 (0.007) –0.140 (0.003)
Kalman	2880.427 (0.000) 2250.789 (0.001) 5221.875 (0.007) 0.300 (0.003)
Potter	2879.576 (0.016) 2249.693 (0.033) 5220.846 (0.082) 0.598 (0.052)

Appendicies present the PASCAL computational programs for the Kalman and Potter techniques.

APPENDIX A: KALMAN PROCEDURE

```
PROCEDURE Kalman ;
VAR            i , j            :            INTEGER ;
    TYPE
    VECTOR = ARRAY [1 .. 10]          OF     REAL ;
    VAR
        S, Sigma, Delta
        : REAL ;
        V                               : VECTOR ;
        BEGIN
        Sigma : = 1 .;
    Delta : = z ;                            { z – input}
    FOR  i : = 1 TO N DO BEGIN
       V[i] : = 0. ;
          FOR j : = 1 TO N DO
             V[i] : = V[i] + P[i, j]*A[ j] ;      {P, A –
input}
             Delta : = Delta – A[i]*X[i] ;   {X – input}
             Sigma : = Sigma + A[i]*V[i] ;
        END ;
        Sigma : = 1. /Sigma ;
        FOR  i : = 1 TO N DO BEGIN
        K[i] : = V[i]*Sigma ;                {K – output}
        X[i] : = X[i] + K[i]*Delta ;
{X output estimate}
                FOR  j : = 1 TO N DO BEGIN
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          P[i, j] : = P[i, j] – K[i]*V[ j] ;
          P[ j, i] : = P[i, j] ;
        END ;
    END ;
    FOR  i : = 1 TO N DO BEGIN
    V[i] : = 0. ;
        FOR j : = 1 TO N DO
          V[i] : = V[i] + P[i, j]*A[ j] ;
    END ;
    FOR j : = 1 TO N DO
      FOR  i : = 1 TO j DO BEGIN
       S:= 0.5*(P[i, j]–V[i]*K[ j] + P[i, j] –
V[ j]*K[i]) ;
        P[i, j] : = S + K[i]*K[ j] ; {P – output}
        P[ j, i] : = P[i, j] ;
    END ;
    FOR  i : = 1 TO N DO
      FOR  j : = 1 TO N DO BEGIN
        PklOut [i, j] : = Pkl [i, j] ;
    END ;
    Xout : = X0 + X[1] ;                    {X – output
coordinate}
    Yout : = Y0 + X[2] ;  {Y – output coordinate}
    Zout ; = Z0 + X[3] ;  {Z – output coordinate}
  END ;
(* . . . . . . . END of the Kalman procedure . . . . . . . *)
```

APPENDIX B: POTTER PROCEDURE

```
PROCEDURE Potter ;
  TYPE
    VECTOR = ARRAY[1 .. 10]               of REAL;

  VAR
    Sigma,   Delta,   Gamma,   Alfa        : extended;
    V    :                                   VECTOR;
  VAR i , j : INTEGER ;
  BEGIN
    Sigma : = 1. ;
    Delta : = Z ;                             {z – input}
    FOR i : = 1 TO N DO BEGIN
      V[i] : = 0.;
        FOR  j : = 1 TO N DO
          V[i] : = V[ i] + S[ j, i]*A[ j] ;  {S, A  –
input}
      Delta : = Delta – A[i]*X[i] ;        {X – input}
      Sigma : = Sigma : + V[i]*V[i] ;
    END ;
    Sigma : = 1./Sigma ;
    Delta : = Delta*Sigma ;
    Gamma : = Sigma/(1. + SQRT(Sigma)) ;
    Alfa : = 0 ;
    FOR i : = 1 TO N DO
      Alfa := Alfa + S[i, j]*V[ j] ;
    X[i] : = X[i] + Alfa*Delta ;              {X – output}
    Alfa := Alfa*Gamma ;
    FOR j ; =1 TO N DO
     S[i, j] : = S[i, j] – Alfa*V[ j] ;                {X –
output estimate}
     Xout : = X0 + X[1] ;      (X – output coordinate)
     Yout : = Y0 + X[2] ;      (Y – output coordinate)
     Zout : = Z0 + X[3] ;      (Z – output coordinate)
END ;  {***  END of the Potter procedure  ***}
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