INVESTIGATION INTO SPONTANEOUS EMISSION AMPLIFICATION IN AN ACTIVE REFRACTIVE MEDIUM BY THE TRANSFER EQUATION METHOD

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The ray tracing technique based on the radiative transfer equation is used to describe the spontaneous emission amplification in an active medium. Using this approach, analytical solutions for the intensity distribution and coherence function in the exit plane of the active medium with parabolic profiles of dielectric constant and emission coefficient in the transverse cross section have been derived. Applicability of the approximation when contribution to output emission is made only by spontaneous sources adjacent to the far end of the active medium is analyzed. It is demonstrated that this approximation is inapplicable in many real situations, and it is necessary to take into account the sources in the whole active medium.

Investigation of the spontaneous emission amplification in active media having nonuniform distribution of population density inversion is important for developing the noncavity lasers, that is, lasers without cavity reflectors (mirrors). Unlike conventional lasers, for which the coherent properties of output radiation are determined to a great extent by the type and quality of a cavity, noncavity laser output radiation is completely determined by the spatial distribution of refractive and amplifying properties of an active medium.

There are several approaches to the theoretical description of output radiation of the lasers of interest. All of them are based on paraxial approximation of the wave equation. The approach based on the expansion of the wave field into an infinite series in transverse modes[1,2] is most widely used.

This approach is close to conventional methods used in laser optics. However, it can be practically used for a limited number of dielectric constant and gain profiles in an active medium for which transverse modes can be derived analytically. The second approach[3] is based on the numerical solution of the paraxial wave equation by the fast Fourier transform and splitting operator methods. Spontaneous emission is simulated by the Monte Carlo method by introducing of a random initial phase. The coherence function is calculated by averaging over 100–200 temporal steps.

The third approach is based on a solution of the equation for the transverse coherence function.[4,5] Using this approach, the authors succeeded in solving analytically the problem for a number of dielectric constant profiles including the case of saturation of the emission amplification.

The approach based on a solution of the radiative transfer equation is presented in this paper. This equation is an approximate consequence of the equation for the transverse coherence function and allows one not only to implement effective numerical algorithms for solution of the given problem but also to derive analytical solutions for nonuniform distributions of dielectric constant and gain in an active medium.

1. The paraxial wave equation

$$2\,i\,\kappa\frac{\partial E}{\partial z} + \nabla_{\perp}^{2} E + \kappa^{2}\Delta\varepsilon(z,\rho)\,E(z,\rho) = P_{\rm sp}(z,\rho) \qquad (1)$$

is considered as an initial one, where k is the wave number, $\Delta\varepsilon$ is the relative perturbation of the complex dielectric constant, $P_{\rm sp}$ is the term caused by the presence of spontaneous polarization in a medium, and $\mathbf{r} = (z, \rho)$.

Let us consider an active medium with relative distribution of dielectric constant of the form

$$\Delta\varepsilon(z,\rho) = \varepsilon(z,\rho) + i\,\sigma(z,\rho)\,, \qquad (2)$$

where ε is the real component and σ is the imaginary component of dielectric constant connected with the gain of the medium g by the following relation:

$$\sigma(z,\rho) = -\,\kappa^{-1}\,g(z,\rho)\,.$$

The form of the functions ε and σ is determined by spatial distribution of population density inversion in the medium. Spontaneous emission is caused by the presence of random polarization, which is supposed to follow the Gaussian statistics and satisfies the condition

$$\langle P_{\rm sp}(\mathbf{r})P^{*}_{\rm sp}(\mathbf{r}')\rangle = W_{\rm eff}(\mathbf{r})\,g_0\,\delta(\mathbf{r}-\mathbf{r}')\,, \qquad (3)$$

where $W_{\rm eff}$ is the effective intensity of spontaneous emission, and g_0 is the gain at the origin of coordinates. In this case we can write the equation for the coherence function in the approximation of paraxial optics

$$2\,i\,\kappa\frac{\partial\Gamma_2}{\partial z} + 2\,\nabla_{\mathbf{R}}\nabla_{\rho}\Gamma_2 + \kappa^2\,[\Delta\varepsilon(z,\mathbf{R}+\rho/2) -$$

$$-\,\Delta\varepsilon^{*}(z,\mathbf{R}-\rho/2)]\,\Gamma_2(z,\mathbf{R},\rho) = \frac{ig_0}{k}\,W_{\rm eff}(z,\mathbf{R})\,\delta(\rho)\,, \qquad (4)$$

where summed $(\mathbf{R} = (\rho_1 + \rho_2)/2)$ and difference $(\rho = \rho_1 - \rho_2)$ transverse coordinates are used.

The coherence function vanishes for ρ larger than the coherence length ρ_c. Then for $\rho_c < a_{\perp}$ ($a_{\perp}$ is the characteristic scale of $\Delta\varepsilon$ variation along the transverse coordinate ρ) we can use an approximate expansion in the Taylor series

$$\Delta\varepsilon(z,\mathbf{R}+\rho/2) - \Delta\varepsilon^{*}(z,\mathbf{R}-\rho/2) \approx \rho\nabla_{\mathbf{R}}\varepsilon(z,\mathbf{R}) + 2\,i\sigma(z,\mathbf{R}). \qquad (5)$$

Substituting Eq. (5) in Eq. (4), we derive the equation

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$$\frac{\partial \Gamma_2}{\partial z} + \left[\frac{1}{i\,\kappa} \nabla_\rho \nabla_{\mathbf{R}} + \frac{\kappa}{2\,i}\, \mathbf{r}\, \nabla_{\mathbf{R}}\, \varepsilon(z,\, \mathbf{R}) + \kappa\, \sigma\,(z,\, \mathbf{R}) \right] \Gamma_2(z,\, \mathbf{R},\, \rho) = = \frac{g_0}{2\,\kappa^2}\, W_{\text{eff}}(z,\, \mathbf{R})\, \delta(\rho)\,. \qquad (6)$$

Equation (6) differs from the corresponding equation derived in Ref. 4 by the presence of the source function in its right–hand side.

Furthermore, taking the Fourier transform with respect to ρ, we obtain the equation

$$\frac{\partial J}{\partial z} + \left[\frac{\mathbf{n}_\perp}{\kappa} \nabla_{\mathbf{R}} + \frac{\kappa}{2} \nabla_{\mathbf{R}}\, \varepsilon\, \nabla_{\mathbf{n}_\perp} + \kappa\, \sigma\,(z,\, \mathbf{R}) \right] J(z,\, \mathbf{R},\, \mathbf{n}_\perp) = = \frac{g_0}{8\pi^2 \kappa^2}\, W_{\text{eff}}(z,\, \mathbf{R})\,, \qquad (7)$$

where J is the emission brightness which is defined as a Fourier transform of the coherence function

$$J(z,\, \mathbf{R},\, \mathbf{n}_\perp) = (2\pi)^{-2} \int\!\!\int_{-\infty}^{\infty} \Gamma_2(z,\, \mathbf{R},\, \rho)\, \exp(-i\, \mathbf{n}_\perp\, \rho)\, \mathrm{d}\rho\,. \qquad (8)$$

The solution of Eq. (7) can be written in the form

$$J(z,\, \mathbf{R},\, \mathbf{n}_\perp) = = \frac{g_0}{8\pi^2 \kappa^2} \int_0^z \mathrm{d}z'\, W_{\text{eff}}(z',\, \mathbf{R}(z'))\, \exp\left[\int_{z'}^{z} \mathrm{d}z''\, g(z'',\, \mathbf{R}(z'')) \right], \qquad (9)$$

where the characteristic $\mathbf{R} = \mathbf{R}(z')$ obeys the equation:

$$\frac{\mathrm{d}^2\mathbf{R}}{\mathrm{d}z^2} = \frac{1}{2} \nabla_{\mathbf{R}}\, \varepsilon(z, \mathbf{R}(z)) \qquad (10)$$

with the initial conditions $\mathbf{R}(z' = z) = \mathbf{R}$, $\mathrm{d}\mathbf{R}(z' = z)/\mathrm{d}z' = \mathbf{n}_\perp$.

The system of equations (9) and (10) can be solved numerically for an arbitrary form of the functions ε and g. The algorithm for its numerical solution and analysis of the accuracy of the given technique are presented in Refs. 6–8.

2. Let us consider the analytical solution for parabolic distribution of population density inversion. We suppose that the functions ε and g can be written for this case in the form

$$\varepsilon(\mathbf{R}) = 1 + (R^2 - a^2)/L_R^2\,,\ |\mathbf{R}|<a\,,\ \ \varepsilon(\mathbf{R}) = 1\,,\ |\mathbf{R}| > a\,,$$
$$g(\mathbf{R}) = g_0(1 - R^2/a^2)\,,\ |\mathbf{R}|<a\,,\quad g(\mathbf{R}) = 0\,,\ |\mathbf{R}| > a\,. \qquad (11)$$

At first, let us determine the contribution to the output brightness from an infinitely thin layer of emitters located in the plane $z = z_0$, i.e., assume that

$$W_{\text{eff}}(z,\, \mathbf{R}) = W_{\delta\,\text{eff}}(\mathbf{R})\, \delta(z - z_0)\,.$$

Then it follows from Eq. (9) that

$$J_\delta(z, \mathbf{R}, \mathbf{n}_\perp) = \frac{g_0}{8\pi^2 \kappa^2}\, W_{\delta\,\text{eff}}(\mathbf{R}_0)\, \exp\left[\int_{z_0}^{z} \mathrm{d}z'\, g(z, \mathbf{R}(z')) \right], \qquad (12)$$

where the characteristic $\mathbf{R} = \mathbf{R}(z')$ is determined by the expression

$$\mathbf{R}(z') = \mathbf{R}_0/\cosh((z' - z_0)/L_R) + \frac{\mathbf{n}_\perp}{\kappa}\, L_R\, \frac{\sinh((z' - z)/L_R)}{\cosh((z' - z_0)/L_R)} \qquad (13)$$

and satisfies the boundary conditions: $\mathbf{R}(z' = z_0) = \mathbf{R}_0$, $\mathbf{R}(z' = z) = \mathbf{R}$. Substituting Eqs. (11) and (13) into Eq. (12) and integrating it over z, for $|\mathbf{R}| < a$ we obtain

$$J_\delta(z,\, \mathbf{R},\, \mathbf{n}_\perp) = \frac{W_{\delta\,\text{eff}}(\mathbf{R}_0)}{8\pi^2 \kappa^2}\, \exp\,(g_0(z - z_0)) \times$$
$$\times \exp\left\{ -\frac{g_0 L_R}{4a^2} \left[\mathbf{R}^2(\sinh(2\bar{z}) + 2\bar{z}) + \frac{\mathbf{n}_\perp^2 L_R^2}{\kappa^2}(\sinh(2\bar{z}) - 2\bar{z}) - \right.\right.$$
$$\left.\left. - \frac{2\mathbf{R}\mathbf{n}_\perp L_R}{\kappa}(\cosh(2\bar{z}) - 1 \right\}, \qquad (14)$$

where $\bar{z} = \dfrac{z - z_0}{L_R}$. Then for the coherence function we can write

$$\Gamma_{2\delta}(z,\, \mathbf{R},\, \rho) = \int\!\!\int_{-\infty}^{\infty} \mathrm{d}\mathbf{n}_\perp\, J(z,\, \mathbf{R},\, \mathbf{n}_\perp)\, \exp\,(i\, \mathbf{n}_\perp \rho) =$$
$$= \int\!\!\int_{-\infty}^{\infty} \mathrm{d}\mathbf{R}_0 \left| \frac{\mathrm{d}\mathbf{n}_\perp}{\mathrm{d}\mathbf{R}_0} \right| J(z,\, \mathbf{R},\, \mathbf{n}_\perp(z - z_0,\, \mathbf{R},\, \mathbf{R}_0) \times$$
$$\times \exp\,[i\, \rho\, \mathbf{n}_\perp\, (z - z_0,\, \mathbf{R},\, \mathbf{R}_0)]\,, \qquad (15)$$

where we carry out the integration over spatial coordinates in the emission plane instead of integration over angular coordinates in the exit plane. Determinant of such a transition is equal to

$$\left| \frac{\mathrm{d}\mathbf{n}_\perp}{\mathrm{d}\mathbf{R}_0} \right| = \frac{\kappa^2}{L_R^2\, \sinh^2(\bar{z})}\,, \qquad (16)$$

where the expression

$$\mathbf{n}_\perp = \kappa\, [\mathbf{R}\, \cosh(\bar{z}) - \mathbf{R}_0]\ /\ [L_R\, \sinh(\bar{z})]\,, \qquad (17)$$

which follows from Eq. (13), is taken into account. Let us assume that the effective intensity of sources in the emission plane $z = z_0$ is distributed by the law

$$W_{\delta\,\text{eff}}(\mathbf{R}_0) = W_{\delta 0}\, \exp\,(-\, R_0^2\ /\ a^2)\,. \qquad (18)$$

This expression should be considered as an approximation, since for rigorous formulation of the problem, the source intensity distribution should duplicate the gain distribution, i.e., it should have the parabolic profile. Substituting Eqs. (16), (17), and (18) into Eq. (15), we obtain

$$\Gamma_{2\delta}(z,\, \mathbf{R},\, \rho) =$$
$$= \frac{W_{\delta 0}\, g_0\, \exp\,(g_0(z - z_0))}{8\pi^2\, L_R^2\, \sinh^2(\bar{z})} \exp\left(\frac{i\, \kappa\, \mathbf{R}\, \mathbf{r}}{L_R\, \tanh(\bar{z})} \right) \exp\left(-\frac{R^2}{a^2} A \right) \times$$
$$\times \int\!\!\int_{-\infty}^{\infty} \mathrm{d}\mathbf{R}_0 \exp\left(-\frac{R_0^2}{a^2}\, [1 + A] \right) \exp\left(-\frac{\mathbf{R}_0}{a^2}\, [2\mathbf{R}B - i\, \rho C] \right), \qquad (19)$$

where $A = \dfrac{g_0 L_R}{2} \dfrac{\sinh(\overline{z})\cosh(\overline{z}) - \overline{z}}{\sinh^2(\overline{z})}$, $B = \dfrac{g_0 L_R}{2} \dfrac{\sinh(\overline{z}) - \overline{z}\cosh(\overline{z})}{\sinh^2(\overline{z})}$,

and $C = \dfrac{L_D}{L_R \sinh(\overline{z})}$. The integral over $\mathbf{R}_0$ in Eq. (19) is readily calculated analytically. Then we can write the following expressions for the output emission intensity:

$$W_\delta(z, \mathbf{R}) = \Gamma_{2\delta}(z, \mathbf{R}, \rho=0) = \frac{W_{\delta 0}}{8\pi} \frac{a^2 g_0 \exp(g_0 z)}{L_R^2 \sinh^2(\overline{z})(1 + A)} \exp\left(-\frac{\mathbf{R}^2}{a_w^2}\right) \tag{20}$$

and for the output coherence function on the axis of the active medium

$$\Gamma_{2\delta}(z, \mathbf{R} = 0, \rho) = \frac{W_{\delta 0}}{8\pi} \frac{g_0 \exp(g_0 z)\, a^2}{L_R^2 \sinh^2(\overline{z})\,(1 + A)} \exp\left(-\frac{\rho^2}{4\, a_\rho^2}\right), \tag{21}$$

where $a_w^2 = a^2(1 + A) / (A + A^2 - B^2)$, and $a_\rho^2 = a^2(1 + A) / C^2$.

Thus to obtain the final result, it is necessary to integrate over all emitting planes, i.e., over z_0:

$$\Gamma_2(z, \mathbf{R}, \rho) = \int_0^z \mathrm{d}\, z_0\, \Gamma_{2\delta}(z - z_0, \mathbf{R}, \rho)\ . \tag{22}$$

However, for active media with the high gain $G = g_0 z > 10$ (this gain is typical of X–ray lasers) the main contribution to the output emission is made by the thin end region of the active medium $0 < z_0 < z_{\mathrm{eff}}$. In this case, we finally obtain

$$\Gamma_2(z, \mathbf{R}, \rho) = z_{\mathrm{eff}}\, \Gamma_{2\delta}(z - z_0, \mathbf{R}, \rho)\Big|_{z_0=0}\ , \tag{23}$$

where z_{eff} is determined from the expression

$$z_{\mathrm{eff}} = \int_0^z \exp(-\, g_0\, z_0)\, \mathrm{d}z_0 = 1/g_0 \ll z\ . \tag{24}$$

Strictly speaking, the approximation given by Eqs. (23) and (24) is valid when all functions in the solution given by Eqs. (21) and (22) vary slowly with increasing z in comparison with $\exp(g_0 z)$. However, the function $\sinh(\overline{z})$ enters into the denominators of Eqs. (20) and (21), whose variations are comparable with $\exp(g_0 z)$ at small values of L_R ($L_R \approx 2/g_0$). In this case, the effect of emission sources in the whole active medium should be taken into account.

3. To determine how much is the contribution from emission sources located beyond the end region of the active medium, it is necessary to integrate over z_0 in accordance with Eq. (22) and compare the result obtained with the approximation given by Eqs. (23) and (24). However, when we are interested only in the transverse coherence length of output emission, the problem is essentially simplified. Let us introduce the following definition for the transverse coherence length ρ_c:

$$\rho_c^2 = \frac{1}{4\pi} \int\!\!\int_{-\infty}^{\infty} \mathrm{d}\rho\ \mu(\rho)\ , \tag{25}$$

where μ is the modulus of the degree of coherence on the axis

$$\mu(\rho) = \frac{|\Gamma_2(z, \mathbf{R} = 0, \rho)|}{W(z, \mathbf{R} = 0)} = \frac{|\Gamma_2(z, \mathbf{R} = 0, \rho)|}{\Gamma_2(z, \mathbf{R} = 0, \rho = 0)}\ . \tag{26}$$

For the approximation given by Eq. (23) we obtain the following expression from Eqs. (21), (25), and (26):

$$\rho_c = a_\rho(z - z_0)\Big|_{z_0=0} =$$

$$= \frac{1}{F_x\, \overline{z}} \sqrt{\sinh(\overline{z}) + \frac{G}{2\overline{z}} (\sinh(\overline{z})\cosh(\overline{z}) - \overline{z})}\ , \tag{27}$$

where $F_x = \kappa a^2/z$ is the Fresnel number. This result is in good agreement with estimates made within the geometric optics approximation.[9] Using Eq. (22), from Eqs. (25) and (26) we have

$$\rho_c^2 = \frac{1}{4\pi} \frac{\int\!\!\int_{-\infty}^{\infty} \mathrm{d}\rho \int_0^z \mathrm{d}\, z_0\, \Gamma_{2\delta}(z - z_0, \mathbf{R} = 0, \rho)}{\int_0^z \mathrm{d}\, z_0\, W_\delta(z - z_0)}\ . \tag{28}$$

It follows from Eqs. (21) and (22) that

$$\int\!\!\int_{-\infty}^{\infty} \mathrm{d}\rho\ \Gamma_{2\delta}(z - z_0, \mathbf{R} = 0, \rho) = a_r^2(z - z_0) W_\delta(z - z_0, \mathbf{R} = 0) =$$

$$= P_{c0} \exp(-\, g_0\, z_0)\ , \tag{29}$$

where $P_{c0} = \dfrac{W_{\delta 0}\, g_0}{8\pi\kappa^2} \exp(g_0\, z)$. It should be noted that P_{c0} does not depend on the transverse size of the active medium. Then taking into account Eq. (29), from Eq. (28) we obtain the expression for the transverse coherence length

$$\rho_c = \left[g_0 \int_0^z \mathrm{d}\, z_0\, a_\rho^2(z - z_0) \exp(-\, g_0\, z_0) \right]^{-1/2}\ . \tag{30}$$

The transverse coherence length calculated using formulas (27) and (30) is plotted in Fig. 1 as a function of the parameter $G_R = g_0\, L_R$.

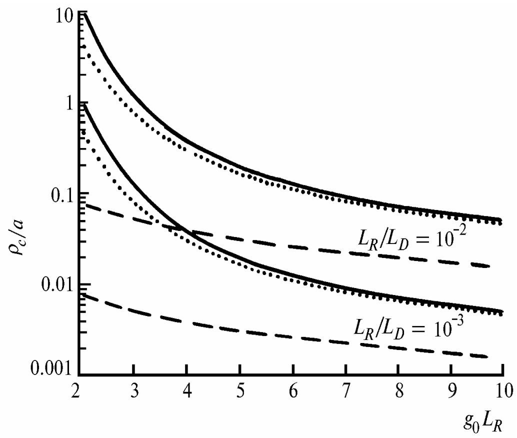

FIG. 1. Normalized transverse coherence radius versus the parameter $G_R = g_0\, L_R$ calculated using Eq. (27) (solid lines), Eq. (30) (dotted lines), and Eq. (31) (dashed lines). Calculations were carried out for the gain $G = g_0\, z = 15$.

Values of the coherence length calculated in accordance with the van Cittert–Zernike theorem

$$\rho_c = z/ka \tag{31}$$

are shown by dashed lines. These values correspond to the active medium with uniform distribution of population inversion in a cylindric region $|\mathbf{R}| \le a$. As is seen, calculations according to formulas (27) and (30) practically coincide at high values of G_R when $z_{\text{eff}} \ll z$. As the value of G_R decreases, the calculations become different. At $G_R \approx 2$ the calculated results differ more than twice. This indicates that the approximation given by Eqs. (23) and (24) is inapplicable in these situations being often encountered for X–ray lasers.

CONCLUSION

Thus the analytical solution to the problem of spontaneous emission amplification has been obtained for active medium with parabolic transverse profiles of dielectric constant and gain. The obtained solutions for the intensity distribution given by Eq. (20) and the coherence function given by Eq. (21) in the exit plane of the active medium are sufficiently rigorous solutions to radiative transfer equation (7) and hence to equation (6) for the coherence function. Only one approximation was made in the derivation of Eqs. (20) and (21), namely, the parabolic profile of spontaneous source intensity was approximated by Gaussian distribution (18).

However, the equations (6) and (7) themselves were derived from the rigorous equation for coherence function (4) using the approximation given by Eq. (5). It is easy to verify that for parabolic profiles of ε and g given by Eq. (11), approximation (5) is rigorous for the real component of dielectric constant ε and approximate for its imaginary component σ. Using Eq. (5), we omit the term $i\frac{\rho^2}{4}\nabla^2_{\mathbf{R}}\sigma(z, \mathbf{R})$, which describes "excess" diffraction caused by the wave distortion due to inhomogeneous absorption of the energy in the transverse cross section. The role played by this term can be found by comparison of solutions (20) and (21) with those of rigorous equation (4) for parabolic profiles of ε and σ given by Eq. (11). The solution of equation (4) exists, but its consideration is beyond the scope of the present paper. We will only point out that approximation (5) can be used if the condition $z \ll 2L_{\text{D}}/(g_0 L_R)$ is valid, and this condition is valid in most real situations. Moreover, using approximation (5) for determination of the coherence length, we should make calculations by Eq. (26) instead of the rigorous definition

$$\mu(\rho) = \frac{|\Gamma_2(z, \mathbf{R} = 0, \rho)|}{\sqrt{W(z, \mathbf{R} = \rho/2)\, W(z, \mathbf{R} = -\rho/2)}}, \tag{26}$$

since it leads to incorrect results in the given case.

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