INFORMATION CONTENT OF THE PRIOR ESTIMATES IN SOLVING THE INVERSE PROBLEM OF LIGHT SCATTERING

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A vector method for analyzing the accuracy of a solution of an inverse, problem has been developed. The method proposed provides for the presence of a Joint complex of optical data and a priori estimates of a given portion of the parameters to be reconstructed. The relations derived are given in terms of the optical signal response and signal correlation with the variations in the parameters and are analyzed as functions of the quality of a priori estimates and limiting values of the factors.

The study of the information content of optical characteristics is very significant for solving inverse problems of light scattering. Being performed at the preliminary stage of model calculations it would contribute to an optimal design of optical experiments and provide reliability of operational information about the object of interest.

We will proceed from a linear model for the relation of sets of measurements of optical characteristics $\sigma_j^*(j = 1.2,\ldots,m)$ and unknown parameters $a_1(I = 1,2,\ldots,n)$

$$\sigma^* = Ua + \Delta_0, \tag{1}$$

where U is the m$m\times n$ matrix, and the vectors $\sigma*$ and a are defined by the components σ_j^* and a_1. We will assume also that the errors Δ_0 in the actual realization of σ are mainly caused by random measurement errors, which are normally distributed and statistically independent with the covariance matrix $C^2(\sigma - \sigma^* = \Delta_0) = I_m\varepsilon_0^2$, where I_m is the unit $m\times n$ matrix.

The major factors responsible for the efficiency of the solution of the inverse problem are included in the Fisher matrix

$$\Phi = \frac{U^TU}{\varepsilon_0^2}. \tag{2}$$

Its diagonal elements that characterize the average sensitivity (or the conditional information content) of the vector σ to a_1, have been examined elsewhere[1-5]

$$\Phi_{ii} = \frac{\|U_1\|^2}{\varepsilon_0^2}, \tag{3}$$

where $\|U_i\|$ is the norm of the ith column vector of the matrix U. Note that in Refs. 1 and 2 Φ is replaced by the "matrix of informational coverage" $S = A_0\Phi A_0$, where A_0 is the diagonal matrix of the parameters a_0 of a model solution, to account for the actual contribution of the unknown parameter to σ. In estimating the accuracy in Ref. 5 this has been achieved by substituting logarithms for a_1.

However, all the salient features of the inverse problem solution are, as a matter of fact, in the nondiagonal elements of Φ, responsible for the correlation between sensitivities (3) for different pairs of parameters. In Refs. 1 and 2 this fact has been accounted for by considering eigenvectors and eigenvalues of the matrix S that implies a search for linear combinations of the starting parameters for which the vector cr provides an "independent" information. Estimates of norms or condition numbers of the matrix Φ, that have been reported in Ref. 3, make it possible to evaluate correctly the information content of the input data of the inverse problem that is averaged over the components a_1. Finally, in Refs. 6 and 7 it is the covariance error matrix

$$C^2(\Delta\tilde{a}) = (U^TU)^{-1}\varepsilon_0^2 \tag{4}$$

of the vector of estimates

$$\tilde{a} = (U^TU)^{-1}U^T\sigma^*. \tag{5}$$

obtained with the use of the least-squares technique ($m \geq n$), which is to be regarded. In this representation the correlation between the sensitivities is ;taken into account in error variances in reconstructing the parameters. The values of the nondiagonal elements $C^2(\Delta\tilde{a})$ are a measure of correspondence of the chosen system of parameters a to linear combinations $\left[0 \leq |C_{1k}^2|/\sqrt{C_{11}^2\, C_{kk}^2} \leq 1\right]$ independent of σ. The conversion to the latter can be realized by orthogonization of columns of the matrix U using, for instance, the well–known Gram–Schmidt procedure.

The relation for the error variance of the parameter reconstruction under the assumption that it separates the response and correlation factors was derived in Ref. 7

$$D^2(\Delta\tilde{a}_1) = \frac{\varepsilon_0^2}{\|U_1\|^2}\,\frac{1}{1 - \rho^2_{1,1\ldots(1-1)(1+1)\ldots n}}. \tag{6}$$

Here $\rho_{\ldots}$ in terms of the statistical relationship theory is the multiple correlation coefficient between variations in σ_j caused by variations in the parameter a_1 and the remaining $n - 1$ parameters $a_1,\ldots, a_{i-1}, a_{i+1},\ldots$, and a_n. Geometrically it is determined solely by the angles included between the vectors U_i. For instance, for $n = 2$ $\rho_{12} = \cos(U_1, U_2)$ and for $n = 3$ $\rho_{1,23}$ is the cosine of the angle of U_1 with the plane in which the vectors U_2 and U_3 lie. It It should be noted that this kind of representation proves very convenient for model computations and makes it possible to reveal the reasons for the changes in the information content of optical measurements depending on one or another set of the parameters to be measured and reconstructed. In particular, the reduced number of the parameters being reconstructed simultaneously results only in the diminishment of the term $\rho^2_{\ldots}$ in Eq. (6). In the investigated method this is realized by diminishing the dimensions of the columns of the matrix U. The latter remark, in fact, means that the corresponding portion of the parameters determined from the data independently of σ is known exactly *a priori*. The question of the existence of the *a priori* estimates is quite legitimate, especially due to limited potentialities of an optical experiment or the development of routine procedures for its processing. Unfortunately, these procedures suffer from errors, therefore the corresponding analytical tools used for the evaluation of the information content are to take into account the quality of *a priori* estimates. The present paper is devoted to this question.

Let us assume that the optical measurements σ^* are supplemented with an independent concomitant information on the q parameters $a_1, \ldots, a_k$, which belong to a set of all n of the parameters a_i, which are significant in describing the optical properties of the dispersed system ($\|U_i\| \gg \|U_{n+1}\|$). In addition the errors in the concomitant measurement are also normally distributed and statistically independent with the covariance matrix C^2 ($\Delta_q = \Delta_{i=1,\ldots,k}$) $= I_q\varepsilon_q^2$. Evidently, such an information will improve the quality of reconstruction of the rth parameter a_r that does not belong to the collection, of the parameters $a_1, \ldots, a_k$. This fact is expressed mathematically by means of the information transmission in terms of correlations described by nondiagonal matrix elements of the second moments (4), i.e., in terms of $\langle\Delta a_r\Delta_1\rangle,\ldots,\langle\Delta a_r\Delta a_k\rangle$. For this reason to obtain the relations- for such a procedure, we make use of the formalism of the statistical relationship theory. The error vector $\Delta\tilde{a}$ of the estimates will be regarded as the vector of randomly correlated quantities with zero expectations. Then, for an absolutely exact assignment of the values of $\Delta\tilde{a}_1\rangle,\ldots,\langle\Delta\tilde{a}_k$ (i.e., the parameters $a_1,\ldots, a_k$) the error $\Delta\tilde{a}_r$ will be described by a certain quantity $\Delta\tilde{a}^{\perp}_{r|1\ldots k}$, which has conditional probability distribution with respect to $\Delta\tilde{a}_1,\ldots,\Delta\tilde{a}_k$. In this case, according to Ref. 8, $\Delta\tilde{a}^{\perp}_{r|1\ldots k}$ will represent $\Delta\tilde{a}_r$, minus the linear regression $\Delta\tilde{a}'_{r|1\ldots k} = \sum\limits_{i=1,\ldots,k} \beta_{r_i}\Delta\tilde{a}_i$ of this quantity over $\Delta\tilde{a}_1,\ldots,\Delta\tilde{a}_k$, where β_{r_j} are partial regression coefficients, i.e., we can write

$$\Delta\tilde{a}_r = \Delta\tilde{a}^{\perp}_{r|1\ldots k} + \Delta\tilde{a}'_{r|1\ldots k}. \tag{7}$$

In this case $\Delta\tilde{a}_r$ is governed solely by the optical measurement errors $\Delta\tilde{a}_r = F_r\,\Delta_0$ (F_0 is the vector formed by the rth row vector of the matrix $F = (U^TU)^{-1}U^T$ in Eq. (5)). The relation (7) only stresses the fact that the estimate of $\Delta\tilde{a}^{\perp}_{r|1\ldots k}$, which appears in Eq. (13) below, is independent of $\Delta\tilde{a}_1,\ldots,\Delta\tilde{a}_k$. Therefore, the expression for the estimate of the error in reconstructing the rth parameter $\tilde{a}_{r,1\ldots k}$ with any order accuracy of the assigned values $a_1,\ldots,a_k$ will be similar to Eq. (7), where the second term can be regarded as an efficient estimate of regression from an independent optical and *a priori* data. On account of the normal distribution of probability of the errors in all the measurements and the fact that the data sources (Δ_0 and Δ_q) are independent, the estimate based on the weighted least–squares technique will be efficient. We then obtain

$$\Delta\tilde{a}_{r,1\ldots k} = \Delta\tilde{a}'_{r|1\ldots k} + \frac{P_0\Delta\tilde{a}'_{0(r|1\ldots k)} + P_1\Delta\tilde{a}'_{1(r|1\ldots k)}}{P_0 + P_1} \tag{8}$$

with the variance

$$D^2\left(\Delta\tilde{a}_{r,1\ldots k}\right) = D^2\left(\Delta\tilde{a}^{\perp}_{r|1\ldots k}\right) + \frac{\varepsilon_0^2}{P_0 + P_1}, \tag{9}$$

where P_0 and P_1 are statistical weights of the estimates

$$P_0 = \frac{\varepsilon_0^2}{D^2\left[\Delta\tilde{a}'_{0(r|1\ldots k)}\right]}, \quad P_1 = \frac{\varepsilon_0^2}{D^2\left[\Delta\tilde{a}'_{1(r|1\ldots k)}\right]} \tag{10}$$

Here $\Delta\tilde{a}'_{0(r|1\ldots k)}$ and $\Delta\tilde{a}'_{1(r|1\ldots k)}$are the estimates of the regression error in reconstructing the parameter a_r in terms of the errors involved in optical and concomitant measurements, respectively

$$\Delta\tilde{a}'_{0(r|1\ldots k)} = \sum_{i=1,\ldots,k} \beta_{r_i} F_i \Delta_0 \tag{11}$$

and

$$\Delta\tilde{a}'_{1(r|1\ldots k)} = \sum_{i=1,\ldots,k} \beta_{r_i} \Delta_i \tag{12}$$

and according to Eq. (7),

$$\Delta\tilde{a}^{\perp}_{r|1\ldots k} = F_r\Delta_0 - \sum_{i=1,\ldots,k} \beta_{r_i} F_i \Delta_0. \tag{13}$$

For the sake of simplicity the subscripts in parenthesis adjacent to Дa will be omitted below. Accounting for Eqs. (11)–(13) and the well–known representations of the regression coefficients in terms of the minors of the covariance matrix of errors in random variables[8,9] it can be shown that the variances have of the form

$$D^2(\Delta\tilde{a}'_0) = C^2\binom{r}{r} - \frac{C^2\begin{pmatrix} r, & l, & \dots, & k \\ r, & l, & \dots, & k \end{pmatrix}}{C^2\begin{pmatrix} l, & \dots, & k \\ l, & \dots, & k \end{pmatrix}}, \tag{14}$$

$$D^2(\Delta\tilde{a}'_1) = \varepsilon_q^2 \sum_{i=1,\dots,k} \left[\frac{C^2_{ir}\begin{pmatrix} l, & \dots, & k \\ l, & \dots, & k \end{pmatrix}}{C^2\begin{pmatrix} l, & \dots, & k \\ l, & \dots, & k \end{pmatrix}} \right]^2, \tag{15}$$

and

$$D^2(\Delta\tilde{a}^{\perp}) = \frac{C^2\begin{pmatrix} r, & l, & \dots, & k \\ r, & l, & \dots, & k \end{pmatrix}}{C^2\begin{pmatrix} l, & \dots, & k \\ l, & \dots, & k \end{pmatrix}}, \tag{16}$$

where $C^2_{ir}\begin{pmatrix} l,\dots,k \\ l,\dots,k \end{pmatrix}$ is the minor of the matrix $C^2(\Delta\tilde{a})$ (Eq. (2)), in which, unlike its principle minor $C^2\begin{pmatrix} l,\dots,k \\ l,\dots,k \end{pmatrix}$ the elements of intersection of the ith column of the set $I = l,\dots,k$ are replaced by the elements of intersection of the rth column with the rows numbered $l,\dots,k$.

For a subsequent representation of the accuracy inherent in the solution of the inverse problem in terms of norms and angles included between the vectors U_i, we replace the minors of matrix (4) by the minors of the matrix $B = U^T U$ in Eqs. (14)–(16). Using the well–known relation between the minors of the inverse matrix and the initial matrix[10] and taking into account the symmetry of B and $C^2(\Delta\tilde{a})$ we obtain

$$D^2(\Delta\tilde{a}'_0) = \varepsilon_0^2 \left[\frac{B\begin{pmatrix} l, & \dots, & k, & l', & \dots, & k' \\ l, & \dots, & k, & l', & \dots, & k' \end{pmatrix}}{\det B} - \frac{B\begin{pmatrix} l', & \dots, & k' \\ l', & \dots, & k' \end{pmatrix}}{B\begin{pmatrix} r, & l', & \dots, & k' \\ r, & l', & \dots, & k' \end{pmatrix}} \right], \tag{17}$$

$$D^2(\Delta\tilde{a}'_1) = \varepsilon_q^2 \sum_{i=1,\dots,k} \left[\frac{B\begin{pmatrix} r, & l', & \dots, & k' \\ i, & l', & \dots, & k' \end{pmatrix}}{B\begin{pmatrix} r, & l', & \dots, & k' \\ r, & l', & \dots, & k' \end{pmatrix}} \right]^2, \tag{18}$$

$$D^2(\Delta\tilde{a}^{\perp}) = \varepsilon_0^2 \frac{B\begin{pmatrix} l', & \dots, & k' \\ l', & \dots, & k' \end{pmatrix}}{B\begin{pmatrix} r, & l', & \dots, & k' \\ r, & l', & \dots, & k' \end{pmatrix}}, \tag{19}$$

where l', ..., k' are the numbers of the remaining set of $n - (q + 1)$ columns and rows of the matrix B that was not covered by the set with the numbers r, $l,\dots,k$. The former set represents a portion of the parameters a'_1, ..., a'_k for which neither a_r nor *a priori* estimates are available. On the basis of the relation $B_{ij} = \|U_i\|\,\|U_j\|\cos(U_i, U_j)$ for the elements of the matrix B we will take the norms $\|U_i\|$ out in Eq. (17)–(18) and express the remaining angular functions in terms of the coefficients, which are similar to those used in the statistical relationship theory. We then obtain

$$D^2(\Delta\tilde{a}'_0) = \frac{\varepsilon_0^2}{\|\vec{U}_r\|^2} \left[\frac{1}{1 - \rho^2_{r,1\dots k1'\dots k'}} - \frac{1}{1 - \rho^2_{r,1'\dots k'}} \right], \tag{20}$$

$$D^2(\Delta\tilde{a}'_1) = \varepsilon_q^2 \sum_{i=1,\dots,k} \frac{\|U_i\|^2}{\|U_r\|^2} \times \frac{1 - \rho^2_{i,1'\dots k'}}{1 - \rho^2_{r,1'\dots k'}} \rho^2_{r_i,1'\dots k'}, \tag{21}$$

and

$$D^2(\Delta\tilde{a}^{\perp}) = \frac{\varepsilon_0^2}{\|U_r\|^2} \frac{1}{1 - \rho^2_{r,1'\dots k'}}. \tag{22}$$

Here

$$\rho^2_{i,1'\dots k'} = \frac{\hat{B}\begin{pmatrix} i, & l', & \dots, & k' \\ i, & l', & \dots, & k' \end{pmatrix}}{\hat{B}\begin{pmatrix} l', & \dots, & k' \\ l', & \dots, & k' \end{pmatrix}} \tag{23}$$

and

$$\rho^2_{r_i,1'\dots k'} = \frac{\left[\hat{B}\begin{pmatrix} r, & l', & \dots, & k' \\ i, & l', & \dots, & k' \end{pmatrix}\right]^2}{\hat{B}\begin{pmatrix} i, & l', & \dots, & k' \\ i, & l', & \dots, & k' \end{pmatrix} \hat{B}\begin{pmatrix} r, & l', & \dots, & k' \\ r, & l', & \dots, & k' \end{pmatrix}} \tag{24}$$

and $\hat{B}(:::)$ are the minors of the matrix $\hat{B}_{ij} = \cos(U_i, U_j)$. Regarding $\hat{B}$ as the matrix of the lowest–order correlation coefficients for U_i and U_j we can conclude with the use of Eqs. (23) and (24) that $\rho_{i,l',\dots,k'}$ and $\rho_{r_i,l',\dots,k'}$ are the multiple and partial correlation coefficients, respectively. The partial correlation coefficient for $n = 3$ is geometrically interpreted as the cosine of the angle included between the projections of the vectors U_r and U_i onto the plane perpendicular to the $U_{l'}$.

Substituting Eqs. (20)–(22) into Eq. (9) we finally obtain

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|\vec{U}_r\|^2 (1 - \rho^2_{r,1'\dots k'})} \times \frac{\left(\dfrac{\varepsilon_q}{\varepsilon_0}\right)^2 \tilde{U}^2 + R^2_{r,1\dots k,1'\dots k'}}{\left(\dfrac{\varepsilon_q}{\varepsilon_0}\right)^2 \tilde{U}^2 (1 - R^2_{r,1\dots k,1'\dots k'}) + R^2_{r,1\dots k,1'\dots k'}}, \tag{25}$$

where

$$\tilde{U}^2 = \sum_{i=1,\dots,k} \|U_i\|^2 (1 - \rho^2_{i,1'\dots k'}) \rho^2_{ri,1'\dots k'}, \tag{26}$$

and the use of the quantity

$$R^2_{r,1\ldots k,1'\ldots k'} = 1 - \frac{1-\rho^2_{r,1\ldots k,1'\ldots k'}}{1-\rho^2_{r,1'\ldots k'}} \quad (27)$$

is the generalization of the multiple correlation coefficient to the case of fixed values of $a'_1,\ldots, a'_k$ ($\rho^2_{r,1',\ldots,k'} = 0$) or the partial correlation coefficient between the vector U_r and the set of the vectors $U_1,\ldots,U_k$. Relation (25) is drastically simplified for a number of the particular cases being of practical value. Given accurate data on the parameters $\left(\frac{\varepsilon_q}{\varepsilon_0} \to 0\right)$ we obtain

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2}\,\frac{1}{1-\rho^2_{r,1'\ldots k'}}. \quad (28)$$

Conversely, for the lack of the relevant *a priori* information on the parameters $\left(\frac{\varepsilon_q}{\varepsilon_0} \to \infty\right)$ we shall have

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2}\,\frac{1}{1-\rho^2_{r,1\ldots k,1'\ldots k'}}. \quad (29)$$

In both cases we essentially arrive at Eq. (6) that have been obtained earlier, where the limiting transition from Eq. (29) to Eq. (28) appears as the diminishing of the number of the columns of matrix U at the set of vectors $U_1,\ldots,U_k$, which corresponds to accurate data on the parameters $a_1,\ldots,a_k$ (Ref. 7) or to a weak response of optical characteristics to the parameters $\|U_1\|^2,\ldots,\|U_k\|^2 \to 0$ (Eq. (26)). However, it is evident from Eqs. (25) and (26) that the transition to Eq. (28) will be valid also under the conditions

$$\rho^2_{1,1'\ldots k'};\ \ldots;\ \rho^2_{k,1'\ldots k'} \longrightarrow 1, \quad (30)$$

$$\rho^2_{r1,1'\ldots k'};\ \ldots;\ \rho^2_{rk,1'\ldots k'} \longrightarrow 0, \quad (31)$$

and

$$R^2_{r,1\ldots k,1'\ldots k'} \longrightarrow 0. \quad (32)$$

The last means that U_r correlates with the remaining set of columns of the matrix U just in the same way as with the set of the vectors U_1, ,U_k, (Eq. (27)). Every of these conditions shuts off *a priori* information channel but makes it possible to reduce the maximum variance of an optical estimate (Eq. (29)) down to the value given by Eq. (28). Its minimum value

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2}, \quad (33)$$

will be found by supplementing one of the conditions (30)–(32) with the condition

$$\rho^2_{r,1'\ldots k'} \longrightarrow 0. \quad (34)$$

As $R^2_{r,1,\ldots,k,1',\ldots,k'} \to 1$ and the condition (30) or (31) is satisfied, we will arrive at formula (28) again. As

$$\rho^2_{1,1'\ldots k'};\ \ldots;\ \rho^2_{k,1'\ldots k'} \longrightarrow 0 \quad (35)$$

and

$$\rho^2_{r1,1'\ldots k'};\ \ldots;\ \rho^2_{rk,1'\ldots k'} \longrightarrow 1, \quad (36)$$

a priori information plays a very important role in solving the inverse problem

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2(1-\rho^2_{r,1'\ldots k'})} \times \left[\left(\frac{\varepsilon_q}{\varepsilon_0}\right)^2 \sum_{i=1,\ldots k} \|U_i\|^2 + 1\right]. \quad (37)$$

For $\left(\frac{\varepsilon_q}{\varepsilon_0}\right)^2 \gg 1$ or $\|U_1\|^2,\ldots,\|U_k\|^2 \gg 1$ the error variance in reconstructing the parameter a_r is determined solely by the accuracy of *a priori* estimates.

Let us assume that in solving the inverse problem either all the parameters, except that a_r, are redefined, or the parameter a_1 alone is redefined. This is very typical of the situations where comparison is made of the a priori information on the particle size distribution function with the refractive index of the material of the particles. In these cases we obtain from Eq. (25)

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2} \times \frac{\left[\frac{\varepsilon_q}{\varepsilon_0}\right]^2 \sum_{i=1,\ldots,k} \|U_i\|^2\rho^2_{ri} + \rho^2_{r,1\ldots k}}{\left[\frac{\varepsilon_q}{\varepsilon_0}\right]^2 (1-\rho^2_{r,1\ldots k}) \sum_{i=1,\ldots k} \|U_i\|^2\rho^2_{ri} + \rho^2_{r,1\ldots k}}; \quad (38)$$

and

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2(1-\rho^2_{r,1'\ldots k'})} \times \frac{\left[\frac{\varepsilon_q}{\varepsilon_0}\right]^2 \|U_1\|^2(1-\rho^2_{1,1'\ldots k'}) + 1}{\left[\frac{\varepsilon_q}{\varepsilon_0}\right]^2 \|U_1\|^2(1-\rho^2_{1,1'\ldots k'})(1-\rho^2_{r1,1'\ldots k'}) + 1}, \quad (39)$$

respectively.

For $n = 2(\rho_{1,1',\ldots k'} = 0,\ \rho_{r1,1',\ldots,k'} = \rho_{r1})$ we derive the relation

$$\frac{D^2(\Delta\tilde{a}_r)}{\varepsilon_0^2} = \frac{1}{\|U_r\|^2} \frac{\left(\frac{\varepsilon_q}{\varepsilon_0}\right)^2 \|U_1\|^2 + 1}{\left(\frac{\varepsilon_q}{\varepsilon_0}\right)^2 \|U_1\|^2 (1 - \rho_{r1}^2) + 1}, \tag{40}$$

which may be used conveniently in solving the simplest and wide–spread problem of determining the particle number density a_r with the *a priori* estimates of the particle characteristic size a_l.

The developed techniques are naturally supplemented with an expression which provides for the *a priori* information on the parameter a_r as well. Regarding the estimates of a_r obtained from *a priori* data and from the set σ, $a_1,\ldots,a_k$ as being independent and using the weighted least–squares technique, we obtain

$$\frac{D^2(\Delta\tilde{a}_r^*)}{\varepsilon_0^2} = \frac{D^2(\Delta\tilde{a}_r)\left(\frac{\varepsilon_r}{\varepsilon_0}\right)^2}{D^2(\Delta\tilde{a}_r) + \varepsilon_r^2}, \tag{41}$$

where ε_r^2 is the variance of the *a priori* estimate of the parameter a_r and $D^2(\Delta\tilde{a}_r)$ is given by formula (25). Obviously, that for a poor redefinition of the parameter $a_r(\varepsilon_r^2 \gg D^2(\Delta\tilde{a}_r))$, Eq. (41) transforms into Eq. (25), and for $\varepsilon_r^2 \ll D^2(\Delta\tilde{a}_r)$ the quantity $D^2(\Delta\tilde{a}_r^*) = \varepsilon_r^2$ is determined solely by *a priori* estimate of the error in the parameter a_r.

Finally, it should be noted that, given *a priori* estimates, the procedure of reconstructing the vector $\tilde{a}$, in contrast to Eq. (5) is possible for $m > n - q$. It can be shown that, in the particular case $q = n$ and $\varepsilon_r = \varepsilon_q$ this procedure assumes regularization character with a zero–order regulator, where *a priori* estimates appear as "trial solution"[11] and the regularization parameter is determined from the ratio of the error variances of the optical measurement and *a priori* estimates.

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