IMPROVEMENT OF THE COMPUTATION SPEED WHEN CALCULATING AUTOCORRELATION FUNCTION BY THE METHOD OF FAST FOURIER TRANSFORM

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We propose here an operative technique to compute first statistically significant readings of the autocorrelation function based on the known double FFT. The speed-up of computations is reached by corresponding replacement of one initial reversal FFT by several ones of lower dimensionality. Two special cases of the algorithm proposed are discussed. Quantitative estimates of the computation speed enhancement relative to the standard one are presented.

The necessity to obtain complementary spectral–correlative information about the object under study often exists in experimental investigations. The situation when the required spectral resolution is provided only by the whole length of the initial realization T is rather typical. In some cases interpolated values of the spectral density are also necessary. For instance, in calculating integral spectral characteristics bearing quantitative information about wind velocity field in gated volume in acoustic sounding of the atmosphere the interval of frequency readings is recommended to be $1/2T$ (Ref. 1).

Here it is worthwhile applying known algorithm[2] when first the spectral density $G(k/2T)$ is calculated by discrete Fourier transform (DFT) and then the autocorrelation function (ACF) $B(r\Delta t)$ of the initial sample set

$$B(r\Delta t) = \frac{1}{L\,\Delta t}\sum_{k=0}^{L-1} G(k/L\,\Delta t)\,\exp(j\,2\pi\,k\,r/L), \qquad (1)$$

where $r = 0, 1, \ldots, L-1$; $L = 2N$; N is the number of readings, and Δt is the discretization interval of the processed realization of length $T = N\Delta t$, is calculated by the inverse DFT (IDFT). The efficiency of the processing is provided by the application of corresponding calculation algorithms; in particular, the relation (1) is realized by the inverse fast Fourier transform (IFFT). Here one obtains the estimates of all possible ACF values. However, in this situation only its first $R \approx (0.1 - 0.15)N$ readings are statistically significant. Computation of the following values leads to unjustified increase in calculation volume what is undesirable in operative measurements.

Let's transform Eq. (1) for calculating only first R readings of the ACF. To do that, let us divide the sequence $G(k)$ into R equal parts of the reading length $M = L/R$ and, omitting the normalizing factor, write Eq. (1) in the form

$$b(r) = B(r)\,L\,\Delta t = \sum_{m=0}^{R-1}\ \sum_{k=mM}^{(m+1)M-1} G(k)\,\exp(j\,2\pi\,k\,r/L).$$

Substituting $l = k - mM$ in the inner sum and changing the summation order we obtain

$$b(r) = \sum_{k=0}^{M-1} g(r, k)\,\exp(j\,2\pi\,k\,r/L),$$

$$r = 0, 1, \ldots, R-1, \qquad (2)$$

where $g(r, k) = \sum_{m=0}^{R-1} G(m\,M + k)\,\exp(j\,2\pi\,m\,r/L)$ are the kth R–dimensional IDFT of the corresponding spectral readings. In order to calculate them using standard IFFT algorithm we suppose that N and R are integer multiple to 2 in corresponding power. Then the value M providing statistically significant ACF readings can be equal to 16 or 32.

Let's reduce the volume of computations in Eq. (2). Note that the spectral density $G(k)$ of the considered sample is a DFT–even sequence on the interval L (according to terminology from Ref. 3), i.e., the equality

$$G(k) = G(L - k), \quad k = 1, 2, \ldots, L/2 - 1 \qquad (3)$$

is valid. In particular, it follows that the ACF $B(r\Delta t)$ is DFT–even and real in realization of the relation (1). Taking into account Eq. (3), considering the expression for $g(r, M - k)$, and changing the summation index m by $l = R - m - 1$ we obtain

$$g(r, M - k) = g^*(r, k)\exp(-j2\pi r/R),$$

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where $k = 1, 2, \ldots, M/2-1$, asterisk denotes complex conjugate. Then an important practical property follows therefrom, namely, the Hermitian symmetry of the summed sequence $X(r, k) = g(r, k)\exp(j2\pi kr/L)$, in Eq. (2), with respect to k, i.e., the condition $X(r, M-k) = X^*(r, k)$, $k = 1, 2, \ldots, M/2-1$ holds. This makes it possible to reduce Eq. (2) to a simpler form

$$b(r) = g(r, 0) + X(r, M/2) + 2\sum_{k=1}^{M/2-1} \mathrm{Re}\, X(r, k),$$

$r = 0, 1, \ldots, R-1$.

It is also obvious that in the general case $g(r, k)$ are Hermitian symmetric sequences with respect to the index r as IDFT of real spectral readings $G(mM+k)$, i.e., conditions $g(R-r, k) = g^*(r, k)$, $r = 1, 2, \ldots, R/2-1$ is valid. It immediately follows therefrom that

$$X(R-r, k) = X^*(r, k)\exp(j2\pi k/M),\ r = 1, 2, \ldots, R/2-1.$$

Then the latter expression for $b(r)$ can be presented in the form

$$\begin{cases} b(r) = g(r, 0) + X(r, M/2) + 2\sum\limits_{k=1}^{M/2-1} \mathrm{Re}X(r, k), \\ b(R-r) = g^*(r, 0) - X^*(r, M/2) + 2\sum\limits_{k=1}^{M/2-1} \mathrm{Re}X(R-r, k), \end{cases} \tag{4}$$

where

$r = 1, 2, \ldots, R/2-1$,

$\mathrm{Re}\, X(R-r, k) = \mathrm{Re}\{X^*(r, k)\exp(j2\pi k/M)\}$.

Let us concretize some values in Eq. (4). Consider $g(r, 0) = \sum\limits_{m=0}^{R-1} G(mM)\exp(j2\pi mr/R)$. It is easy to demonstrate that DFT-parity of the initial spectral readings (3) leads to that of $G(mM)$, $m = 0, 1, \ldots, R-1$. It follows therefrom that $g(r, 0)$ is real and DFT-even.[3]

Further, let's consider the value

$$X(r, M/2) = g(r, M/2)\exp(j\pi r/R),$$

where

$$g(r, M/2) = \sum_{m=0}^{R-1} G(mM + M/2)\exp(j2\pi mr/R).$$

In the given case Eq. (3) implies evenness of the sequence $G(mM + M/2) = \hat{G}(m)$ in the common sense, i.e., condition $\hat{G}(R-m-1) = \hat{G}(m)$, $m = 0, 1, \ldots, R/2$ holds. So one can avoid straightforward calculations of $g(r, M/2)$ with simultaneous reduction of the dimensionality of the required IFFT by four times. Let's present $g(r, M/2)$ in the form

$$g(r, M/2) = \sum_{i=0}^{3} \sum_{m=0}^{R/4-1} G[(4m+i)M + M/2] \times$$

$$\times \exp[j2\pi r(4m+i)/R] = \sum_{i=0}^{3} g_i(r)\exp(j2\pi r/R), \tag{5}$$

where

$$g_i(r) = \sum_{m=0}^{R/4-1} G[(4m+i)M + M/2]\exp[j2\pi mr/(R/4)]$$

are IDFT of the corresponding spectral sequences; they are periodical with the period of $R/4$ readings, and $g_i(r) = g_i^*(R/4-r)$, $r = 1, 2, \ldots, R/8-1$. By changing the summation index $l = R/4-m-1$ in the expressions for $g_2(r)$ and $g_3(r)$, and taking into account the property (3) we obtain $g_2(r) = g_1^*(r) \times \exp(-j8\pi r/R)$, $g_3(r) = g_0^*(r)\exp(-j8\pi r/R)$. Then Eq. (5) for the main period of $g_i(r)$ takes the form

$$g(r, M/2) = 2[\mathrm{Re}\,\beta_0(r) + \mathrm{Re}\,\beta_1(r)]\exp(-j\pi r/R),$$

where

$$\beta_0(r) = g_0(r)\exp(j\pi r/R),\ \beta_1(r) = g_1(r)\exp(j3\pi r/R),$$

$r = 0, 1, \ldots, R/4-1$.

Therefore, $X(r, M/2) = 2[\mathrm{Re}\beta_0(r) + \mathrm{Re}\beta_1(r)]$ is a purely real sequence. In order to find $X(r, M/2)$ when $r > R/4-1$, we use the above mentioned properties of $g_i(r)$ and their corollaries:

$$g_i(R/2-r) = g_i^*(r),\ r = 1, 2, \ldots, R/8-1;$$

$$\mathrm{Im}\, g_i(0) = \mathrm{Im}\, g_i(R/8) = \mathrm{Im}\, g_i(3R/8) =$$

$$= \mathrm{Im}\, g_i(R/4) = \mathrm{Im}\, g_i(R/2) = 0.$$

At the same time, in order to reduce computation volume, we decrease the domain of IDFT $g_i(r)$. Finally, for $r = 1, 2, \ldots, R/8-1$ we obtain

$$\begin{cases} \mathrm{Re}\, X(r, M/2) = 2[\mathrm{Re}\,\beta_0(r) + \mathrm{Re}\,\beta_1(r)], \\ \mathrm{Re}\, X(R/4-r, M/2) = \\ = 2[\mathrm{Re}\,\{\beta_0^*(r)\exp(j\pi/4)\} + \mathrm{Re}\,\{\beta_1^*(r)\exp(j3\pi/4)\}], \\ \mathrm{Re}\, X(R/4+r, M/2) = \\ = 2[\mathrm{Re}\{\beta_0(r)\exp(j\pi/4)\} + \mathrm{Re}\,\{\beta_1(r)\exp(j3\pi/4)\}], \\ \mathrm{Re}\, X(R/2-r, M/2) = 2[\mathrm{Im}\,\beta_0(r) - \mathrm{Im}\,\beta_1(r)]. \end{cases} \tag{6}$$

And for singular points

$$\begin{cases} \mathrm{Re}X(0, M/2) = 2[\mathrm{Re}g_0(0) + \mathrm{Re}g_1(0)] \\ \mathrm{Re}\, X(R/2, M/2) = 0, \\ \mathrm{Re}X(R/8, M/2) = \\ = 2[\mathrm{Re}g_0(R/8)\cos(\pi/8) + \mathrm{Re}\, g_1(R/8)\cos(3\pi/8)], \\ \mathrm{Re}X(R/4, M/2) = \sqrt{2}\,[\mathrm{Re}\, g_0(0) - \mathrm{Re}g_1(0)], \\ \mathrm{Re}X(3R/8, M/2) = \\ = 2[\mathrm{Re}g_0(R/8)\cos(3\pi/8) - \mathrm{Re}g_1(R/8)\cos(\pi/8)]. \end{cases} \tag{7}$$

With allowance for the above-mentioned facts the final formulas (4) for calculating first R readings of the ACF take the form

$$\begin{cases} b(r) = \mathrm{Re}\, g(r, 0) + \mathrm{Re}\, X(r, M/2) + 2 \sum\limits_{k=1}^{M/2-1} \mathrm{Re}\, X(r, k), \\ b(R-r) = \mathrm{Re}\, g(r, 0) - \mathrm{Re}\, X(\mathrm{r}, \mathrm{M}/2) + 2 \sum\limits_{k=1}^{M/2-1} \mathrm{Re}\, X(R-r, k), \end{cases} \tag{8}$$

where $r = 1, 2, \ldots, R/2 - 1$. And for singular points

$$\begin{cases} b(0) = \mathrm{Re}\, g(0, 0) + \mathrm{Re}\, X(0, M/2) + 2 \sum\limits_{k=1}^{M/2-1} \mathrm{Re}\, g(0, k), \\ b(R/2) = \mathrm{Re}\, g(R/2, 0) + 2 \sum\limits_{k=1}^{M/2-1} \mathrm{Re}\, g(R/2, k) \cos(\pi k/M). \end{cases} \tag{9}$$

Summing up the above statements we present the algorithm in its complete form:

1. Form $M/4$ complex sequences of the length of R readings from the initial spectral values $G(k)$, $k = 0, 1, \ldots, L - 1$, by the following rule

$$Z_i(m) = G(2i + mM) + j\, G[(2i + 1) + mM],$$

where $i = 0, 1, \ldots, M/4-1$, $m = 0, 1, \ldots, R-1$.

2. Calculate $M/4$ complex IFFT of the dimensionality R

$$z_i(r) = \sum_{m=0}^{R-1} Z_i(m) \exp(j 2\pi m r/R), \quad r = 0, 1, \ldots, R-1.$$

3. Reconstruct separate IFFT $g(r, k)$, $k=0,1,\ldots, M/2-1$ by full analogy with the same procedure for the direct FFT[2]

$$\mathrm{Re}\, g(r, 2i) = [\mathrm{Re}\, z_i(r) + \mathrm{Re}\, z_i(R - r)]/2,$$
$$\mathrm{Im}\, g(r, 2i) = [\mathrm{Im}\, z_i(r) - \mathrm{Im}\, z_i(R - r)]/2,$$
$$\mathrm{Re}\, g(r, 2i+1) = [\mathrm{Im}\, z_i(r) + \mathrm{Im}\, z_i(R - r)]/2,$$
$$\mathrm{Im}\, g(r, 2i+1) = [\mathrm{Re}\, z_i(R - r) - \mathrm{Re}\, z_i(r)]/2,$$

where $r = 1, 2, \ldots, R/2-1$;

$$\mathrm{Re}\, g(0, 2i) = \mathrm{Re}\, z_i(0), \quad \mathrm{Re}\, g(R/2, 2i) = \mathrm{Re}\, z_i(R/2),$$
$$\mathrm{Re}\, g(0, 2i+1) = \mathrm{Im}\, z_i(0),$$
$$\mathrm{Re}\, g(R/2, 2i+1) = \mathrm{Im}\, z_i(R/2).$$

4. Multiply $g(r, k)$, $k=0,1,\ldots, M/2-1$ by phase factors, i.e., form

$$X(r, k) = g(r, k) \exp(j 2\pi k r/L), \quad r = 1, 2, \ldots, R/2-1.$$

5. For $k = M/2$ form a complex sequence of the length $R/4$

$$Z(m) = G[4mM + M/2] + j\, G[(4m + 1)M + M/2] =$$
$$= G_0(m) + j\, G_1(m), \quad m = 0, 1, \ldots, R/4-1$$

and perform the IFFT of $R/4$ dimensionality with further reconstruction of $g_0(r)$ and $g_i(r)$, $r = 0, 1, \ldots, R/8$ by analogy with the point 3 and changing R by $R/4$. Then, using the relations (6) and (7), obtain $\mathrm{Re}X(r, M/2)$, $r = 0, 1, \ldots, R/2-1$.

6. Following (8) and (9) calculate first R ACF readings.

In spite of a bulky form of the proposed algorithm it is more effective in computations than the classical method (1) because of reduction of the number of complex multiplications although the number of other (but more elementary) operations is larger. The advantage is provided by the change of a single initial IFFT (1) by several IFFTs of lower dimensionality. It is also supposed that the sequences of complex exponents used in the presented algorithm are calculated by known recurrence relations,[4] i.e., by a single complex multiplication. Using the relations given in Ref. 4 for the volume of computing operations of IFFT we obtain the following estimates for the number of complex multiplications: $P_1 \approx (L/2)\log_2 L$ for the traditional variant (1) and $P \approx (L/8)(\log_2 R + 5)$ for the proposed one. Hence the expected gain in speed $\gamma = P_1/P$ for the values $M = 16$ or $M = 32$ used by the algorithm and characterizing the reduction of the correlation information volume is $\gamma \approx (3.6-4)$ times. And the actual gain obtained on the basis of direct calculations of ACF is less and averages (3.4–3.8) times for both variants.

The particular case of $M = 2$ in the above–considered algorithm is also of practical interest. Here, as it follows from Eq. (2), it is necessary to form a complex sequence $Z(m) = G(2m) + jG(2m + 1)$, $m = 0, 1, \ldots, L/2-1$ and calculate its $L/2$–dimensional IFFT, i.e., to find $z(r)$, $r = 0, 1, \ldots, L/2-1$. Note that $G(2m)$ is a DFT–even sequence, and $G(2m + 1)$ is even in the common sense. Then, using the above stated facts, one can demonstrate that in order to calculate first $R \le L/4$ ACF readings it is sufficient to make the following operations

$$b(r) = \mathrm{Re}\, g(r, 0) + \mathrm{Re}\, \{g(r, 1) \exp(j 2\pi k r/L)\}, \quad r = 0, 1, \ldots, R - 1. \tag{10}$$

where

$$\mathrm{Re}\, g(r, 0) = [\mathrm{Re}\, z(r) + \mathrm{Re}\, z(L/2 - r)]/2,$$
$$\mathrm{Re}\, g(r, 1) = \mathrm{Im}\, z(r),$$
$$\mathrm{Im}\, g(r, 1) = [\mathrm{Re}\, z(L/2 - r) - \mathrm{Re}\, z(r)]/2,$$
$$\mathrm{Re}\, g(0, 0) = \mathrm{Re}\, z(0), \quad \mathrm{Re}\, g(0, 1) = \mathrm{Im}\, z(0)$$

for $r = 1, 2, \ldots, R-1$.

The main advantage of this method is in its simplicity and flexibility in the choice of the number of ACF points to be computed. But it is inferior to the main variant (8), (9) in operativeness. Straightforward calculations show that it is 1.9 times slower than the algorithm considered above and at the same time faster than the standard variant (1) by the same factor.

A larger gain can be reached if there is a necessity to compute first R points of ACF with rarefying, i.e. when $r = nR_r$, $n = 0, 1, \ldots, R/R_r - 1$, R_r is the step of rarefying. This will work good, for instance, in the case of sufficiently smooth ACFs. Then the relation (2) is valid for $L = M_r R_r$ again. But in contrast to the previous case we consider the value $g(r, k)$ beyond its Nyquist uniqueness interval, namely, at the points $r = nR_r$ multiple to its period R_r. Since $g(nR_r/k) = g(0, k)$, the relation (2) takes the form

$$b(n R_r) = \sum_{k=0}^{M_r-1} g(k) \exp(j 2\pi k n/M_r),$$

$$n = 0, 1, \ldots, M_r - 1, \qquad (11)$$

where $g(k) = \sum_{m=0}^{R_r-1} G(m M_r + k)$.

Thus in the general case it is necessary to sum R_r corresponding values of the initial spectral density as a preliminary and calculate $M_r = L/R_r$–dimensional IFFT in order to obtain only one of R_rth ACF readings. In fact, Eq. (11) illustrates the process of overlapping of spectral readings when the discretization interval in the correlation domain extends as compared with the initial one.

It is easy to demonstrate that the sequence $g(k)$ is DFT–even, i.e., $g(M_r - k) = g(k)$, $k = 1, 2, \ldots, M_r/2 - 1$. To do that, it is sufficient to substitute $l = R_r - m - 1$ into the expression for $g(M_r - k)$ and use DFT–evenness of the initial spectral readings (3). Therefore, in order to calculate first R/R_r readings of the rarefied ACF, one can use the algorithms (8), (9), or (10) if L is changed by M_r and R by R/R_r. And it is expedient, when obtaining $g(k)$, to use the initial spectral readings corresponding only to positive frequencies:

$$g(0) = G(0) + G(L/2) + 2 \sum_{m=1}^{R_r/2-1} G(m M_r),$$

$$g(M_r/2) = 2 \sum_{m=0}^{R_r/2-1} G[M_r (m + 1/2)],$$

$$g(k) = \sum_{m=0}^{R_r/2-1} \{G(m M_r + k) + G[(m + 1) M_r - k]\},$$

$$k = 1, 2, \ldots, M_r/2 - 1.$$

The increase in the speed of computations of the rarefied ACF by the proposed method in comparison with the classical variant (1) is characterized by the value $\gamma \approx 4R_r$.

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