# LIMITING EFFICIENCIES AND APPLICABILITY OF DIFFERENT WAYS TO FORM REFERENCE LASER STARS

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*We describe here three possible schemes of the laser reference star formation, namely, monostatic, bistatic, and intermediate. We also determine here limiting potentialities of methods of correction for random tilts of a wavefront from natural star using a signal from a laser star. The monostatic scheme of the laser reference star formation is shown to be totally inapplicable for these purposes. The capabilities of the bistatic correction scheme are estimated. Then we show the possibility of using an intermediate scheme.*

Among the problems that seriously challenge the investigators of adaptive optical telescopes is the need for use of reasonably bright stars as reference sources, since the telescope wavefront sensor, as a rule, needs for a large amount of energy of star radiation to provide its proper functioning. The requirement to the reference source energy as well as the necessity of simultaneous staying in one isoplanar area with the image of a star being studied (or any other space object) and of a reasonably bright reference star, provided allowing for the fact that the atmospheric isoplanatism angle is very small (in the visible range along the direction to zenith this angle is 10″–15″) essentially decreases the percentage of sky coverage with this telescope.

The investigators of adaptive optics solved this problem when using the focused laser radiation guided from the Earth and backscattered by the atmospheric inhomogeneities,[1–3] namely, elastic aerosol scattering at 8–20 km altitude or a re-emission at 80–100 m altitude from the atomic sodium clouds.

The problem of forming of a laser reference star is in fact a combination of many scientific and technical problems to be solved, such as design of a specialized laser system, choice of an optimal altitude for the laser reference star, measurement of the phase of laser radiation reflected by the atmosphere and, finally, the selection of the control algorithm.

In this connection, of particular interest is the publication by Dr. Robert Fugate, a well-known in the U.S. specialist in laser systems, in which the author states that by the time of his publication[4] (February 1996) no laser reference stars, using the scattering from the atomic sodium clouds were operating successfully, were unknown to him. Here it should be noted that it is just sodium layer reference stars which can provide obtaining the best characteristics of the adaptive telescopes.

In addition to the above problems, the use of laser reference stars meets an obstacle, namely, the problem of impossibility of full correction for the random wavefront tilts from a natural star based on measurements of tilts of a wavefront from a laser reference star.

No doubts that the use of the laser reference stars, due to the light backscatter, is connected with the problem on selecting an optimal algorithm for making use of optical measurement data to correct for random jitter of a star image. It is just this problem we deal with in this paper.

Let us consider the following scheme of the optical experiment: formation of the natural star image in the focal plane of a ground-based telescope takes place ($F$ is the focal length of the optical system, $\Sigma$ is the size of the optical system aperture). As was already noted, the star image jitter formed in the focal plane occurs due to the influence of atmospheric turbulence over the telescope. We define this image jitter as a random shift of the position of the center of gravity (provided that these fluctuations are small) of the star image intensity using the vector

$$\rho_{\mathrm{F}}^{\mathrm{pl}} = -\frac{F}{k\,\mathrm{S}} \iint\limits_{\mathrm{S}} \mathrm{d}^2\rho \nabla_{\rho} S^{\mathrm{pl}}(0, \rho)\,, \qquad (1)$$

where $k$ is the radiation wave number; $S^{\mathrm{pl}}(0, \rho)$ are the phase fluctuations in the plane wave from the star formed.

In its turn, the measured random vector of the laser reference star image jitter, formed on the basis of the focused laser radiation, using a ground-based laser system, is given by the expression

$$\rho_m = \rho_c + \rho_{\mathrm{F}}^{\mathrm{sph}}\,, \qquad (2)$$

where

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$$\rho_c = \frac{1}{P_0}\int\limits_0^x d\xi(x-\xi)\iint d^2R\, I(\xi,\mathbf{R})\,\nabla_R\, n_1(\xi,\mathbf{R}) \quad (3)$$

is the position of the center of gravity, focused at a distance $x$ from the source of the Gaussian laser beam; $I(\xi, \mathbf{R})$ is the current value of the optical field intensity; $\nabla_R\, n_1(\xi, \mathbf{R})$ is the gradient of fluctuations of the atmospheric refractive index. We consider that the laser radiation is focused onto a sufficiently small spot, not resolved by a telescope through the atmosphere. The second term in (2) is of the form

$$\rho_F^{sph} = -\frac{F}{k\,S}\iint\limits_S d^2\rho\nabla_\rho S^{sph}(\rho, 0) \quad (4)$$

and represents the point source image jitter in the telescope focal plane.

We construct the correction algorithm of a star image jitter[5,6] $\rho_F^{pl}$ in the form:

$$\rho_F^{pl} - \rho_k\,, \quad (5)$$

where $\rho_k = A\rho_m$, and the coefficient $A$ is chosen to provide the minimal variance of the residual distortions

$$<(\rho_F^{pl} - A\,\rho_m)^2>_{\min} = <e^2>\,. \quad (6)$$

Having found the minimum for variance in the form of Eq. (6), we obtain

$$<e^2>_{\min} = <(\rho_F^{pl})^2> - \frac{<\mathbf{r}_F^{pl}\,\mathbf{r}_m>^2}{<(\mathbf{r}_m)^2>}\,, \quad (7)$$

where the correcting coefficient $A$ is expressed only in terms of the determinant functions as

$$A = <\rho_F^{pl}\,\rho_m>/<(\rho_m)^2>\,. \quad (8)$$

It should be noted that the traditional correction algorithm in the form of Eq. (5), where $A \equiv 1$, does not provide minimum (6) to the variance and therefore cannot be considered as any serious alternative.

In a real experiment we have only the measurement data $\rho_m$, since the vector $\rho_F^{pl}$, characterizing the real star jitter, whose image should be corrected, cannot be measured, since the real star emits only little light for the measurements with the wavefront sensor to be feasible.

At the same time, the coefficient $A$ can be calculated using a model description of the altitude behavior of the turbulence intensity $C_n^2(\xi)$. Taking into account Eqs. (2) and (8), the variance and correlation, as components of Eq. (7) can be written in the form

$$<(\rho_m)^2> = <(\rho_c)^2> + <(\rho_F^{sph})^2> + 2<\rho_F^{sph}\,\rho_c>, \quad (9)$$

$$<\rho_F^{pl}\,\rho_m> = <\rho_F^{pl}\,\rho_c> + <\rho_F^{pl}\,\rho_F^{sph}>\,. \quad (10)$$

Now the question arises on how can the algorithm (5) be useful for correction? First of all, based on the knowledge of a model of the altitude turbulence profile one can:

1) estimate the limiting level of correction of the general wavefront tilt, $\rho_F^{pl}$, by the following expression:

$$<\beta^2>_{\min} = <(\varphi_F^{pl})^2>\left\{1 - \frac{<\mathbf{j}_F^{pl}\,\mathbf{j}_m>^2}{<(\mathbf{j}_F^{pl})^2><(\mathbf{j}_m)>^2}\right\},$$

where the second term is estimated using the models of turbulent atmosphere.

2) calculate the scaling factor $A$ of measured values of $\rho_m$ in the control algorithm, expressed in terms of the average values

$$A = <\varphi_F^{pl}\,\varphi_m>/<\varphi_m^2>\,.$$

There are several schemes of laser reference star formation. From the viewpoint of calculation of variance and correlation from Eqs. (9) and (10) only two schemes can be mentioned as limiting ones, namely the monostatic and bistatic schemes. In the monostatic scheme the star image formation in the telescope and the formation of the laser reference star image take place through one and the same atmospheric inhomogeneities. In the bistatic scheme the reference star is formed in the region isoplanar with the image of a natural star, but the propagation of a focused laser beam itself, forming the reference star, occurs through turbulent inhomogeneities, uncorrelated with those on the way from the natural star.

## MONOSTATIC SCHEME

Thus, for the monostatic scheme the correction coefficient $A = A_M$ in Eq. (8) and its components are calculated by formulas (9) and (10), respectively. Using normalization and changing the characteristics for angular ones we obtain

$$<\varphi_m^2> = \frac{<r_m^2>}{x^2} = (2^{7/6}\,\pi^2\,0.033\,\Gamma(1/6))\times$$
$$\times\left[R_0^{-1/3} + a_0^{-1/3} - 2^{7/6}\,(R_0^2 + a_0^2)^{-1/6}\right]\times$$
$$\times\int\limits_0^x d\xi(1-\xi/x)^{5/3}\,C_n^2(\xi)\,, \quad (11)$$

provided that the focused $(x = f)$ laser beam is sufficiently wide $((ka_0^2)/x \gg 1)$ and the turbulent laser beam broadening does not exceed focusing (i.e., $\Omega^{-2}(1/2\,D_S(2a_0))^{6/5} \ll 1$);

$$<\varphi_F^{pl}\,\varphi_c> = (-\,2^{4/3}\,\pi^2\,0.033\,\Gamma(1/6))\times$$
$$\times\int\limits_0^x d\xi\ C_n^2(\xi)(1-\xi/x)\left[(R_0^2 + a_0^2(1-\xi/x)^2\right]^{-1/6}. \quad (12)$$

When making these calculations we consider the radiation from a natural star as an infinite plane wave, propagating from zenith, and a laser beam is formed coaxially with the main telescope, forming the image, then

$$<\varphi_F^{pl}\,\varphi_F^{sph}> = (2^{4/3}\,\pi^2\,0.033\,\Gamma(1/6))\,R_0^{-1/3}\times$$
$$\times\int_0^x d\xi\;C_n^2(\xi)(1-\xi/x)[1+(1-\xi/x)^2]^{-1/6}\,. \qquad (13)$$

The latter expression represents the correlation between the plane wave image jitter and the point source image jitter (measured in the telescope focal plane), with the source being at a distance $x$ from the telescope.

The variance of the star image jitter is calculated by the following formula:

$$<(\varphi_F^{pl})^2> = (2^{7/6}\,\pi^2\,0.033\,\Gamma(1/6))R_0^{-1/3}\int_0^\infty d\xi\;C_n^2(\xi)\,. \qquad (14)$$

Since the star is far out of the atmosphere, the upper integration limit in Eq. (14) tends to $\infty$. Using these designations the minimal variance of residual fluctuations of angular shifts of the star image for a monostatic scheme is given in the form

$$\frac{<e^2>_{\min}}{F^2} = <\beta^2>_{\min} =$$
$$= <(\varphi_F^{pl})^2>\left\{1-\frac{2^{1/3}\,f_M(x,\,C_n^2)}{[1+b^{-1/3}-2^{7/6}(1+b^2)^{-1/6}]}\right\}, \qquad (15)$$

where $b = a_0/R_0$,

$$f_M(x,\,C_n^2) =$$
$$=\frac{\left\{\int_0^x dx\,C_n^2(\mathrm{x})\left(\left[1+b^2\left(1-\frac{\mathrm{x}}{x}\right)^2\right]^{-1/6}-\left[1+\left(1-\frac{\mathrm{x}}{x}\right)\right]^{-1/6}\right)\right\}^2}{\int_0^x dx\,C_n^2(\mathrm{x})\left(1-\frac{\mathrm{x}}{x}\right)^{5/3}\int_0^\infty dx\,C_n^2(\mathrm{x})}. \qquad (16)$$

From Eqs. (11), (15), and (16) it is clear that $b = 1$ ($a_0 = R_0$) the signal $\varphi_m$ becomes noninformative because $<\varphi_m^2(R_0 = a_0)> \equiv 0$. At the same time, the function $f_M(x,\,C_n^2)$ vanishes. Therefore, for the monostatic scheme of the laser reference star formation, from the standpoint of information content of $\varphi_m$ as well as from the power standpoint, the domain of admissible values of $b = a_0/R_0$ is the interval (0, 1), i.e., $b < 1$. For very small values of the parameter $b$ the estimate of minimal value of the variance of the residual star image jitter is expressed as

$$<\beta^2>_{\min}=<(\varphi_F^{pl})^2>\left\{1-\frac{2^{1/3}\,\hat{f}_M(x,\,C_n^2)}{[1+b^{-1/3}-2^{7/6}(1+b^2)^{-1/6}]}\right\}, \qquad (17)$$

where the function

$$\hat{f}_M(x,\,C_n^2) =$$
$$= \frac{\left\{\int_0^x dx\;C_n^2(\mathrm{x})(1-\mathrm{x}/x)(1-[1+(1-\mathrm{x}/x)^2]^{-1/6})\right\}^2}{\int_0^x dx\;C_n^2(\mathrm{x})(1-\mathrm{x}/x)^{5/3}\int_0^\infty dx\;C_n^2(\mathrm{x})} \qquad (18)$$

is the limit for the function $f_M(x,\,C_n^2)$ from Eq. (16) at the parameter $b \to 0$.

Table I gives the calculated values of all the parameter of a monostatic scheme interesting for us. The calculations have been done for different values of the parameter $b$ ($b$ = 0; 0.75; 0.80; 0.85; 0.90; 0.95) with the use of the model of $C_n^2(\xi)$ from Ref. 7 for the mean conditions of vision through the turbulent atmosphere and the altitudes of location of a reference source $x$ from 5 to 100 km. Table I shows that the value of the function $\hat{f}_M(x,\,C_n^2)$ varies from 6.48 to 11.2. In the same table the values of the quantity

$$A_m = <\rho_F^{pl}\,\rho_m>/<(\rho_m)^2>\,,$$

are given, calculated by the formula (8) for the monostatic scheme of the laser reference star formation. Thus, for the parameter $b = 0.95$ the values of $A_m$ vary from $-15$ to $-16.1$. Here the value of $C_M = <\beta^2>/<(\varphi_F^{pl})^2>$ is given, characterizing the ratio of the value of variance of residual fluctuations to the value of variance of the natural star jitter signal. The calculational data show that the values of $C_M$ vary from 0.9197 to 0.87. These results clearly demonstrate that because of small value of $f_M(x,\,C_n^2)$ and owing to the fact that $C_M$ only slightly differs from 1, no efficient correction of random tilts with the use of the monostatic scheme of the laser reference star formation can be expected. It should be noted that this result has been obtained for the case of optimal correction, therefore the use of "direct' correction algorithm and the optical measurement data (at $A \equiv 1$) the correction (6) is much less effective.

Owing to the fact that $f_M(x,\,C_n^2)$ is small, the optimal value of the ratio $b$, minimizing the variance $<\beta^2>_{\min}$, given by Eq. (17), turns out to be comparable with the dimension of the telescope aperture $R_0$. It is known that in this case (when $a_0 \to R_0$) the measured signal $\varphi_m$ decreases, and its variance $<\varphi_m^2>$ vanishes. Therefore, a compromise should exist in the choice of such a ratio $b = a_0/R_0$, which, on the one hand, minimizes the variance (17), and, on the other hand, provides the measurement a measurable signal $\varphi_m$, i.e., ensures a reasonable level of the variance (11). For example, one can select the value $b = a_0/R_0$, such that

$$1 + b^{-1/3} - 2^{7/6}(1 + b^2)^{-1/6} \le 0.01\,,$$

i.e., the signal of the reference star jitter proved to be ten times smaller than real star jitter (although this

*TABLE I.*

| $x$, km | $f_M \cdot 10^3$ | | | | | | $A_M$ | | | | | $C_M$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 |
| 5 | 6.48 | 0.8096 | 0.522 | 0.295 | 0.1314 | 0.03286 | −2.4 | −3.22 | −4.59 | −7.33 | −15.6 | 0.9387 | 0.934 | 0.9292 | 0.9245 | 0.9197 |
| 10 | 7.82 | 0.9644 | 0.6211 | 0.3506 | 0.156 | 0.03896 | −2.42 | −3.24 | −4.62 | −7.38 | −15.7 | 0.927 | 0.9214 | 0.9159 | 0.9103 | 0.9048 |
| 15 | 8.58 | 1.05 | 0.6784 | 0.3828 | 0.1703 | 0.04251 | −2.43 | −3.26 | −4.64 | −7.41 | −15.7 | 0.9202 | 0.9142 | 0.9082 | 0.9022 | 0.8961 |
| 20 | 9.14 | 1.12 | 0.7199 | 0.406 | 0.1805 | 0.04505 | −2.44 | −3.28 | −4.67 | −7.45 | −15.8 | 0.9153 | 0.9089 | 0.9026 | 0.8963 | 0.8899 |
| 85 | 11.1 | 1.32 | 0.8494 | 0.4779 | 0.212 | 0.05277 | −2.5 | −3.35 | −4.77 | −7.61 | −16.1 | 0.8998 | 0.8926 | 0.8853 | 0.8782 | 0.8711 |
| 90 | 11.1 | 1.33 | 0.8522 | 0.4795 | 0.2126 | 0.05293 | −2.5 | −3.35 | −4.77 | −7.61 | −16.1 | 0.8995 | 0.8922 | 0.885 | 0.8778 | 0.8707 |
| 95 | 11.2 | 1.33 | 0.8548 | 0.4809 | 0.2133 | 0.05308 | −2.51 | −3.35 | −4.77 | −7.61 | −16.1 | 0.8992 | 0.8919 | 0.8846 | 0.8774 | 0.8703 |
| 100 | 11.2 | 1.34 | 0.8572 | 0.4822 | 0.2138 | 0.05322 | −2.51 | −3.35 | −4.77 | −7.61 | −16.1 | 0.8989 | 0.8916 | 0.8843 | 0.8771 | 0.87 |

*TABLE II.*

| $x$, km | $f_b$ | $A_b$ | | | | $C_b$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1.0 | 3.0 | 0.1 | 0.5 | 1.0 | 3.0 |
| 5 | 0.6284 | 0.3524 | 0.4918 | 0.5557 | 0.6564 | 0.749 | 0.6497 | 0.6041 | 0.5324 |
| 10 | 0.7127 | 0.3465 | 0.4837 | 0.5465 | 0.6455 | 0.7153 | 0.6027 | 0.551 | 0.4697 |
| 15 | 0.7523 | 0.3424 | 0.4779 | 0.54 | 0.6378 | 0.6995 | 0.5806 | 0.5261 | 0.4403 |
| 20 | 0.77 | 0.3383 | 0.4722 | 0.5335 | 0.6301 | 0.6925 | 0.5707 | 0.5149 | 0.4271 |
| 85 | 0.7919 | 0.3231 | 0.451 | 0.5096 | 0.6018 | 0.6837 | 0.5585 | 0.5011 | 0.4108 |
| 90 | 0.7922 | 0.3228 | 0.4505 | 0.5091 | 0.6012 | 0.6836 | 0.5583 | 0.5009 | 0.4106 |
| 95 | 0.7926 | 0.3225 | 0.4501 | 0.5086 | 0.6007 | 0.6834 | 0.5581 | 0.5007 | 0.4103 |
| 100 | 0.7929 | 0.3222 | 0.4497 | 0.5082 | 0.6002 | 0.6833 | 0.558 | 0.5005 | 0.4101 |

opinion is too optimistic), then the optimal ratio $b$ should be equal to $b_{opt} \geq 0.86$. However, it is known that it is very difficult to perform accurate measurements of small signals under conditions of noise and background fluctuations. This shows once more that the correction of wavefront tilts in the monostatic scheme is ineffective since any effective correction should be expected for the parameter value $b \to 1$, i.e., for the case when the measured value itself becomes small and, hence, it is measured with larger error.

## BISTATIC SCHEME

According to the bistatic scheme the laser star formation is performed through the turbulent inhomogeneities uncorrelated with those inhomogeneities, through which the natural star image is formed with the telescope. This can be done using lateral irradiation (at a sufficiently large spacing between the optical axes of the laser beam propagation and the telescope). Using the same procedure of search for minimum of variance of residual fluctuations of image jitter, we obtain for the residual level of fluctuations from Eq. (7), respectively,

$$<\beta^2>_{\min} = <(\varphi_F^{pl})^2> - \frac{<\mathbf{j}_F^{pl}\,\mathbf{j}_F^{sph}>^2}{<(\mathbf{j}_b)^2>}, \qquad (19)$$

where

$$<(\varphi_b)^2> = <\varphi_c^2> + <(\varphi_F^{sph})^2> . \qquad (20)$$

Having made the same calculations as for the monostatic scheme, we obtain the following expressions for the correcting coefficient $A = A_b$ and for the residual level of corrected variance (7), where

$$A_b = \frac{<\mathbf{j}_F^{pl}\,\mathbf{j}_F^{sph}>}{<\mathbf{j}_c^2> + <(\mathbf{j}_F^{sph})^2>} =$$

$$= \frac{2^{1/6}\int\limits_0^x dx\, C_n^2(x)(1 - x/x)[1 + (1 - x/x)^2]^{-1/6}}{(1 + b^{-1/3})\int\limits_0^x dx\, C_n^2(x)(1 - x/x)^{5/3}}, \qquad (21)$$

$$\frac{<e^2>_{\min}}{F^2} = <\beta^2>_{\min} = <(\varphi_F^{pl})^2>\left\{1 - \frac{2^{1/3} f_b(x, C_n^2)}{(1 + b^{-1/3})}\right\}, \qquad (22)$$

$$f_b(x, C_n^2) =$$

$$= \frac{\left(\int\limits_0^x dx\, C_n^2(x)(1 - x/x)[1 + (1 - x/x)^2]^{-1/6}\right)^2}{\int\limits_0^x dx\, C_n^2(x)(1 - x/x)^{5/3}\int\limits_0^x dx\, C_n^2(x)}. \qquad (23)$$

As the analysis of the latter expressions has shown, the effective correction with the bistatic scheme of the reference star formation ensures the minimum variance of residual image jitter (22) with the correction in the form

$$\varphi_F^{pl} - A_b\,\varphi_b ,$$

where $\varphi_b$ is the signal of the reference bistatic star image jitter, $A_b$ is given by formula (21). It is clear that in contrast to the monostatic scheme the correction within the bistatic scheme is possible at any ratio $b = a_0/R_0$, it is evident that the correction is the better the larger is the value $b$ (see (22). If $b = 1$, from Eq. (22) we find that

$$<\beta^2>_{\min} = <(\varphi_F^{pl})^2> \{1 - 2^{-2/3} f_b(x, C_n^2)\} .$$

The functions $f_b(x, C_n^2)$, $A_b$, $C_b = 1 - \dfrac{2^{1/3} f_b(x, C_n^2)}{1 + b^{-1/3}}$ are represented in Table II for the model of the turbulent atmosphere[7] at different altitudes of the reference source formation $x \in [1, 100]$ km, and for the values of the parameter $b = a_0/R_0$, 0.1; 0.5; 1.0; 3.0, respectively. The values of the function $f_b(x, C_n^2)$ vary from 0.628 to 0.7930. Therefore a more effective correction should be expected from the bistatic scheme as compared with the monostatic one. Besides, in the bistatic scheme we do not face the situation when the measured signal or its variance $<(\rho_b)^2>$ vanishes. Of course, the bistatic scheme has a limiting correction level, and the variance of residual distortions (for example, for $b = 1$) as a result of such correction proved to be equal to

$$<\beta^2>_{\min} \approx <(\varphi_F^{pl})^2> \{1 - 2^{-2/3} f_b(x, C_n^2)\} .$$

The two limiting schemes of laser reference star formation can be compared only by means of concrete estimates. It should be noted that it is necessary to make the estimation not for a separate telescope (with adaptive optics) but for the whole observatory, for example, the Mauna Kiya observatory on Hawaian Islands, where the three largest telescopes (Keck I, Keck II, and CHFT) are located, operating with the adaptive correction of turbulent distortions. The first two telescopes are with the 10 m aperture, and the CHFT telescope (Canada, Hawaian Islands, France) has the 3.6 m aperture.

Thus, when investigating, with the use of the monostatic scheme, equals the variance of residual distortions for every telescope

$$<\beta^2>_{\min} = <(\varphi_F^{pl})^2> \left\{1 - \frac{2^{1/3} f_M(x, C_n^2)}{[1 + b^{-1/3} - 2^{7/6}(1 + b^2)^{-1/6}]}\right\} .$$

If the Keck I telescope produces the bistatic star for the Keck II telescope (the distance between the telescopes is 85 m), then we have

$$<\beta^2>_{\min} = <(\varphi_F^{pl})^2> \{1 - 2^{-2/3} f_b(x, C_n^2)\},$$

if the Keck I telescope produces the star for the CHFT telescope, then

$$\langle\beta^2\rangle_{\min} = \langle(\varphi_F^{pl})^2\rangle \left\{1 - \frac{2^{1/3} f_b(x, C_n^2)}{1 + (10/3.6)^{-1/3}}\right\} .$$

If CHFT produces the star for the pair of Keck I and Keck II telescopes, then

$$\langle\beta^2\rangle_{\min} = \langle(\varphi_F^{pl})^2\rangle \left\{1 - \frac{2^{1/3} f_b(x, C_n^2)}{1 + (3.6/10)^{-1/3}}\right\} .$$

Before making the final conclusions we consider the so-called intermediate scheme of the laser reference star formation.

## INTERMEDIATE SCHEME OF THE LASER REFERENCE STAR FORMATION

Let us consider the bistatic scheme of laser reference star formation in detail as is stated below. Let we have got two telescopes whose axes are spaced by the distance (vector) $\rho_0$. For simplicity we consider that one of these telescopes is focused to zenith and forms a natural star image, and the second telescope, forming the laser reference star, is inclined at an elevation angle $\theta$ relative to the first telescope so that the elevation angle equals numerically $\theta = \pi/2 - \rho_0/x$, where $x$ is the altitude at which the laser reference star is formed.

Let us first consider cross-correlations of random shifts of the center of gravity $\rho_c(\rho_0)$ of the laser beam formed with the second telescope whose directional pattern axis is shifted by the vector $\rho_0$ and is inclined at an angle $\theta = \pi/2 - \rho_0/x$ to the horizon, as well as the shifts of the center of gravity of the plane wave image $\rho_F^{pl}$ and the spherical wave image $\rho_F^{sph}$ formed by the first telescope , i.e., the correlations

$$\langle\rho_F^{pl}\,\rho_c(\rho_0)\rangle\ , \quad \langle\rho_F^{sph}\,\rho_c(\rho_0)\rangle\ .$$

It is important to understand that the first correlation, $\langle\rho_F^{pl}\,\rho_c\,(\rho_0)\rangle$ for the plane wave, decreases faster with the increase of the value of spacing between the telescope optical axes $\rho_0$ than the second one, $\langle\rho_F^{sph}\,\rho_c\,(\rho_0)\rangle$, for the spherical wave. We try to prove this on the basis of analytical and numerical calculations. Let us write the expression for the vector of energy center of gravity of a laser beam, formed with the second telescope from the ground surface, in the form

$$\rho_c(\rho_0) = \frac{1}{P_0}\int\limits_0^x d\xi(x-\xi) \int\int d^2R \langle I(\xi, \mathbf{R})\rangle \nabla_R\, n_1(\xi, \mathbf{R}), \tag{24}$$

where

$$\nabla_R\, n_1(\xi, \mathbf{R}) = i \int\int d^2n(\xi, \boldsymbol{\kappa})\boldsymbol{\kappa} \exp(i\, \boldsymbol{\kappa}\, \mathbf{R}) , \tag{25}$$

and the mean intensity distribution of a laser beam, shifted to the vector $\rho_0$ and tilted at an angle $\theta$ to the Earth ($\theta = \pi/2 - \rho_0/x$), is given by the expression

$$\langle I(\xi, \mathbf{R})\rangle = \frac{a_0^2}{a_{\rm eff}^2(x)} \exp\{-(\mathbf{R} - \rho_0(1-\xi/x))^2/a_{\rm eff}^2(\xi)\} . \tag{26}$$

Having made the calculations, we obtain

$$\rho_c(\rho_0) = i\int\limits_0^x d\xi(x-\xi) \int\int d^2n(\xi, \boldsymbol{\kappa})\boldsymbol{\kappa} \times$$

$$\times \exp\{-\kappa^2 a_{\rm eff}^2(\xi)/4\} \exp[i\kappa\rho_0(1 - \xi/x)]. \tag{27}$$

As a result we have the following expressions for the variance $\rho_c(\rho_0)$ and cross-correlations $\langle\rho_F^{pl}\,\rho_c\,(\rho_0)\rangle$, $\langle\rho_F^{sph}\,\rho_c\,(\rho_0)\rangle$:

$$\langle(\varphi_c(\rho_0))^2\rangle = \frac{\langle(\mathbf{r}_c(\mathbf{r}_0))^2\rangle}{x^2} =$$

$$= (2\pi^2\, 0.033\, \Gamma(1/6))\, 2^{1/6}\, (R_0^2 + a_0^2)^{-1/6} \times$$

$$\times {}_1F_1\left(1/6;\, 1;\, -\frac{r_0^2}{R_0^2 + a_0^2}\right) \int\limits_0^x d\xi\, C_n^2(\xi)(1-\xi/x)^{5/3} , \tag{28}$$

$$\langle\varphi_c(\rho_0)\,\varphi_F^{sph}\rangle = \frac{\langle\mathbf{r}_c(\mathbf{r}_0)\,\mathbf{r}_F^{sph}\rangle}{x\, F} =$$

$$= (-\,2\pi^2\, 0.033\, \Gamma(1/6))\, 2^{1/3}\, (R_0^2 + a_0^2)^{-1/6} \times$$

$$\times {}_1F_1\left(1/6;\, 1;\, -\frac{r_0^2}{R_0^2 + a_0^2}\right) \int\limits_0^x d\xi\, C_n^2(\xi)(1-\xi/x)^{5/3} , \tag{29}$$

$$\langle\varphi_c(\rho_0)\,\varphi_F^{pl}\rangle = \frac{\langle\mathbf{r}_c(\mathbf{r}_0)\,\mathbf{r}_F^{pl}\rangle}{x\, F} =$$

$$= [-\,2\pi^2\, 0.033\, \Gamma(1/6)]\, 2^{1/3} \int\limits_0^x d\xi\, C_n^2(\xi)(1-\xi/x) \times$$

$$\times \{R_0^2 + a_0^2(1 - \xi/x)^2\}^{-1/6} \times$$

$$\times {}_1F_1\left(1/6;\, 1;\, -\frac{r_0^2(1 - x/x)^2}{(R_0^2 + a_0^2(1 - x/x)^2)}\right) . \tag{30}$$

Having analyzed the latter expressions we can state that the correlation between the slant beam and a spherical wave decreases slower than the correlation between this beam and a plane wave.

All the values calculated for this intermediate case are marked by the subscript "i". Besides, together with the parameter $b = a_0/R_0$ the parameter $d = \rho_0/R_0$ is introduced, characterizing the spacing between the laser beam axis, forming the star and the axis of the main telescope. In the general case (for arbitrary values of the parameters $b = a_0/R_0$ and $d = \rho_0/R_0$) we have, using Eqs. (28), (29), and (30) for a correcting factor $A = A_i$ and the values $C_i$, $\langle\beta^2\rangle_i$, characterizing the variance of residual distortions, the following expressions:

$$A_i = 2^{1/6} \left[\int\limits_0^x dx\, C_n^2(x)(1 - x/x) \left\{(1 + (1-\xi/x)^2)^{-1/6} - \right.\right.$$

$$- (1 + b^2(1 - \xi/x)^2)^{-1/6} \times$$

$$\times\, {}_1F_1\left(1/6;\, 1;\, -\frac{r_0^2(1-x/x)^2}{(1+b^2(1-x/x)^2)}\right)\Bigg\}\Bigg]\times$$

$$\times\left(\Big[1+b^{-1/3}-2^{-7/6}(1+b^2)^{-1/6}\times\right.$$

$$\left.\times\, {}_1F_1\left(1/6;\, 1;\, -\frac{d^2}{(1+b)^2}\right)\Big]\int\limits_0^x d\xi\, C_n^2(\xi)(1-\xi/x)^{5/3}\right)^{-1},$$

$$\langle\beta^2\rangle_i = \langle(\varphi_F^{pl})^2\rangle\, C_i\,,$$

$$C_i = 1-2^{1/3}\left[\int\limits_0^x dx\, C_n^2(x)(1-x/x)\times\right.$$

$$\times\Big\{(1+(1-\xi/x)^2)^{-1/6}-(1+b^2(1-\xi/x)^2)^{-1/6}\times$$

$$\times {}_1F_1\left(1/6;\, 1;\, -\frac{r_0^2(1-x/x)^2}{(1+b^2(1-x/x)^2)}\right)\Bigg\}\Bigg]\times$$

$$\times\left(\Big[1+b^{-1/3}-2^{-7/6}(1+b^2)^{-1/6}\times\right.$$

$$\times\, {}_1F_1\left(1/6;\, 1;\, -\frac{d^2}{(1+b)^2}\right)\Big]\times$$

$$\left.\times\int\limits_0^x d\xi\, C_n^2(\xi)(1-\xi/x)^{5/3}\int\limits_0^\infty d\xi\, C_n^2(\xi)\right)^{-1}.$$

It can easily be seen that at $d\to 0$ we obtain the expressions for the monostatic scheme, and at $d\to\infty$ we obtain the expressions for a bistatic scheme. For an arbitrary parameter $d=\rho_0/R_0$ the values of $A_i$, $C_i$, $\langle\beta^2\rangle_i$ exist even at $b=1$ $(a_0=R_0)$ in contrast to the monostatic scheme. At $b=1$ we obtain

$$\langle\beta^2\rangle_i = \langle(\varphi_F^{pl})^2\rangle\left\{1-2^{-2/3}\frac{\left(\int\limits_0^x dx\, C_n^2\left(1-\frac{x}{x}\right)\left(\left[1+\left(1-\frac{x}{x}\right)\right]^2\right)^{-1/6}\left[1-{}_1F_1\left(1/6;\,1;\,-\frac{d^2\left(1-\frac{x}{x}\right)^2}{\left(1+b^2\left(1-\frac{x}{x}\right)^2\right)}\right)\right]\right)^2}{\left[1-{}_1F_1\left(1/6;\,1;\,1-\frac{d^2}{2}\right)\right]\int\limits_0^x dx\, C_n^2\left(1-\frac{x}{x}\right)^{5/3}\int\limits_0^x dx\, C_n^2(x)}\right\}. \quad (31)$$

Let us consider the asymptotic behavior of the expression for the correcting factor $A=A_i$ and the values of $C_i$, $\langle\beta^2\rangle_i$ at the parameter $d\to\infty$. In this case we use the analytical continuation for a hypergeometric function

$${}_1F_1(1/6;\,1;\,z)=\frac{(-z)^{-1/6}}{G(5/6)}\left(1+\frac{1}{36(-z)}+\ldots\right).$$

As a result, the denominator in $A_i$ equals

$$\left[1+b^{-1/3}-\frac{2^{7/6}\, d^{-1/3}}{G(5/6)}\right]\int\limits_0^x d\xi\, C_n^2(\xi)\,(1-\xi/x)^{5/3}$$

and the numerator equals

$$1-{}_1F_1\left(1/6;\,1;\,-d^2\frac{(1-x/x)^2}{1+b^2(1-x/x)^2}\right)=\frac{(1-x/x)^{-1/3}}{G(5/6)}(1+(1-\xi/x))^{1/6}.$$

We obtain that

$$\langle\beta^2\rangle = \langle(\varphi_F^{pl})^2\rangle\; C_i = \langle(\varphi_F^{pl})^2\rangle\left\{1-\frac{2^{1/3}\, F^2}{\left[1+b^{-1/3}-\frac{2^{7/6}\, F^2}{G(5/6)}\right]\int\limits_0^x dx\, C_n^2(x)(1-\frac{x}{x})^{5/3}\int\limits_0^x dx\, C_n^2(x)}\right\} \quad (32)$$

and the function

$$F=\int\limits_0^x d\xi\, C_n^2(\xi)\left\{\frac{(1-x/x)^2}{[1+(1-x/x)^2]^{-1/6}}-\frac{(1-x/x)^{2/3}}{G(5/6)\, d^{1/3}}\right\}. \quad (33)$$

It turns out that the numerator of the function $C_i$ (see Eq. (32)) at $d\to\infty$ does not depend on the parameter $b=a_0/R_0$, whereas the denominator depends on $b=a_0/R_0$ as follows:

$$\left[1+b^{-1/3}-\frac{2^{7/6}\, d^{-1/3}}{G(5/6)}\right]\int\limits_0^x d\xi\, C_n^2(\xi)\,(1-\xi/x)^{5/3}.$$

Thus, one can decrease the variance (32) of residual fluctuations of the random star position by increasing $b=a_0/R_0$.

Tables III and IV give the results of calculations of the functions $A_i$ and $C_i$ by formula (31) at $b = 1$, the altitude of beacon from 5 to 100 km, the spacing between the first and second telescopes $d = 2^3, 3^3, \ldots, 10^3$. When comparing the data of these two tables with the column (at $b = 1$) from Table II one can see that the values of $A_i$ and $C_i$ approach the values of $A_b$ and $C_b$ when $d \to \infty$.

*TABLE III.*

| $x$, km | $d^{1/3}$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 0.5372 | 0.5465 | 0.5496 | 0.5511 | 0.552 | 0.5526 | 0.5531 | 0.5534 | 0.5537 |
| 10 | 0.5299 | 0.5382 | 0.541 | 0.5423 | 0.5432 | 0.5437 | 0.5441 | 0.5444 | 0.5447 |
| 15 | 0.5271 | 0.5335 | 0.5357 | 0.5368 | 0.5374 | 0.5378 | 0.5381 | 0.5384 | 0.5386 |
| 20 | 0.5237 | 0.5286 | 0.5302 | 0.5311 | 0.5316 | 0.5319 | 0.5321 | 0.5323 | 0.5324 |
| 85 | 0.5075 | 0.5085 | 0.5089 | 0.509 | 0.5092 | 0.5092 | 0.5093 | 0.5093 | 0.5093 |
| 90 | 0.5071 | 0.5081 | 0.5084 | 0.5086 | 0.5087 | 0.5087 | 0.5088 | 0.5088 | 0.5088 |
| 95 | 0.5067 | 0.5077 | 0.508 | 0.5081 | 0.5082 | 0.5083 | 0.5083 | 0.5084 | 0.5084 |
| 100 | 0.5064 | 0.5073 | 0.5076 | 0.5077 | 0.5078 | 0.5079 | 0.5079 | 0.508 | 0.508 |

*TABLE IV.*

| $x$, km | $d^{1/3}$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 0.814 | 0.7441 | 0.7091 | 0.6881 | 0.6741 | 0.6641 | 0.6566 | 0.6508 | 0.6461 |
| 10 | 0.7877 | 0.7089 | 0.6695 | 0.6458 | 0.63 | 0.6187 | 0.6102 | 0.6037 | 0.5984 |
| 15 | 0.7729 | 0.6907 | 0.6496 | 0.6249 | 0.6084 | 0.5966 | 0.5878 | 0.581 | 0.5755 |
| 20 | 0.765 | 0.6817 | 0.64 | 0.615 | 0.5983 | 0.5864 | 0.5775 | 0.5705 | 0.565 |
| 85 | 0.7512 | 0.6679 | 0.6262 | 0.6012 | 0.5845 | 0.5726 | 0.5637 | 0.5567 | 0.5512 |
| 90 | 0.751 | 0.6676 | 0.626 | 0.601 | 0.5843 | 0.5724 | 0.5634 | 0.5565 | 0.5509 |
| 95 | 0.7508 | 0.6674 | 0.6257 | 0.6007 | 0.5841 | 0.5722 | 0.5632 | 0.5563 | 0.5507 |
| 100 | 0.7506 | 0.76672 | 0.6255 | 0.6005 | 0.5839 | 0.572 | 0.563 | 0.5561 | 0.5505 |

As a practical outcome of the above considerations we can state that one can, based on the results obtained, quantitatively characterize the spacing between the optical axes of the main and star forming telescopes characteristic of the so-called bistatic scheme.

Thus, the following conclusions can be drawn:

1. Monostatic scheme does not remove the wavefront tilts.
2. Bistatic and intermediate schemes (at spacing between the axes of the two telescopes $d > 40$) are practically identical.
3. Bistatic scheme, where the two telescopes are used for correction of random tilts is more efficient at larger values of the parameter $b$.

## REFERENCES

1. C. Primmerman, D. Murphy, et al., Nature (London) **353**, 141–143 (1991).
2. R. Fugate, J. Ellerbroek, et al., J. Opt. Soc. Am. **A11**, 310–324 (1994).
3. D. Fried and J. Belsher, J. Opt. Soc. Am. **A11**, 277–286 (1994).
4. R. Fugate, Optics and Photonics News **4**, No. 6, 14–19 (1993).
5. V.P. Lukin, *Atmospheric Adaptive Optics* (Nauka, Novosibirsk, 1986), 275 pp.
6. V.P. Lukin, *Laser Beacons and Full Aperture Tilt Measurements*. Proc. “Adaptive Optics”, OSA, 1996, p. AMB35-1–AMB35-5.
7. M.E.Gracheva and A.S.Gurvich, Izv. Akad. Nauk SSSR, Fiz. Atmos. Okeana **16**, No. 10, 1107–1111 (1980).