ANALYTICAL FORMULAS FOR DETERMINING OPTICAL PARAMETERS OF A CLOUDY LAYER BASED ON MEASURED CHARACTERISTICS OF THE SOLAR RADIATION FIELD. I. THEORY

I.N. Mel'nikova
State University, St. Petersburg
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Additional terms of asymptotic expansions of the functions describing the scattered light in weakly media and the approximation relations for the reflective coefficient that essentially widen the applicability limits of asymptotics and simplify computations of these functions are obtained. Based on asymptotic formulas of the radiation transfer theory for the intensity of diffuse radiation at the boundaries of an optically thick scattering layer an expression for the parameter which describes actual absorption of light ($s^2 = (1 - \Lambda)/(3 - x_1)$) is derived. This relation, in combination with the formula for optical thickness of the layer, enables one to determine the volume scattering and absorption coefficients from measured intensity of radiation outgoing from the layer. The accuracy and applicability limits of the proposed formulas are investigated.

INTRODUCTION

Optical properties of cloud layers is a subject of interest in many problems of atmospheric physics, e.g., for construction of optimal models for climatic calculations and indication of possible pollutions of the atmosphere in ecological monitoring. Measurements of the scattered solar radiation field in the atmosphere and solution of the so–called "inverse problem" of the radiation transfer theory seem to be quite applicable to reconstruction of the optical parameters of cloud layers. Earlier such problems were solved using, in particular, the asymptotic formulas of the radiation transfer theory (see Refs. 1 and 2) by fitting the values of optical parameters of a cloud layer so that the best agreement between the radiation field characteristics calculated using asymptotic formulas (or other method) and those measured in the atmosphere was achieved.

Further development of the method proposed by the author in Ref. 3 and based on derivation of analytical relations between the scattering and absorption coefficients and measurable characteristics of solar radiation in the visible spectral range is the subject of this study. In some experiments they measure not the radiation fluxes but the intensity of solar radiation[2], therefore in this paper we shall propose relevant formulas relating the scattering and absorption coefficients of a cloud layer to the intensity of scattered radiation. To do this, we shall have a need for analytical expressions for the functions describing reflection of diffuse radiation from a semi–infinite medium. Such expressions are also being derived bellow.

APPROXIMATION FORMULAS

The radiation transfer equation is used for describing multiple scattering of light in a diffuse medium. In the case of an optically thick medium such as cloud layers the so–called depth or asymptotic regime is established and a solution of the radiation transfer equation is given by asymptotic formulas. The applicability limits of the asymptotic formulas of the radiation transfer theory has been studied elsewhere.[4,5] The authors infer that the asymptotics errors do not exeed 3% when $\tau_0 \geq 7$. Radiation reflected from a plane layer of a large optical depth τ_0 is described with the reflection coefficient by the formula[6]

$$\rho(\zeta, \eta, \tau_0) = \rho(\zeta, \eta) - \frac{MNu(\zeta)\, u(\eta)\, e^{-2k\tau_0}}{1 - N^2 e^{-2k\tau_0}}, \tag{1}$$

where ζ and η are the cosines of the angles of incidence and reflection, the function $u(\zeta)$ describes the angular distribution of radiation intensity emanating from the medium, $\rho(\zeta, \eta)$ is the azimuth–independent term of the reflective coefficient of a semi–infinite atmosphere, the values M and N are defined by integral relations which involve the function $u(\zeta)$, and the value k is called the diffuse length.

For a weak actual absorption of light in a medium (which is true, e.g., for clouds in the visible spectral region) the asymptotic values and functions can be represented by series expansions of small parameter powers characterizing the absorption. Different authors use different quantities to do this. In this paper we use the value $s = \sqrt{(1 - \Lambda)/(3 - x_1)}$, where Λ is the probability of a photon survival at single scattering, x_1 is the first coefficient in the series expansion of the scattering phase function over the Legendre polynomials. The scattering phase function is often described using the Henny–Greenstein formula $x(\gamma) = \dfrac{1 - g}{(1 + g^2 - 2g\cos\gamma)^{3/2}}$, where the parameter g determines the degree of the scattering phase function forward peakedness and coincides with the mean cosine of the scattering angle. In this case the relations $g = \cos\gamma$, $x_1 = 3g$, and $x_2 = 5g^2$ hold.

In the series expansions of M, N, and k values the coefficients at terms are known for up to the third term that is quite sufficient for solving some applied problems of atmospheric optics (see, e.g., Ref. 7). As to the series expansions of functions $\rho(\zeta)$ and $u(\zeta)$, only the coefficient at the first power of parameter s has been obtained, and the first and second coefficients have been determined for the plane albedo $a(\zeta) = 2\int_0^1 \rho(\eta, \zeta)\, \eta\, d\eta$. The series expansions of these functions have the form[6,7]

$$a(\zeta) = 1 - 4u_0(\zeta)\, s + a_2(\zeta)\, s^2 + a_3(\zeta)\, s^3\,; \tag{2}$$

$$u(\zeta) = u_0(\zeta)\,(1 - 3/2\delta s) + u_2(\zeta)\, s^2\,; \tag{3}$$

$$\rho(\zeta, \eta) = \rho_0(\zeta, \eta) - 4u_0(\zeta)\, u_0(\eta)\, s + \rho_2(\zeta, \eta)\, s^2 + \rho_3(\zeta, \eta)\, s^3, \quad (4)$$

where $u_0(\zeta)$ is the value of the function $u(\zeta)$ for a purely scattering $\Lambda = 1$ and $\delta = 4 \int_0^1 u_0(\zeta)\, \zeta^2\, d\zeta = 1.427$. The relation for the expansion coefficient $a_2(\zeta)$ has been derived in Ref. 8

$$a_2(\zeta) = \left[6\delta u_0(\zeta) + \frac{15(3 - x_1)}{5 - x_2}\, \upsilon_0(\zeta) \right] \text{ and}$$

$$\upsilon_0(\zeta) = \zeta^2 - 2 \int_0^1 \rho_0(\zeta, \eta)\, \eta^3\, d\eta\ , \ \varepsilon = 6 \int_0^1 u_0(\zeta)\, \zeta^3\, d\zeta = 1.667\ . \quad (5)$$

Let us also represent here an expression for the spherical albedo of a semi–infinite atmosphere $a^\infty = 2 \int_0^1 a(\zeta)\, \zeta\, d\zeta$ from Ref. 9 and for the value $Q = 2 \int_0^1 u(\zeta)\, \zeta\, d\zeta$

$$Q = 1 - 3/2\delta s + \left[9/4\delta^2 - \frac{(5\varepsilon - x_2)(3 - x_1)}{(5 - x_2)} \right] s^2 ; \quad (6)$$

$$a^\infty = 1 - 4s + 6\delta s^2 -$$

$$- 3 \left[2(2 - x_1) + 3\delta^2 + \frac{4(3 - x_1)}{3(5 - x_2)} (11 + x_2 - 10\varepsilon) \right] s^3$$

where x_2 is the second coefficient in the series expansion of the scattering phase function, $x_2 = 5\, g^2$. As is shown in Ref. 3 based on the analysis of numerical values of the functions $u_0(\zeta)$ and $\upsilon_0(\zeta)$ listed in the tables,[9] the ratio $u_0(\zeta)/\upsilon_0(\zeta)$ is well approximated by the formula $\upsilon_0(\zeta)/u_0(\zeta) = u_0(1)\, \zeta - 0.9$, with the deviations from this representation being less than fractions of a per cent. Then the expression for the expansion coefficient $a_2(\zeta)$ takes the form

$$a_2(\zeta) = 3u_0(\zeta) \left[\frac{5(3 - x_1)}{5 - x_2} (u_0(1) \cdot \zeta - 0.9) + 2\delta \right]. \quad (7)$$

The integration of this equation over ζ yields the value which differs only by 0.4% from $a_2^\infty = 6\,\delta$.

Let us now consider the function $u_2(\zeta)$ which is the second coefficient in series expansion (3)

$$u_2(\zeta) = u(\zeta) - u_0(\zeta)\,(1 - 3/2\delta) - 0(s^3)\ .$$

Using the values of the function $u(\zeta)$ obtained in Ref. 9, and examining numerically the difference, while neglecting the term $\sim s^3$ one can see that the function u_2 is well described by a quadratic function $u_2(\zeta) = Q_2\, \varepsilon(\zeta^2 + 0.1)$. Taking into account that $Q_2 = 2 \int_0^1 u_2(\zeta)\, \zeta\, d\zeta$ it is easy to check that the proposed representation of the function $u_2(\zeta)$ provides an approximate equality $Q_2 = Q_2 \cdot 1.0002$. A numerical verification of the approximation for the function $u_2(\zeta)$ by comparing it with the tabulated values[7] shows that the error does not exceed 1% up to the values $\Lambda = 0.990$ (Table I).

TABLE I. Errors in calculating the functions $u(\zeta)$.

Λ	0.999		0.995		0.990	
g	0.5	0.9	0.5	0.9	0.5	0.9
ζ/s	0.0258	0.05774	0.05774	0.12910	0.08165	0.18257
0.1	0.1	0.2	0.4	1.0	0.5	2.0
0.5	0.1	0.4	0.1	2.0	0.1	4.0
0.7	0.03	0.5	0.3	0.8	0.4	3.0
1.0	0.2	0.6	0.6	2.0	1.0	4.0

Moreover, taking into account the relation between the functions $u_2(\zeta)$ and $a_3(\zeta)$, pointed out in Ref. 7, where $a_3(\zeta)$ is the coefficient at s^3 in series expansion (2) of the plane albedo it is possible to find an analytical representation for the function $a_3(\zeta)$

$$a_3(\zeta) = 4 \left\{ u_0(\zeta) \left[\frac{(5\varepsilon - 11)(3 - x_1)}{(5 - x_2)} - \varepsilon Q_2(\zeta^2 + 0.1) \right] \right\}. \quad (8)$$

It is of practical interest for the problems on calculating the intensity of reflected radiation to improve the accuracy of asymptotic series expansion (4) of the function $\rho(\eta, \zeta)$ of the reflective coefficient of a semi–infinite layer. To do this, it is necessary to derive the additional terms of the series expansion. We assume, by analogy with the expression for $\rho_1(\eta, \zeta)$, that the function $\rho_2(\eta, \zeta)$ can be represented as $\rho_2(\eta, \zeta) = f(\eta) \cdot f(\zeta)$, where $f(\zeta)$ is an arbitrary function. Then taking into account that $2 \int_0^1 \rho_2(\eta, \zeta)\, \eta\, d\eta = a_2(\zeta)$ as well as the second coefficient in series expansion (6) for the spherical albedo $a_2 = 2 \int_0^1 a_2(\zeta)\, \zeta\, d\zeta = 6\delta$ it is possible to write the following expressions for the function $f(\zeta)$:

$$2 \int_0^1 \eta\, d\eta\ 2 \int_0^1 f(\eta)\, f(\zeta)\, \zeta\, d\zeta = \left[2 \int_0^1 f(\zeta)\, \zeta\, d\zeta \right]^2 = 6\,\delta; \quad (9)$$

$$2 f(\zeta) \int_0^1 f(\eta)\, \eta\, d\eta = a_2(\zeta)\ ,$$

from where $f(\zeta) = a_2(\zeta)/\sqrt{6\,\delta}$, and for the function $\rho(\eta, \zeta)$ we have

$$\rho_2(\eta, \zeta) = \frac{a_2(\eta)\, a_2(\zeta)}{6\delta}\ . \quad (10)$$

From analogous consideration of the third coefficient of series expansion (4) one can notice that the following relation

$$\rho_3(\eta, \zeta) = \frac{a_3(\eta)\, a_3(\zeta)}{a_3^\infty} \quad (11)$$

is also valid. The expressions for $a_3(\zeta)$ and a_3^∞ have been given above. The values of the coefficients $\rho_2(\eta, \zeta)$ and $\rho_3(\eta, \zeta)$ for $\eta = \zeta = 1$ are on the average $\rho_2 \sim 20$ and $\rho_3 \sim 80$, the values s^2 and s^3 are determined by the actual light absorption and by the scattering phase function, for instance, the values $s^2 \sim 0.03$, $s^3 \sim 0.006$, and $\rho_0(\eta, \zeta) \sim 1$ are characteristic of the earth's clouds, therefore the second and third terms of series expansion (4) amount 10 – 50% of the value $\rho(\eta, \zeta)$ and an account of these values significantly improves the accuracy of calculating the coefficient of reflection from a semi–infinite layer. The errors in calculating the function $\rho(\zeta, \zeta)$ based on asymptotic series expansion (4) with an account of formulas (10) and (11) are less than 1% for $\Lambda \geq 0.990$ and $g \leq 0.85$ (Table II).

TABLE II. Errors in calculating the function $\rho(\zeta, \zeta)$ based on asymptotic series expansion (4) with an account of formulas (10) and (11).

Λ	0.999		0.995		0.990	
g	0.5	0.9	0.5	0.9	0.5	0.9
ζ/s	0.0258	0.05774	0.05774	0.12910	0.08165	0.18257
0.1	0.2	0.6	0.2	1.0	0.3	2.6
0.5	0.2	0.3	0.4	1.0	1.0	3.0
1.0	0.1	0.3	0.5	1.0	0.7	3.0

Let us now consider the reflection coefficient of a conservative scattering ($\Lambda = 1$) $\rho_0(\eta, \zeta)$. As is well known[7] the azimuth–independent part of the reflective coefficient is represented by a linear combination of the Ambartsumyan functions, e.g., for an isotropic scattering ($x_1 = 0$ or $g = 0$)

$$\rho_0(\eta, \zeta, g = 0) = 1/4\,\frac{\varphi_0(\eta)\,\varphi_0(\zeta)}{\eta + \zeta}\,. \tag{12}$$

The Ambartsumyan function $\varphi_0(\eta)$ at an arbitrary value of a photon survival probability Λ is described by a complicated formula.[7] A brief table of values of the function $\varphi_0(\eta)$ and its moments can also be found in Ref. 7. Numerical analysis of the function $\varphi_0(\eta)$ made in spite of its cumbersome analytical representation reveals the fact that the Ambartsumyan function can be approximated by a linear function with the error of approximation not exceeding 0.25%

$$\varphi_0(\eta) = 1.81\,\eta + 1.10,\ \eta \geq 0.15\,,$$
$$\varphi_0(\eta) = 2.25\,\eta + 1.0,\qquad \eta < 0.15\,. \tag{13}$$

In the below discussion we shall meet again a limitation on the η value but the matter is that at small angles of incidence and reflection more complicated methods are to be used since the model of a plane layer is not applicable, and one has to take into account the refraction etc. When the above formulas are applied to the models of a plane atmosphere the limitation $\eta \geq 0.2$ actually does not impose any restriction. Thus in the case of a conservative isotropic scattering the relation

$$\rho_0(\eta, \zeta) = \frac{0.810\eta\zeta + 0.308}{\eta + \zeta} + 0.5 \tag{14}$$

is valid for the function $\varphi_0(\eta)$. Inaccuracy of this approximation by Eq. (14) is about 0.4%.

In the case of anisotropic light scattering the analytical view of the function $\rho_0(\eta, \zeta))$ is more cumbersome but the author has succeeded in deriving an analytical representation which well approximates the reflective coefficient.

Given below are the principal stages of the analysis of the problem for anisotropic scattering. We use here the tables of the function $\rho(\zeta, \zeta)$ values which have been obtained using rigorous numerical calculations[9] for a wide set of parameters of the Henny–Greenstein scattering phase function g and the single scattering albedo. The result presented in Ref. 9 for a two–term scattering phase function and the analysis of the numerical values show that the function $\rho(\zeta, \zeta)$ linearly depends on $g = 0.5$–0.9 (such values are characteristic of different cloud forms of the Earth and Venus). It should be noted that the parameters of linear approximation are in fact functions of the angle ζ. In other words, it is possible to employ the representation

$$\rho_0(\zeta, \zeta) = \rho_0(\zeta, \zeta, g = 0) + \frac{\hat{f}_1(\zeta)}{2z} + g\,\frac{\hat{f}_2(\zeta)}{2\zeta}\,, \tag{15}$$

where $\hat{f}_1$ and $\hat{f}_2$ are the functions satisfying the same condition, following from the symmetry of the function $\rho(\eta, \zeta)$ with respect to the variables η and ζ. This leads to the requirement that the functions $\hat{f}_1(\zeta)$ and $\hat{f}_2(\zeta)$ be represented by complete squares. Subsequent numerical analysis of the functions $\hat{f}_1$ and $\hat{f}_2$ and their derivatives, according to the tables,[7] gives the results

$$\hat{f}_1(\zeta) = \hat{f}_1^2(\zeta) = (0.386 - 0.237\zeta)^2\,,$$

$$\hat{f}_2(\zeta) = 0.730\zeta^2 - 0.350\zeta - 0.118\,, \tag{16}$$

where the function $\hat{f}_2$ can be easily expressed in the form of a linear combination of complete squares. There exist many versions in this case, but one of them is sufficient here

$$\hat{f}_2(\zeta) = f_2^2(\zeta) - f_3^2(\zeta) = (1.414-1.907\zeta)^2 - (1.479-1.705\zeta)^2. \tag{17}$$

In the case of a conservative anisotropic scattering ($g = 0.5 - 0.9$) we finally obtain an approximation formula

$$\rho_0(\eta, \zeta) = \frac{f_0(\eta)f_0(\zeta)}{\eta + \zeta} - \frac{f_1(\eta)f_1(\zeta)}{\eta + \zeta} + g\,\frac{f_2(\eta)\,f_2(\zeta) - f_3(\eta)f_3(\zeta)}{\eta + \zeta} \tag{18}$$

for the reflective coefficient. The functions f_1, f_2, and f_3 are calculated using Eqs. (16) and (17), and the function $f_0(\eta) = \varphi_0(\eta)/2$. It can be seen from the comparison between the values of the function $\rho_0(\zeta, \zeta)$ obtained from formula (18) and those calculated using the rigorous method[9] that the relative error for the value $\zeta = 0.2$ is less than 3% and for $\zeta > 0.2$ it is within a fraction of a per cent (Table III).

TABLE III. Errors of approximation of the function $\rho_0(\eta, \zeta)$ using formula (16), %.

ζ/g	0.5	0.75	0.8	0.85	0.9
0.1	12	15	17	19	27
0.2	3.0	0.3	0.9	1.5	3.0
0.4	0.1	0.5	0.7	0.8	0.4
0.6	0.1	<0.1	<0.1	<0.1	<0.1
0.8	0.1	0.05	0.2	0.1	0.3
1.0	≤0.1	<0.1	<0.1	<0.1	0.1

One can assume that the accuracy of calculations by formula (18) also holds for other sets of η and ζ values. But for $\zeta \leq 0.15$ the error increases to 15–20% and the above–proposed formulas are invalid. The situation can be improved by a proper selection of the parameters of the linear dependences for the functions f_2 and f_3, if any the form of the function f_1 being unchanged. It should be noted that Eq. (18) is not reduced to the formula for isotropic scattering (12) at $g = 0$, since it has been obtained under condition $g \geq 0.5$.

A rigorous numerical method for calculating the function $\rho(\zeta, \eta)$ requires long computer time and is not useful, e.g., in radiative climatic models. The approximate analytical formulas ensuring a sufficiently high–precision calculations are more convenient for use in applied problems of atmospheric optics.

FORMULAS FOR SCATTERING AND ABSORPTION COEFFICIENTS OF A CLOUD LAYER EXPRESSED IN TERMS OF DIFFUSE RADIATION INTENSITY

Let us consider a plane horizontally infinite homogeneous cloud layer. Optical thickness of this layer $\tau_0 = \varepsilon z$, where $\varepsilon = \sigma + \kappa$ is the volume extinction coefficient, σ is the volume scattering coefficient, and κ is the volume absorption coefficient, z is the geometric thickness of the layer, and $\Lambda = \sigma/(\sigma + \kappa)$ is the albedo of single scattering. In the cloud layers in the visible spectral region $1 - \Lambda \ll 1$.

Let a parallel flux of solar radiation be incident on the upper boundary of the layer at the angle across ζ. To describe the reflected and transmitted by the layer diffuse radiation we introduce the intensities averaged over azimuth

$$I(0, \eta, \zeta) = S\zeta\rho(0, \eta, \zeta) ;$$
$$I(\tau_0, \eta, \zeta) = S\zeta\sigma(\tau_0, \eta, \zeta) ; \quad (19)$$

where $\rho(0, \eta, \zeta)$ and $\sigma(\tau_0, \eta, \zeta)$ are the brightness coefficients for diffuse reflection and transmission which for a large optical thickness of the layer, are[7]

$$\rho(0, \eta, \zeta) = \ ,$$

$$\sigma(\tau_0, \eta, \zeta) = \frac{u(z)\, \overline{u}(\eta)\, M\, e^{-\kappa\tau_0}}{1 - N\overline{N}\, e^{-2\kappa\tau_0}} . \quad (20)$$

Functions $I(0, \eta, \zeta)$ and $I(\tau_0, \eta, \zeta)$ can be measured in experiment, the function $u(\zeta)$ and the values M, N, k, and Q depend on actual absorption in the cloud and on the scattering phase function that are described with the parameters Λ and g; the dependence on optical thickness is the exponential one. Thus there are two equations and three unknown values Λ, g, and τ_0. Assuming the parameter g to be known we shall attempt to determine the values Λ and τ_0 from independent measurements or model calculations. The effect of the light reflected from the underlying surface is taken into account as follows:

$$\left.\begin{aligned} \overline{N} &= N - AMQ^2/(1 - Aa^{\infty}) \\ \overline{u}(\eta) &= u(\eta) + AQa(\eta)/(1 - Aa^{\infty}) \end{aligned}\right\} , \quad (21)$$

where A is the albedo of the surface.

M.D. King in his paper[10] when solving the problem on determining optical thickness of the layer from the measured brightness of reflected radiation has obtained for optical thickness the relation $\tau' = \tau_0\,(3 - x_1)$ by transforming the first equation of a system of equations (19). Here we represent an intermediate relation which will be used in the below considerations

$$e^{2\kappa\tau_0} - \overline{N}N = \frac{MNu(\eta)u(\zeta)}{\rho(\eta, \zeta) - \rho(0, \eta, \zeta)} . \quad (22)$$

We should like to note that in contrast to this paper in Ref. 8 it is assumed that a weak absorption in the clouds as well as the spectral dependence of the albedo of the underlying surface can be neglected and the problem should be solved for $\Lambda = 1$ and $A = 0.2$.

To solve a system of equations (19), Eq. (22) is substituted into a system of equations (19) instead of the second equation and after obvious transformations the formula is given by

$$\sigma^2(\tau_0, \eta, \zeta)\overline{N}\, u^2(\eta) = \overline{u}^2(\eta)\, \{Mu(\eta)\, u(\zeta)[\rho(\eta, \zeta) - \rho(0, \eta, \zeta)] -$$
$$- N\, [\rho(\eta, \zeta) - \rho(0, \eta, \zeta)]^2 . \quad (23)$$

As was noted above at weak absorption the series expansions of the quantities and functions entering into this formula over small parameter powers are known. The series expansions for values M and N over the parameter s are given by

$$M = 8\, s + O(s^3) ,$$

$$N = 1 - 3\,\delta s + 9/2\, \delta^2 s^2 + O(s^3) . \quad (24)$$

Let us substitute series expansions (2) – (4) and (24) into Eq. (23) and denote the difference $[\rho_0(\eta, \zeta) - \rho(0, \eta, \zeta)] = \rho$, and then making necessary transformations and neglecting the terms of the order of s^3 and higher we obtain the equation for s^2 whose solution is rather simple

$$s^2 = \frac{(1 - A)^2[u_0^2(\eta)\, \rho^2 - u_0^2(\eta)\, \sigma^2(\tau_0)}{16\, u_0^2(\eta)\, [u_0^2(\zeta)\overline{u}_0^2(\eta) - A^2\sigma^2(\tau_0)] - \overline{u}_0^2\, a_2(\eta)\, a_2(\zeta)\, \rho/(3\delta) - 12\, \delta\sigma^2(\tau_0)\, u_0^2(\eta)\, A(1 - A) + 2A\, \overline{u}_0^2(\eta)\, (a_2(\eta) - 6\delta u_0(\eta))\, \rho} . \quad (25)$$

The values of the functions $\rho(0, \eta, \zeta)$ and $\sigma(\tau_0, \eta, \zeta)$ are obtained from the measurements of the intensity of reflected and transmitted through the cloud layer radiation. The values of the functions $u_0(\eta)$, $a_2(\zeta)$, and $\rho_0(\eta, \zeta)$ for η, and ζ corresponding to measurement conditions are taken from tables[5,7] or calculated using the above–proposed formulas.

The relation for $\tau' = \tau_0\,(3 - x_1)$ derived in Ref. 8 is

$$\tau' = (2s)^{-1} \ln \overline{N}\left[\frac{Mu(\eta)\, u(\zeta)}{\rho(\eta, \zeta) - \rho(0, \eta, \zeta)} + N \right] . \quad (26)$$

By substituting the series expansions of the asymptotic constants and truncating the series by terms containing the third power of the parameter s

$$\tau' = (2s)^{-1}\left\{2 \ln N + \ln \frac{1 - A - 4As - 6A\delta s^2}{1 - A + 4As - 6A\delta s^2} + \ln \frac{\rho + 4u_0(\eta)u_0(\zeta)s + a_2(\eta)a_2(\zeta)s^2/(6\delta) - 18u_0(\eta)u_0(\zeta)s^3}{\rho - 4u_0(\eta)\, u_0(\zeta)\, s + a_2(\eta)\, a_2(\zeta)s^2/(6\delta)}\right\} . \quad (27)$$

In addition, the formulas for the volume absorption and scattering coefficients $\kappa = (1 - \Lambda)\, \tau_0/z$ are valid or, taking into account the definition of the parameter s, $\kappa = \tau' s^2/z$. For the volume scattering coefficient we have $\sigma = \tau'[(3 - x_1)^{-1} - s^2]/z$.

SYSTEMATIC ERRORS AND RANGE OF APPLICABILITY

In our previous paper[3] the formulas for determining the relative errors δs and $\delta\tau_0$ were proposed. A more comprehensive analysis of this problem leads to the other relations which follow from the formulas for s^2 and τ_0 according to the theory of errors. Thus, when the radiation fluxes $F^{\uparrow}$ and $F^{\downarrow}$ are measured, we have

$$\delta s \leq \frac{\Delta F}{1 - F\uparrow - F\downarrow} + \frac{\Delta F a_2(\zeta) + 16 u_0 \Delta u_0 + (1 - F\uparrow)\,\Delta a_2}{16 u_0(\zeta) - 2a_2(\zeta)\,(1 - F\uparrow)}\,; \quad (28)$$

$$\delta\tau_0 \leq \left((3\delta + 25)\,\Delta s + \frac{\Delta F}{(1 - F\uparrow)^2}\right) \Big/ \,\tau_0 + \frac{\Delta x_1}{3 - x_1} + \frac{\Delta s}{s}\,, \quad (29)$$

where ΔF is the absolute error of measurements of the fluxes, the stronger is the radiation absorption the larger is the value $(1 - F^{\uparrow} - F)$ determining the relative value of the energy influx into the layer. Typically, this value is in the range from 0.04 to 0.08 (small difference between the close values). Thus, the first term of the sum in Eq. (28) determines the error in the value s, i.e., $\delta_s \leq 4\%$ for $\Delta F = 0.002$. The absolute errors Δu_0 and Δa_2 are caused by the flux of radiation partially scattered in the overlying atmosphere that is incident on the upper boundary of a cloud layer. In Ref. 7 it is shown that the fraction of scattered light in a clear atmosphere can reach 0.1. In the case of a completely diffuse rafraction we should use the value $\int_0^1 u_0(\zeta)\,\zeta\, d\zeta = 1$ instead of $u_0(\zeta)$ and $6\sigma = 8.5$ instead of $a_2(\zeta)$ and so, taking into account the above said, we have

$$\Delta u_0(\zeta) = 0.1\,(u_0(\zeta) - 1) \sim 0.02\,,$$

$$\Delta a_2(\zeta) = 0.1\,(a_2(\zeta) - 6\delta) \sim 0.2\,. \quad (30)$$

The differences in relations (30) depend on the cosine of the zenith angle ζ and are minimal at the zenith angles $\zeta \sim 0.6 - 0.7$. So it is advisable to provide the fulfilment of this condition during measurements.

The error in the value τ_0 is primarily determined by the error in s value and by the uncertainty in the scattering phase function x_1. For the clouds of large optical thickness the first term in the sum of Eq. (28) can be small and weakly affects the value of the error. It can easily be shown that for intensities the errors will be the same since the relevant formulas have analogous structure and are expressed in terms of the same asymptotic constants.

When using the relations obtained for τ_0 and s it is necessary to take into account the fact that they are valid only within the applicability limits of asymptotic formulas and series expansions, that is $(\tau_0 \gg 1)$, $(1 - \Lambda \ll 1)$. The applicability limits of the proposed method concerning the optical thicknesses of the scattering layer and actual light absorption in it has been thoroughly analyzed in Ref. 3

CONCLUSIONS

It should be noted that the method proposed in Ref. 3 and developed in this paper which is used for obtaining the volume coefficients of scattering and absorption based on measurements of the diffuse radiation field at the boundaries of a scattering layer is useful in studying the cloud layers with the parameters satisfying the applicability limits of asymptotic formulas. The formulas are inapplicable to the clouds of a small optical thickness or in the region of strong absorption.

The approximation formulas and above–derived additional terms of asymptotic series expansions significantly improve the calculational accuracy, widen the applicability range, and allow one to use the analytical representations for all functions what is particularly useful in solving inverse problems.

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