Solution of the problem of light scattering by small-scale inhomogeneities of maritime environment by the method of modified approximation of abnormal diffraction

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A new approximation method for solving the problem of light scattering by optical inhomogeneities (modified approximation of abnormal diffraction) has been considered. A rigorous mathematical validation of the method is presented, the ways of its further correction (with allowance for the volume diffraction with the use of Rytov phase) are given, the possibility for rigorous formulation of the problem of determination of the field of applicability and the velocity of convergence of the approximate solution are validated. It is shown that the field of applicability of the approximation is substantially expanded as compared to the Hulst approximation. The new method is suitable for calculation of the scattering phase function at any values of the scattering angle including the backscattering.

Interpretation of results of optical measurements of physical-chemical parameters of the sea medium becomes significantly simpler, if, solving the inverse problem, it is possible to describe the light field propagation in the medium in the framework of a simple analytical approximation. Thus, the purpose of investigation of the sea medium by the laser sensing method is determination of its optical characteristics (for example, number density of weighted particles, their size and refractive index distributions).[1] Relation between the physical characteristics of the medium and its primary optical characteristics is set in the single scattering approximation (Born) through quite simple analytical formulas.[2,5–7]

Unfortunately, the well-known approximate methods for solving the problem of light scattering are poorly applicable to sea suspension. In practice, the only method capable of calculating the characteristics of light scattering by particles with $\rho_a \in [1, 200]$ ($\rho_a = 2\pi a / \lambda$, where a is the characteristic particle size and λ is the wavelength), is the abnormal diffraction approximation (ADA) proposed by Hulst.[2] Regretfully, it is applicable only to optically soft particles and small scattering angles.

A new method for solving the Maxwell equations with boundary conditions corresponding to the electromagnetic wave scattering by an inhomogeneity of dielectric permittivity is proposed in Ref. 3. The method allows a new solution of the problem of light scattering by a particle – modified abnormal diffraction approximation (MADA) – even in the first approximation.

We describe briefly the essence of the MADA method. Let monochromatic electromagnetic field propagates in a medium with magnetic permittivity $\mu = 1$ and zero conductivity. Then the following formula holds for the complex amplitude of the intensity of electromagnetic field at the point $\mathbf{r}$[4]:

$$\Delta\mathbf{E}(\mathbf{r}) + k^2 \, [1 + \varepsilon_1(\mathbf{r})] \, \mathbf{E}(\mathbf{r}) = \nabla \, [\nabla\mathbf{E}(\mathbf{r})], \qquad (1)$$

where

$$k = 2\pi / \lambda; \; \varepsilon_1 = \varepsilon / \varepsilon_0 - 1 = (n^2 - \chi^2 - 1) + i(2n\chi)$$

is the relative dielectric permittivity of the optical inhomogeneity, ($n + i\chi = m$ is its relative complex refractive index[5]).

Let us seek a solution of Eq. (1) in the form $\mathbf{E} = \mathbf{e}u$, where the scalar function $u(\mathbf{r})$ fulfills the Helmholtz equation

$$\Delta u(\mathbf{r}) + k^2 \, u(\mathbf{r}) = - k^2 \, \varepsilon_1(\mathbf{r}) \, u(\mathbf{r}), \qquad (2)$$

and the vector of polarization $\mathbf{e}(\mathbf{r})$ fulfills the equation

$$\begin{aligned} &\Delta\mathbf{e}(\mathbf{r}) + 2 \, [\nabla \, \ln(u(\mathbf{r}) \, \nabla)] \, \mathbf{e}(\mathbf{r}) = \\ &= - \nabla \, \ln\{u(\mathbf{r}) \, \mathbf{e}(\mathbf{r}) \, \nabla \, \ln[1 + \varepsilon_1(\mathbf{r})] - \\ &- \nabla \, [\mathbf{e}(\mathbf{r}) \, \ln(1 + \varepsilon_1(\mathbf{r}))]\}. \end{aligned} \qquad (3)$$

Within the above approximations, we calculate the intensity of light, scattered by a spherical particle of radius a, and relative dielectric permittivity $\varepsilon_1 \equiv \varepsilon_a$ for the case, when the particle center is at the origin ($\mathbf{r}_{0a} = 0$), the point of observation lies in the Fraunhofer zone ($\mathbf{r} \gg ka^2$), and the primary field is a plane wave

$$\mathbf{E}_0(\mathbf{r}) = u_0 \, \mathbf{e}_0 \exp \, (ik\mathbf{n}_0\mathbf{r}). \qquad (4)$$

Here u_0 = const is the primary field amplitude, $\mathbf{e}_0$ is its polarization vector, $(\mathbf{e}_0, \mathbf{e}_0^*) = 1$; $\mathbf{n}_0$ is the unit vector in the direction of the primary wave propagation. We consider the integral equation equivalent to Eq. (1), taking into account the boundary condition

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) - \int G(\mathbf{r}, \mathbf{r}') \times$$

$$\times \{k^2 \varepsilon_a(\mathbf{r}') \mathbf{E}(\mathbf{r}') + \nabla [\ln(1 + \varepsilon_a(\mathbf{r}'))]\} \, d\mathbf{r}', \quad (5)$$

where $G(\mathbf{r}, \mathbf{r}')$ is the Green's function.

Substituting the approximate solution of Eq. (1) into the right part of Eq. (5), we obtain a new approximate solution. If the initial approximate method converges to the exact solution, then the obtained «daughter» approximation is convergent as well.

Let us write the scalar part of $\mathbf{E}$ in the form

$$u(\mathbf{r}) = u_0(\mathbf{r}) \exp [\varphi(\mathbf{r})], \quad (6)$$

then, substituting (6) into (2), we obtain the equation for the complex phase $\varphi(\mathbf{r})$

$$\Delta\varphi(\mathbf{r}) + [\nabla\varphi(\mathbf{r})]^2 + k^2 + k^2\varepsilon_a = 0, \quad (7)$$

which can be solved by the method of consecutive approximations.

The proposed method for constructing new approximations is the following. The approximate solution (7) is substituted to the right part of Eq. (5). For example, the solution (7) in the first approximation is the Rytov phase[4]:

$$\varphi_R(\mathbf{r}) = - k^2 \int G(\mathbf{r}, \mathbf{r}') \frac{u_0(\mathbf{r}')}{u_0(\mathbf{r})} \varepsilon_a(\mathbf{r}) \, d\mathbf{r}'. \quad (8)$$

Following the above method for constructing approximations and restricting ourselves to the expressions linear in ε_a, when solving (3), we obtain the new approximation (MADA):

$$\mathbf{E}_{\mathrm{MADA}}(\mathbf{r}) = \mathbf{E}_0(\mathbf{r}) - k^2 \int G(\mathbf{r}, \mathbf{r}') \exp [\varphi(\mathbf{r}')] \times$$

$$\times \{\mathbf{n}(\mathbf{rr}') [\mathbf{E}_0(\mathbf{r}') \mathbf{n}(\mathbf{rr}')]\} \, d\mathbf{r}', \quad (9)$$

where $\mathbf{n}(\mathbf{rr}') = (\mathbf{r} - \mathbf{r}') / |\mathbf{r} - \mathbf{r}'|$.

Then for the scattered radiation we have

$$\mathbf{E}^s(\mathbf{r}) = \frac{u_0 \, k^2 \exp (ikr)}{4\pi r} [\mathbf{n}_s(\mathbf{e}_0 \, \mathbf{n}_s)] \times$$

$$\times \int \varepsilon_a(\mathbf{r}') \exp [\varphi_R(\mathbf{r}') - i\mathbf{qr}'] \, d\mathbf{r}'. \quad (10)$$

Here $\mathbf{n}_s = \mathbf{r} / |\mathbf{r}|$; $\mathbf{q} = k (\mathbf{n}_s - \mathbf{n}_0)$.

Equation (10) can be simplified by substitution into it of the phase obtained in the first approximation of geometric optics[4] for the Rytov phase $\varphi_R(\mathbf{r})$

$$\varphi_g(\mathbf{r}) = \varphi_g(z, \zeta, \varphi) = \frac{ik}{2} \int_0^z \varepsilon_a(\xi, \zeta, \varphi) \, d\xi. \quad (11)$$

The cylindrical coordinate system $\mathbf{r} = (z, \zeta, \varphi)$ was used when deriving Eq. (11), and it was assumed that the considered spherical particles lie in the region $z \ge 0$, and $\mathbf{E}_0 = u_0 \, \mathbf{e} \exp (ikz)$. In this case

$$\mathbf{E}^s_{\mathrm{MADA}}(\mathbf{r},\theta) = \frac{u_0 \, S(\theta,\rho_a,\varepsilon_a) \exp (ikr)}{kr} [\mathbf{n}_s(\mathbf{e}_0\mathbf{n}_s)]. \quad (12)$$

Here

$$S(\theta,\rho_a,\varepsilon_a) = \frac{\rho_a^2 \varepsilon_a}{(1-\cos\theta+\varepsilon_a / 2)} \times$$

$$\times \int_0^1 y\sqrt{1-y^2} \, J_0\left(\rho_a \sqrt{1-y^2} \sin\theta\right) \exp\left(\frac{i\varepsilon_a \rho_a}{2} y\right) \times$$

$$\times \sin [\rho_a (1-\cos\theta+\varepsilon_a / 2) y] \, dy, \quad (13)$$

$J_0(x)$ is the Bessel function of the first type, θ is the scattering angle, $\varepsilon_a = (n^2 - \chi^2 - 1) + i(2n\chi)$, $n + i\chi = m$ (see Eq. (1)).

It is convenient to rewrite formula (12) containing the vector factor in the matrix form[5]:

$$\begin{pmatrix} \mathbf{E}^s_{\parallel} \\ \mathbf{E}^s_{\perp} \end{pmatrix} = \frac{\exp (ikr)}{kr} \begin{pmatrix} S_2 & 0 \\ 0 & S_1 \end{pmatrix} \begin{pmatrix} \mathbf{E}^0_{\parallel} \\ \mathbf{E}^0_{\perp} \end{pmatrix}, \quad (14)$$

where $\mathbf{E}^0_{\parallel}$, $\mathbf{E}^s_{\parallel}$, $\mathbf{E}^0_{\perp}$, $\mathbf{E}^s_{\perp}$ are the parallel and the perpendicular to scattering plane components of vectors of primary and scattered light fields. Then

$$S_1 = S; \quad S_2 = S \cos\theta, \quad (15)$$

where $S = S(\theta, \rho_a, \varepsilon_a)$ is determined by Eq. (13) in our approximation. The same value for spherical particle can be exactly calculated by means of summing the Mie series, and in the Born approximation it is expressed through an elementary formula[5]

$$S = - 2 \, (m - 1) \, \rho_a^3 \, (\sin u - u \cos u) / u^3;$$

$$u = 2\rho_a \sin (\theta / 2). \quad (16)$$

Having calculated elements S_1 and S_2, one can easily determine the relationship between the intensities of primary and scattered fields for different cases of polarization of the incident light field $\mathbf{E}_0$ (Ref. 5). For example, if the incident light is unpolarized, then this relationship has the form

$$S_{11}(\theta) = (|S_1|^2 + |S_2|^2) / 2 = I_s(\theta) \, k^2 \, r^2 / I_i, \quad (17)$$

where $S_{11}(\theta)$ is the element of the Stokes matrix, I_i is the intensity of the incident radiation, $I_s(\theta)$ is the intensity of radiation scattered in the direction determined by the angle θ; $k = 2\pi / \lambda$ is the wave number, r is the distance from the point of observation to the particle center.

Let us correlate the range of applicability of the obtained approximation (MADA) with that of the Born approximation and the Hulst approximation by comparing the radiation intensities calculated by the exact Mie formulae[5] for a spherical particle with the results obtained in the framework of three considered approximations at different values of θ, ρ_a, and ε_a (m).

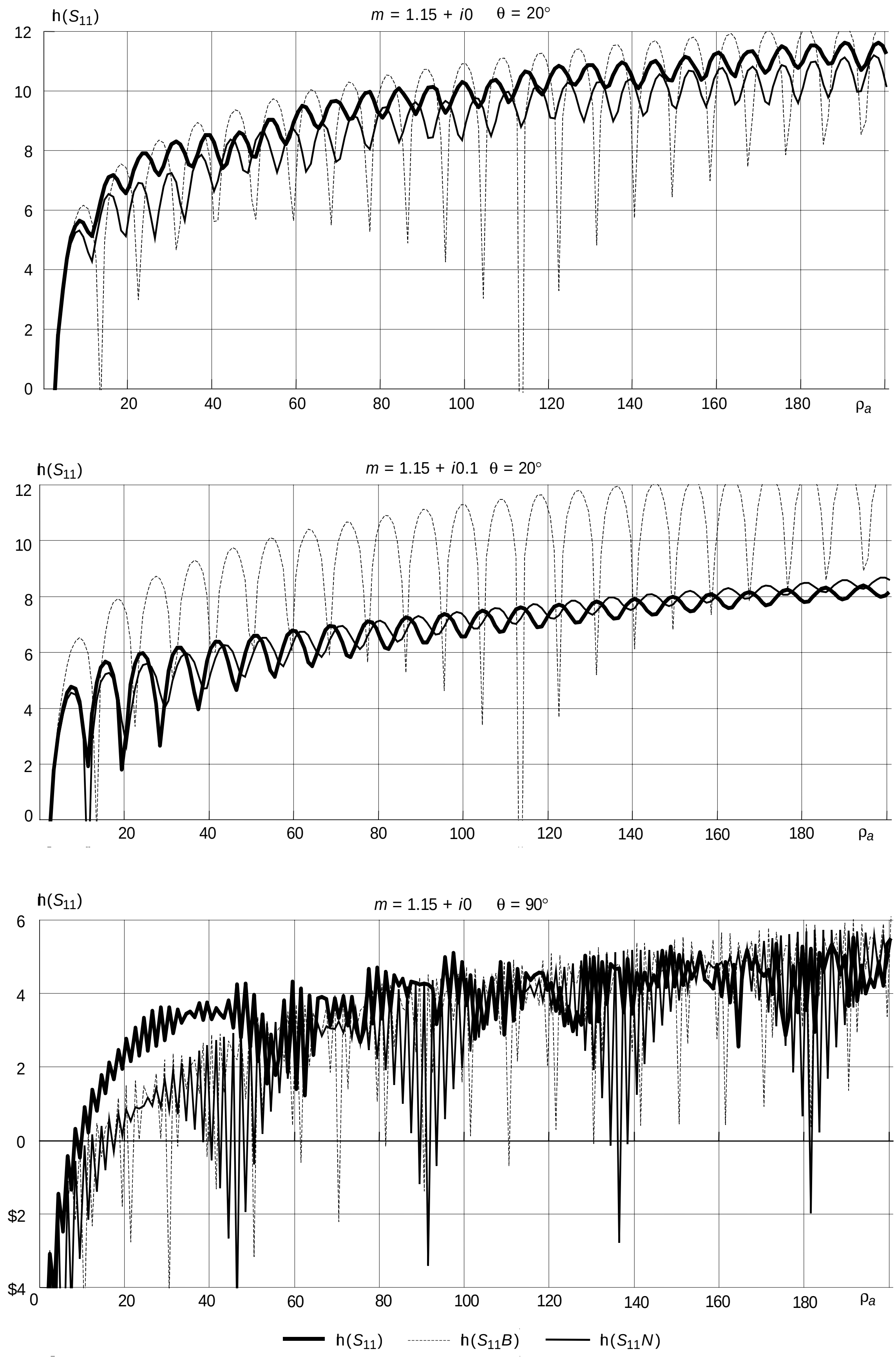

Fig. 1.

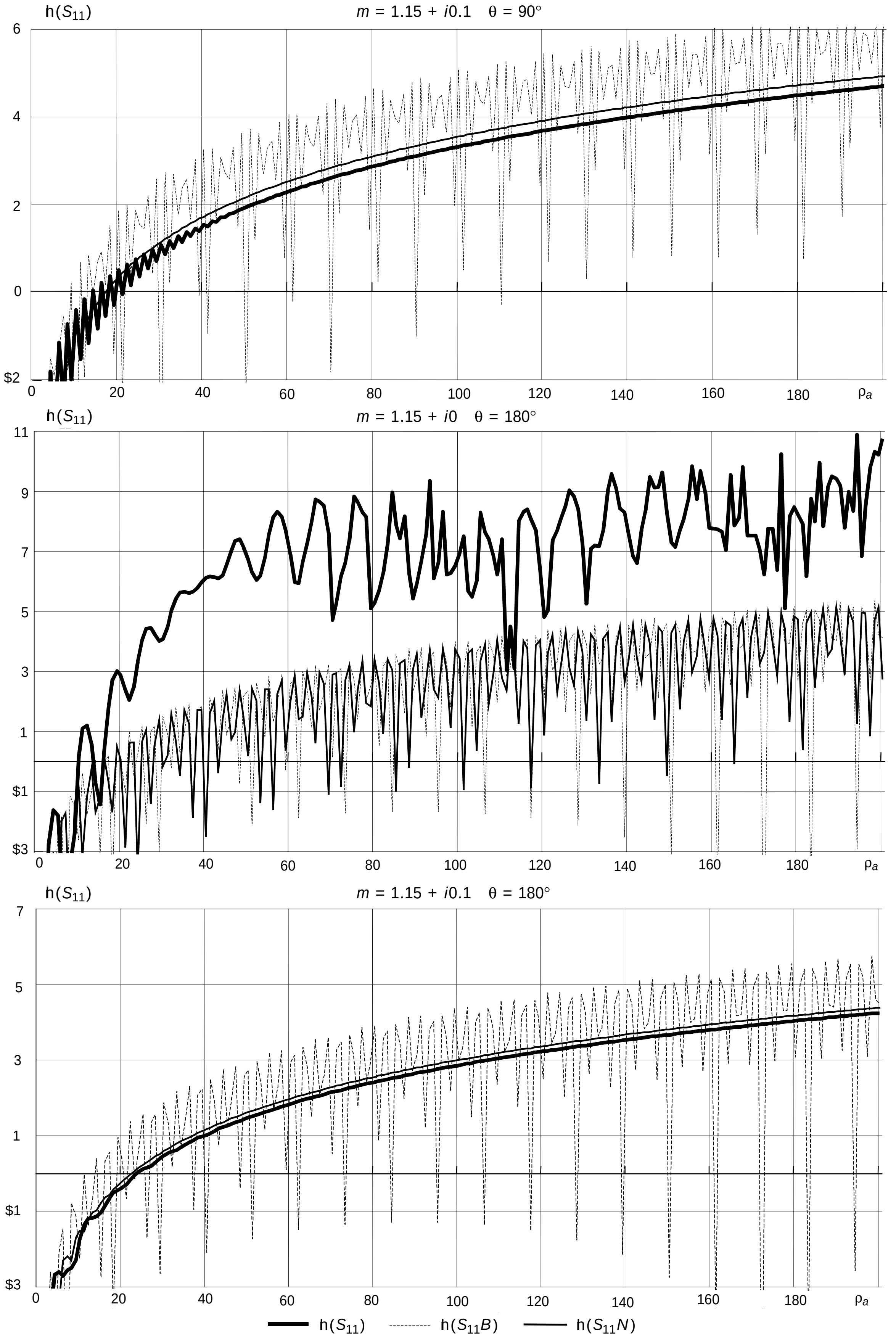

Fig. 1 (continued)

The dependences of $\ln(S_{11})$ on $\rho_a = 2\pi a / \lambda$ (in this case a is the particle radius, $\rho_a \in [1, 200]$) are shown in Fig. 1 for three scattering angles (θ = 20, 90, 180°) and two values of the relative refractive index of particle $m = 1.15 + i0$ («non-absorbing» particle), and $m = 1.15 + i0.1$. As it was mentioned above, the value $\ln(S_{11})$ was calculated by exact Mie formulae $\ln(S_{11})$, in the Born approximation[5] $\ln(S_{11}\mathrm{B})$, and in the new modification of the abnormal diffraction approximation (MADA) $\ln(S_{11}\mathrm{N})$. It is seen that MADA provides for a good agreement for real absorbing particles of the sea suspension with $\mathrm{Im}(\varepsilon_a) \neq 0$ at any values θ up to 180°, while the Hulst approximation is inapplicable at $\theta > 20°$, because it uses the $\cos\theta = 1$ approximation.

Thus, significant expansion of the range of applicability is reached due to taking into account the volume spatial arrangement of oscillators, while the Hulst ADA is applicable only to description of scattering by optically soft inhomogeneities at small angles θ (up to 20°). MADA is suitable (just in the cases of real complex values m) for calculation of the scattering phase function at any θ, including the backscattering case.

In conclusion it should be noted that MADA is a step forward as compared to the Hulst approximation and it is, in fact, a new approximation method for solving the problem of light scattering by optical inhomogeneities. In this paper we presented the rigorous mathematical validation of the method, showed the ways for its subsequent improvement (taking into account the volume diffraction when using the Rytov phase), and proved a possibility of strict setting of the problem of determination of the range of applicability and the rate of convergence of the approximate solution. Averaging over the particle size, when calculating the scattering characteristics on the ensemble of particles, should increase the accuracy of MADA. Moreover, the Born approximation[7] can be used in this case for satisfactory qualitative description (this is seen from the figure). The proof of this statement will be presented in a separate paper.

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