A NEW BISTABILITY MECHANISM OF THE SECOND–HARMONIC GENERATION INSIDE A CAVITY WITH EXTERNAL PUMPING

A.V. Gainer, N.P. Konoplyova, and G.I. Surdutovich

Institute of Automation and Electrometry,
Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk
Institute of Semiconductor Physics,
Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk

Received August 11, 1990

The process of the second–harmonic generation inside an annular cavity with external pumping at fundamental frequency is considered. It is shown that such a system can be bistable and multistable. To implement these regimes, it is necessary that the normalized depth of the nonlinear medium (the ratio of the depth of the medium to the conversion distance) should be large enough ($\gtrsim 4$).

The second–harmonic generation (SHG) inside a cavity with external pumping into a nonlinear crystal is of interest from two viewpoints. On the one hand, along with an increase of the lasing efficiency, the second harmonic (SH) output intensity may be controlled. On the other hand, a passive cavity containing a nonlinear medium is a nonlinear system with the feedback and a special mechanism of coupling of the pump wave and the SH. Therefore, new stationary and nonstationary lasing regimes may be expected in such a system. The former are Instability1 and multistability, while the latter is self–pulsing.[2,3]

The SHG process inside a passive cavity with a double" feedback has been considered in Refs. 1–3 for the laser radiation fields at the frequencies ω and 2ω. In these studies the approximation of a point medium was used, i.e., the effects of wave propagation were neglected. In the present study we consider the case of a single feedback only for the SH simultaneously taking into account the variations of the SH amplitude as a result of interaction, including the back reaction on the pump wave. As a result we found new previously unknown regimes.

When solving the problem of the SHG inside a cavity, it is necessary to know how the SH behaves in the case of generation in a semi–infinite medium with arbitrary complex amplitudes at the boundary of the medium. In Section 1 we analyze the main features of this process as a function of the initial conditions. Section 2 presents the results of analytical and numerical studies of the SHG inside a passive cavity. Conditions are found for the origin of several stationary states and their stability is investigated.

1. GENERAL FEATURES OF THE SHG WITHOUT A CAVITY

When studying the SHG process inside a cavity with the feedback one needs the results of the solution of the problem on the stationary SHG in a semi-infinite medium with arbitrary initial conditions at the boundary of the medium. This solution is given in Ref. 4 for the case of the exact phase synchronization $\Delta\kappa = 2\kappa_1 - \kappa_2 = 0$. For arbitrary values of $\Delta\kappa$ the variations of the SH amplitude $a_2(z)$ and the phase change of the waves at the frequencies ω, $2\omega - \varphi_1(z)$, and $\varphi_2(z)$ with distance have the form[5]

$$a_2^2(z) = E^2\left[\frac{2+\Delta^2}{3} + e_3 + (e_2 - e_3)\,\mathrm{sn}^2\left[\sqrt{e_1 - e_3}\,\sigma E z + z_0/m\right]\right], \quad (1)$$

$$\varphi_1(z) = \varphi_1(0) - \Delta\sigma E z - \sigma\left(CE^3 + \Delta E^3\right)\int_0^z \frac{dz}{E^2 - a_2^2(z)}, \quad (2)$$

$$\varphi_2(z) = \varphi_2(0) - \Delta\sigma E z - \sigma C E^3\int_0^z \frac{dz}{a_2^2(z)}, \quad (3)$$

where the distance z is counted off from the boundary of the nonlinear medium, z_0 is determined from the boundary condition, σ is the nonlinear coupling constant, $\Delta = \Delta\kappa/2\sigma E$ is the reduced detuning, and the elliptic sine function parameter $m = (e_1 - e_3)/(e_1 - e_3)$ is expressed in terms of the roots of the cubic equation(A1) (see Appendix A) $e_1 > e_2 > e_3$. The integrals in Eqs. (2) and (3) may be expressed in terms of elliptic functions (see Eqs. (B1) and (B2), Appendix B). The values E and C entering into Eqs. (1)–(3), remain unchanged during the wave propagation through the medium[4,6]

$$E^2 = a_1^2(z) + a_2^2(z) = a_1^2(0) + a_2^2(0), \quad (4)$$

$$C = \frac{a_1^2(z)a_2(z)}{E^3}\cos\Psi(z) + \Delta\frac{a_2^2(z)}{E^2} = \frac{a_1^2(0)a_2(0)}{E^3}\cos\Psi(0) + \frac{a_2^2(0)}{E^2}, \quad (5)$$

where $\Psi = 2\varphi_1 - \varphi_2$ is the phase difference. The solution (1) describes periodic conversion of the SH energy into the radiation at the fundamental frequency, and the inverse process, except for two cases. When the waves amplitudes

0235-6880/91/02 115–4 $02.00

and the phase differences at the medium boundary are such that $a_2(0)\cos\Psi(0) = \Delta E(C = \Delta)$, the energy of the system is completely converted into the second harmonic in the limit $z \to \infty$. Another example of nonperiodic behavior is the phase centers,[6] in which the wave amplitudes are independent of the distance. As follows from Eq. (5), the phase difference is also a periodic function of z. Note that the spatial period

$$T = 2K(m)/\sqrt{e_1 - e_3}\,\sigma E, \tag{6}$$

where $K(m)$ is the elliptic integral of the first kind, as well as the maximum and minimum values of the SH amplitude

$$a_{max} = \left[\frac{2+\Delta^2}{3} + e_2\right]^{1/2} E,$$

$$a_{min} = \left[\frac{2+\Delta^2}{3} + e_3\right]^{1/2} E \tag{7}$$

depend on wave amplitudes and phase differences at the boundary in addition to nonlinearity and detuning and are single-valued functions of the values E and C. When looking for a qualitative description of the process, it is more convenient to use a phase picture,[6] in which the value C determines the trajectory (E = const). The minimum value $T_{min} = \pi/\sqrt{e_1 - e_3}\,\sigma E$ is reached in the phase centers, where $m = 0$ and $|C| = |C_{max}|$. The period T increases as $C - \Delta$, i.e., when the trajectory approaches the separatrix ($C - \Delta$, $m = 1$) where it becomes infinity. The degree of modulation ($a_{max} - a_{min}$) is maximum for $C = 0$, and vanishes with increase of $|C|$ to $|C| = |C_{max}|$ at the phase centers.

In the case when $\Delta\kappa = 0$, it follows from Eqs. (2) and (4) that $\varphi_1(z)$ and $\varphi_2(z)$ monotonically increase or decrease depending on the sign of C. If $\Psi = -\pi/2$ ($C = 0$), the SH amplitude $a_2(z)$ decreases vanishing at $\tilde{z} = z/\sigma E$. At this point the SH has phase discontinuity and changes by π (see Appendix C), and the solution transfers to the branch corresponding to $\Psi = \pi/2$. If $\Delta\kappa \neq 0$, wave phases monotonically depend on z, provided the initial conditions are such that $C^{\Phi_2} \le C < 0$, $\Delta < C \le C^{\Phi_1}$ where C^{Φ_1} and C^{Φ_2} denote the values of C at phase centers.

2. SH GENERATION INSIDE A CAVITY WITH EXTERNAL PUMPING

We enclose a nonlinear medium of the depth/ in an annular cavity. An external laser, lasing at a frequency ω is used for pumping. Let the cavity mirrors be transparent for the pump wave, so the resonant conditions take place for the SH:

$$a_2(0,\ t) = \alpha a_2(l,\ t - \Delta t),$$
$$\varphi_2(0,\ t) = \varphi_2(l,\ t - \Delta t) + \delta, \tag{8}$$

where $\Delta t = (l_c - l)/c$, l_c is the cavity length, α is the extinction coefficient of radiation per one transmission through the cavity, and δ is the cavity detuning. In a stationary case the amplitude and phase of the SH $a_S(z)$ and $\varphi_S(z)$ are time independent and satisfy the system of equations

$$a_s(0) - \alpha a_s(l) = 0, \tag{9}$$

$$\varphi_s(0) - \varphi_s(l) - \delta = \pm 2\pi n,\ n = 0,\ 1,\ 2, \ldots \tag{10}$$

Then the amplitude and phase upon exiting the nonlinear medium $a_S(l)$ and $\varphi_S(l)$ are obviously described by the solution of the stationary problem on the SHG without a cavity (1) and (3) with boundary conditions $a_S(0)$ and $\varphi_S(0)$. The values $a_1(0)$ and $\varphi_S(0)$ are constant and are determined by the emission from the external lasing. Since the SHG depends only on the phase difference at the boundary, one may assume without loss of generality that $\varphi_1(0) = 0$ and $\Psi(0, t) = -\varphi_2(0, t)$.

When detuning from the exact phase synchronization is $\Delta\kappa = 0$, and phase velocities of the waves in the medium are matched, the temporal evolution of the system may be studied using Poincare's mapping technique. If $a_n(0)$ and $\varphi_n(0)$ are the corresponding values for the nth passage of radiation through the cavity at the inlet of the nonlinear medium, then $a_n(l)$ and $\varphi_n(l)$ are determined at the outlet from the medium from solutions (1) and (3) of the stationary problem with no cavity while conditions (8) yield the amplitude and phase for the $(n + 1)th$ passage. The mapping thus constructed allows one to find the SH amplitude and phase, in the time intervals being multiple of the time of radiation passage through the cavity. The stationary states (9) and (10) represent the fixed points of the mapping. Note that in the vicinity of a fixed point, where the SH complex amplitude varies insignificantly during the time of radiation passage though the cavity, the solutions of the stationary problem (1) and (3) may be used for approximate description of radiation propagation through the medium provided $\Delta\kappa/\kappa_2 \ll 1$.

To investigate the stability of the fixed point, we find a mapping relating small deviations from this point

$$\delta a_n = a_n(0) - a_s(0),\quad \delta\varphi_n = \varphi_n(0)$$

for the nth and $(n + 1)th$ passages. Appendix C gives the coefficients of the matrix T:

$$\begin{pmatrix} \delta a_{n+1} \\ \delta\varphi_{n+1} \end{pmatrix} = T \begin{pmatrix} \delta a_n \\ \delta\varphi_n \end{pmatrix}. \tag{11}$$

We perform a transformation to the variables $x = a_2\cos\varphi_2$ and $y = a_2\sin\varphi_2$ in Eq. (11). The fixed point remains stable if the eigenvalues of the matrix $\tilde{T}$ satisfy the condition $|\lambda_{1,2}|^2 < 1$ for new variables λ_1 and λ_2.

a) The case of the exact resonance and synchronization ($\sigma = 0$ *and* $\Delta\kappa = 0$).

For the existence of the solution of Eq. (10) corresponding to $n = 0$ the nonlinear phase change of the SH $\varphi = \varphi_2(l) - \varphi_2(0)$ should be equal to zero when the radiation has passed once through the cavity. It is possible only in the case when $\Psi(0) = \pi/2$ and $C = 0$. The stationary value of amplitude at the inlet of the nonlinear medium is given by the expression

$$a_s(0) = \alpha E \frac{E\,\mathrm{th}(\sigma E l) + \alpha_s(0)}{E + \mathrm{th}(\sigma E l)\alpha_s(0)}. \tag{12}$$

This solution holds for arbitrary normalized depth of the nonlinear medium l/L ($L = 1/\sigma a_1(0)$ is the conversion distance) and for arbitrary Q–factors of the cavity.

From Eqs. ($C1$)–($C4$) we may obtain the condition of stability in the form

$$2 - \alpha\frac{l}{L}\sqrt{1-\alpha^2} - (1-\alpha)(1-\alpha^2) < 0 \tag{13}$$

for $C = 0$ and $l \gg L$. A numerical analysis shows that Eq. (13) is satisfied for arbitrary Q—factor of the cavity, if $l < l_{crit} = 3.6\,L$. For the depths exceeding the critical and for "moderate" values of a there appears an instability zone, which increases with increase of the normalized depth. To analyze the reasons of the appearance of such an instability, we consider the corresponding phase picture. The stationary amplitude satisfies the condition $a_2(0)/E \approx \alpha$ for the depths under consideration.

Let the stationary values of the SH amplitude and phase acquire identical increments δa_0 and $\delta\varphi_0$ for various α. It is easy to see that the points in the phase plane corresponding to the perturbed values lie on the trajectories extremely spaced from the separatrlx ($\alpha \approx 1 - \alpha$). For these α the phase change in the SH for one passage will be largest. In the range of small C the value $\Delta\varphi$ is linearly dependent on C (Eq. B4). Therefore, if the depth of the nonlinear medium is such that $|\Delta\varphi| > 2|\delta\varphi_0|$ then the inequality will be satisfied for arbitrary $\delta\varphi$, and the stationary state will become nonstationary. Apparently, the type of this instability is the "saddle", i.e., when only the amplitude is perturbed, the system returns to its stationary point.

The solution of system of equations (9) and (10) holds for $n = \pm 1$ within a certain range of the Q—factors of the cavity if the nonlinear phase change of SH for one passage is $\pm 2\pi$. The value of $|\Delta\varphi|$ increases with increase of $|C|$ during the period T (T is defined by expression (6)) and is at its maximum at the phase centers, where $|C| = |C_{\max}| = \dfrac{1}{3\sqrt{3}}$. Since T is at its minimum at the phase centers, the nonlinear phase changes in the media of depths $l \geq T^{\Phi} = \sqrt{\dfrac{2}{3}}\,\pi L$ would be at its maximum for those initial conditions, which correspond to phase centers $a_2(0) = a_2^{\Phi} = E/\sqrt{3}$ and $\varphi(0) = 0, \pi$. Therefore, the minimal depth may be found from the condition

$$\Delta\varphi| = \sqrt{2}\,\frac{l_{\min}}{L} = 2\pi. \qquad (14)$$

Apparently, if $l = l_{\min}$, Eq. (9) is satisfied only for $\alpha = 1$. With increase of the depth of the nonlinear medium Eqs. (9) and (10) have solutions only within a certain range of the Q—factors of the cavity. Figure 1*a* shows the results of the numerical solutions of Eqs. (9) and (10) for various values of the extinction coefficient for one passage. The zones of stable stationary states are shown by solid curves, and the instabilities — by dashed curves. In case $n = \pm 1$, the matrix $\tilde{T}$ was numerically retrieved. It was found that the fixed point was either a stable or an unstable "focus". Note that two sets of phase differences at the inlet of the medium correspond to this stationary state, because both initial conditions $\Psi(0)$ and $\pi - \Psi(0)$ yield identical amplitude distributions in the medium for $\Delta\kappa = 0$.

If the condition $\sqrt{2}\,l/L \geq 4\pi$ is satisfied a third stationary state appears, which corresponds to $\Delta\varphi = \pm 4\pi$ (see Fig. 2). Thus even in the simplest case $\Delta\kappa = 0$ and $\delta = 0$, we may obtain both bistability and multistabi1ity of the system if only the medium is long enough, of the order of several conversion lengths L.

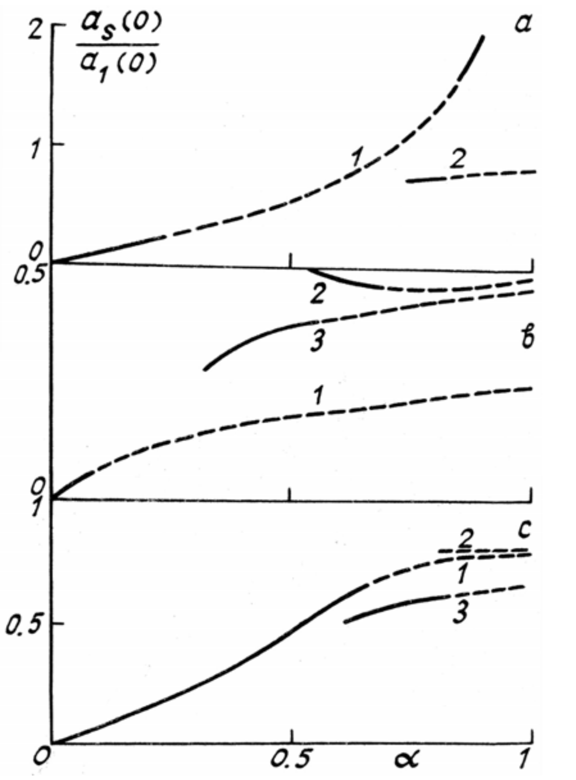

FIG. 1. The dependence of the SH stationary amplitude on the Q—factor of the cavity: a) $\delta = 0$, $\Delta = 0$, *and* $1/L = 4.4$; *b)* $\delta = 0.2$, $\Delta = 0$, $1/L = 5.5$; *c)* $\delta = 0$, $\Delta = 0.05$, $1/L = 4.4$; *Curve 1 corresponds to stationary amplitudes with a nonlinear phase change for one passage* $\Delta\varphi = \delta$, *curves 2 and 3 — to* $\Delta\varphi = \delta \pm 2\pi$ *(for* $\delta = 0$ *and* $\Delta = 0$, *the stationary states with* $\Delta\varphi = \pm 2\pi$ *feature identical amplitudes). The zone of stability and instability are shown by solid and dashed curves, respectively.*

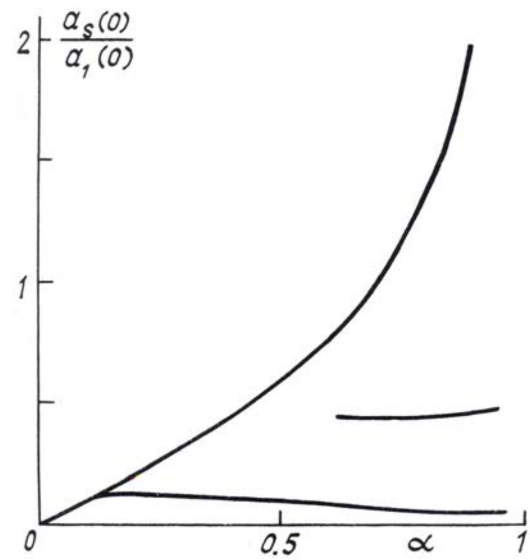

FIG. 2. The SH stationary amplitude as a function of the Q—factor of the cavity for $\delta = 0$, $\Delta = 0$, *and* $l/L = 10$.

b) The case of an untuned cavity ($\delta \neq 0$).

The stationary state with $n = 0$ and $\delta < \pi$ exists within a certain range of the parameter α, in which the nonlinear phase change for one passage through the cavity compensates for the linear phase change produced by detuning, i.e., $\Delta\varphi = \delta$. It is clear that this solution exists for arbitrary depth of the nonlinear medium, even for the small one ($1/L \ll 1$), when the approximation of the prescribed field holds. If $\delta > \pi$ the condition $\Delta\varphi_2 + \delta = 2\pi$ may be satisfied for arbitrary depth $1/L$. Moreover, the stationary amplitude will be the same as when $\delta' = \delta - \pi$. This happens because for the detunings of δ and δ' the values of $\Delta\varphi$ differ only in their sign, so that phase differences at the inlet of the medium are $\Psi(0)$ and $\pi - \Psi(0)$, and the amplitudes are equal to each other. If $\delta = \pi$ the phase difference in a stationary state is $\Psi(0) = -\frac{\pi}{2}$, and the amplitude may be calculated from expression (12) for $z_0 < 0$.

The depth, for which there exist two simultaneous solutions within a certain range of α, may be found from the condition $\Delta\varphi^{\Phi_1} + \delta \geq 2\pi$, where Eq. (C5) is used at the phase center Φ_1 ($\Delta\varphi > 0$). If $1/L$ is increased, so that the condition $\Delta\varphi^{\Phi_1} + \delta \geq -2\pi$ is satisfied, the third stationary state appears (Fig. 1*b*). It can be seen that for the values of the parameters given in this figure legend there exists a range of the Q–factors in which both stationary cases corresponding to $\Delta\varphi^{\Phi_1} + \delta \geq \pm 2\pi$ are simultaneously stable.

When the fixed point corresponding to the stationary state with $n = 0$ becomes unstable, there appears a pair of points *1* and *2* in the phase plane, such that if the system is at point *1* then after one passage it will be at point *2* and then returns back. In our next study we shall demonstrate that such a behavior results in the SHG pulses of rectangular shape.

c) The case of inexact phase synchronization ($\Delta\kappa \neq 0$, $\delta = 0$).

It follows from expression (4) that the single–pass problem has no solutions in the case in which the condition $\varphi_2(z) = \varphi_2(0)$ is satisfied. However, one may find a set of solutions satisfying the condition $\varphi_2(l) = \varphi_2(0)$ for arbitrary depth of the nonlinear medium, i.e., the linear phase change, produced by detuning from the exact synchronization, is completely compensated for by the nonlinear phase change. It means that the solution with $\Delta\varphi = 0$ may be constructed within a certain range of α. The depth at which a solution appears with the condition $\Delta\varphi = \pm 2\pi$, may be estimated using (B5) once again. As demonstrated by the numerical computations, a change in the type of stability of the stationary state with $\Delta\varphi = 0$ may occur at $\Delta\kappa \neq 0$: a "knot" is transformed into a "focus". In this case, there exists a range of the (Q–factor values for which the stationary states are found with $\Delta\varphi = \pm 2\pi$ (Fig. 1*c*, the upper stationary state is stable for $0.8 \leq \alpha \leq 0.81$) and also with $\Delta\varphi = 0$ and $\Delta\varphi = -2\pi$.

CONCLUSION

Thus the use of the SHG cavity system not only results in higher efficiency of conversion of radiation but also produces, new stationary and nonstationary regimes. In particular, the system may be bistable or multistable, and under certain conditions, interrelating the system parameters, the stationary states may generally not exist. At low values of $1/L$ ($L = 1/\sigma\, a_1(0)$ is the conversion distance) using a cavity of arbitrary Q–factor does not result in new regimes. This is explained by the fact that the increase of the intensity of the second harmonic does not increase the nonlinear phase change (due to the cavity effect) which would be enough for the appearance of a second (new) solution. There exists a certain critical (threshold) value of $1/L$, after it being exceeded, new solutions appear in the schemes with cavities of high enough (Q–factors. Following a further increase in nonlinearity, the nonlinear phase change may become a multiple of 2π, which leads to multistable solutions. However, for a fixed value of $1/L$ above the critical one additional solutions appear only after the Q–factor of the cavity reaches its critical value. The regimes may be used in schemes of the SH modulation. One of the variants of these regimes is quite special: it is that of the periodically repeated rectangular pulses. It will be considered elsewhere.

APPENDIX A

The variables $e_1 > e_2 > e_3$ in expressions (1)–(3) denote the roots of the cubic equation

$$t^3 + Bt + D = 0 \qquad \text{(A1)}$$

with the coefficients

$$B = -\frac{1}{3}\left(2 + \Delta^2\right)^2 + 1 + 2\Delta C,$$

$$D = \frac{1}{3}\left(2 + \Delta^2\right)(1 + 2C\Delta) - \frac{2}{27}\left(2 + \Delta^2\right)^3 - C^2. \qquad \text{(A2)}$$

The roots of Eq. (Al) may be expressed in the form proposed by Cardano[7]:

$$e_1 = 2\left(-\frac{B}{3}\right)^{1/2}\cos\frac{\beta}{3}, \quad e_{2,3} = -2\left(-\frac{B}{3}\right)^{1/2}\cos\left(\frac{\beta}{3} \pm \frac{\pi}{3}\right), \qquad \text{(A3)}$$

where

$$\cos\beta = -D/2\left(-\frac{B}{3}\right)^{3/2}. \qquad \text{(A4)}$$

We give the values of those roots for several particular cases. For the separatrix when $C = \Delta$

$$e_1 = e_2 = \frac{1 - \Delta^2}{3}, \quad e_3 = \frac{2 - 2\Delta^2}{3}. \qquad \text{(A5)}$$

When $C = 0$ (this condition gives the trajectory which intersects the origin of the coordinates)

$$e_{1,2} = \frac{2 + \Delta^2}{6} \pm \sqrt{\left(2 + \Delta^2\right)^2 - 4}, \quad e_3 = -\frac{2 + \Delta^2}{3}. \qquad \text{(A6)}$$

For phase centers

$$e_1^{\Phi_1,\Phi_2} = \frac{2}{9}\left[\Delta^2 + 3 \mp 2\Delta\sqrt{\Delta^2 + 3}\right],$$

$$e_2^{\Phi_1,\Phi_2} = e_3^{\Phi_1,\Phi_2} = -\frac{1}{9}\left[\Delta^2 + 3 \mp 2\Delta\sqrt{\Delta^2 + 3}\right]. \qquad \text{(A7)}$$

Apparently, if $\Delta\kappa = 0$ the roots for phase centers Φ_1 and Φ_2 coincide with each other.

APPENDIX B

In the case of arbitrary $\Delta\kappa$, phase changes of the pump wave and the SH with the distance is given by

$$\varphi_1(z) = \varphi_1(0) - \Delta\sigma Ez - \frac{C - \Delta}{\sqrt{e_1 - e_3}\left(\frac{1 - \Delta^2}{3} - e_3\right)} \times$$

$$\times \left[\Pi\left(m_1;\ \sqrt{e_1 - e_3}\,\sigma Ez + z_0 | m\right) - \Pi(m_1;\ z_0|m)\right], \quad \text{(B1)}$$

$$\varphi_2(z) = \varphi_2(0) + \Delta\sigma Ez - \frac{C}{\sqrt{e_1 - e_3}\left(\frac{2 + \Delta^2}{3} + e_3\right)} \times$$

$$\times \left\{ \frac{(e_1 - e_2)\left(\frac{2 + \Delta^2}{3} + e_3\right)}{\left(\frac{2 + \Delta^2}{3} + e_2\right)\left(\frac{2 + \Delta^2}{3} + e_1\right)} \times \right.$$

$$\times \left[\Pi\left(m_2;\ \sqrt{e_1 - e_3}\,\sigma Ez + z_0 | m\right) - \Pi(m_2;\ z_0|m)\right] +$$

$$+ \frac{\frac{2 + \Delta}{3} + e_3}{\frac{2 + \Delta^2}{3} + e_1}\sqrt{e_1 - e_3}\,\sigma Ez +$$

$$+ \frac{\left(\frac{2 + \Delta^2}{3} + e_3\right)^{1/2} (e_1 - e_3)^{1/2}}{\left(\frac{2 + \Delta^2}{3} + e_2\right)^{1/2} \left(\frac{2 + \Delta^2}{3} + e_1\right)^{1/2}} +$$

$$\times \left[\mathrm{arcot}\left[\left[\frac{(e_2 - e_3)^2 \left(\frac{2 + \Delta^2}{3} + e_1\right)}{(e_1 - e_3)\left(\frac{2 + \Delta^2}{3} + e_2\right)\left(\frac{2 + \Delta^2}{3} + e_3\right)} \right]^{1/2} \times \right.\right.$$

$$\times \frac{\mathrm{sn}\left(\sqrt{e_1 - e_3}\,\sigma Ez + z_0 | m\right)}{\mathrm{dn}\left(\sqrt{e_1 - e_3}\,\sigma Ez + z_0 | m\right)} \times$$

$$\left. \times\, \mathrm{cn}\left(\sqrt{e_1 - e_2}\,\sigma Ez + z_0 | m\right) \right] -$$

$$- \mathrm{arcot}\left[\left[\frac{(e_2 - e_3)^2 \left(\frac{2 + \Delta^2}{3} + e_1\right)}{(e_1 - e_3)\left(\frac{2 + \Delta^2}{3} + e_2\right)\left(\frac{2 + \Delta^2}{3} + e_3\right)} \right]^{1/2} \times \right.$$

$$\left.\left.\left. \times \frac{\mathrm{sn}(z_0|m)\ \mathrm{cn}(z_0|m)}{\mathrm{dn}(z_0|m)} \right]\right]\right\}, \quad \text{(B2)}$$

where m is the parameter of the elliptic functions and the parameters of the elliptic integral $\Pi(m_{1,2};\ t/m)$ are given by

$$m_1 = \frac{e_2 - e_3}{\frac{1 - \Delta^2}{3} - e_3}, \quad m_2 = m\,\frac{\frac{2 + \Delta^2}{3} + e_1}{\frac{2 + \Delta^2}{3} + e_2}. \quad \text{(B3)}$$

The latter change within $0 \le m_1 \le m$, $m \le m_2 \le 1$. When $\Delta\kappa = 0$ and $C = 0$ ($\Psi(0) = \pm\pi/2$ or $a_2(0) = 0$) the solution (1) is transformed into the well–known solution in the form of a hyperbolic tangent.[4] As $\Psi(0) \to \pm\pi/2$, the behavior of the SH phase may be found expanding Eq. (B2) in a small parameter $|C| \ll \frac{2}{3\sqrt{3}}$:

$$\varphi_2(z) \approx \varphi_2(0) - C\sigma Ez -$$

$$- \mathrm{sign}(C)\left[\mathrm{arcot}\left[\frac{\mathrm{th}(\sigma Ez \pm z_0)}{|C|}\right] \mp \mathrm{arcot}\left[\frac{\mathrm{th} z_0}{|C|}\right]\right]. \quad \text{(B4)}$$

In the limiting case $\Psi \to -\pi/2$ Eq. (B4) yields a phase discontinuity by π at this point in the medium where the SH amplitude turns to zero. Except for this point the phases of the radiation with fundamental frequency radiation and the SH are constant for the solution corresponding to the initial condition $C = 0$. The functions $\varphi_1(z)$ and $\varphi_2(z)$ are linear functions of z for the case of $\Delta\kappa = 0$ and for $\Delta\kappa \ne 0$:

$$\varphi_1(z) = \varphi_1(0) - \Delta\sigma E^{\Phi} z - \frac{C^{\Phi} - \Delta}{\frac{1 + \Delta^2}{3} - e_3^{\Phi}}\,\sigma E^{\Phi} z,$$

$$\varphi_2^{\Phi}(z) = \varphi_2(0) + \Delta\sigma E^{\Phi} z - \frac{C^{\Phi}}{\frac{2 + \Delta^2}{3} + e_3^{\Phi}}\,\sigma E^{\Phi} z, \quad \text{(B5)}$$

and the phase difference $\Psi(z)$ remains constant.

APPENDIX C

To determine the matrix T, it is necessary to relate the minor deviations $\delta a = a_2(0) - a_s(0)$, $\delta\varphi = \varphi_2(0) - \varphi_s(0)$ at the inlet of the nonlinear medium and the respective deviations at the outlet. To do this, we compute the first terms in the expansions of expressions (1) and (3) over δa and $\delta\varphi$. Using expressions (8), (9), and (10) we find the elements of the matrix

$$T_{11} = \alpha^2\left\{-\frac{\Delta^2}{3}(1 - V(l)) + \frac{a_1^2(0)}{E^2}V(l) + \frac{a_s^2(l)}{E^2} + \right.$$

$$+ E^2\frac{x_3}{2a_s(0)}(1 - V(0)) +$$

$$+ E^2\frac{x_2 - x_3}{2a_s(0)}\left[\mathrm{sn}^2\left(\sqrt{e_1 - e_3}\,\sigma El + z_0|m\right) - \right.$$

$$\left. - V(l)\,\mathrm{sn}^2(z_0|m)\right] + W(l)\sigma El\left(1 + \frac{E^2}{2a_s(0)}\frac{x_1 - x_3}{e_1 - e_3}\right) -$$

$$-\frac{E^2 W(1)\, I(1)}{2a_s(0)\sqrt{e_1-e_3}}\left[\frac{x_2-x_3}{e_1-e_3}-m\,\frac{x_1-x_3}{e_1-e_3}\right], \qquad \text{(C1)}$$

$$T_{12}=\alpha^2E^2\left\{\frac{y_3}{2a_s(0)}\,(1-V(1))+\right.$$

$$+W(1)\,\frac{\sigma E1}{2a_s(0)}\,\frac{y_1-y_3}{e_1-e_3}+$$

$$+\frac{y_2-y_3}{2a_s(0)}\left[\mathrm{sn}^2\left(\sqrt{e_1-e_3}\,\sigma E1+z_0|m\right)-\mathrm{sn}^2(z_0|m)\times\right.$$

$$\left.\times V(1)\right]-\frac{W(1)\, I(1)}{2a_s(0)\sqrt{1_1-1_3}}\left[\frac{y_2-y_3}{e_1-e_3}-m\,\frac{y_1-y_3}{e_1-e_3}\right], \qquad \text{(C2)}$$

$$T_{21}=\sigma E^3C\left\{\frac{2a_s(0)}{3}\,\Delta^2\int_0^1\frac{V(z)-1}{a_s^4(z)}\,dz\;-\right.$$

$$-\frac{1}{a_s(0)}\int_0^1\frac{dz}{a_s^2(z)}+E^2x_3\int_0^1\frac{1-V(z)}{a_s^4(z)}dz+E^2(x_2-x_3)\times$$

$$\times\int_0^1\frac{\mathrm{sn}^2\left(\sqrt{e_1-e_3}\,\sigma Ez+z_0|m\right)-\mathrm{sn}^2(z_0|m)V(z)}{a_s^4(z)}+$$

$$+2E\sigma\int_0^1\frac{W(z)z}{a_s^4(0)}\,dz\left[a_s(0)+\frac{E^2}{2}\,\frac{x_1-x_3}{e_1-e_3}\right]+$$

$$+2\,\frac{a_s(0)a_1^2(0)}{E^2}\int_0^1\frac{V(z)}{a_s^4(z)}\,dz+2\,\frac{a_s(0)}{E^2}\int_0^1\frac{dz}{a_s^2(z)}-$$

$$\left.-\frac{E^2}{\sqrt{e_1-e_3}}\int_0^1\frac{W(z)\, I(z)}{a_s^4(z)}\,dz\left[\frac{x_2-x_3}{e_1-e_3}-m\,\frac{x_1-x_3}{e_1-e_3}\right]\right\}, \qquad \text{(C3)}$$

$$T_{22}=E^3C\left\{\mathrm{tg}\Psi_s(0)\int_0^1\frac{dz}{a_s^2(z)}+E^2y_3\int_0^1\frac{1-V(z)}{a_s^4(z)}dz+\right.$$

$$+E^2(y_2-y_3)\times$$

$$\times\int_0^1\frac{\mathrm{sn}^2\left(\sqrt{e_1-e_3}\,\sigma Ez+z_0|m\right)-V(z)\mathrm{sn}_0^2(z_0|m)}{a_s^4(z)}dz-$$

$$\left.-\frac{E^2}{\sqrt{e_1-e_3}}\int_0^1\frac{W(z)\, I(z)}{a_s^4(z)}\,dz\left[\frac{y_2-y_3}{e_1-e_3}-m\,\frac{y_1-y_3}{e_1-e_3}\right]\right\}, \qquad \text{(C4)}$$

where

$$W(z)=\frac{a_{1s}^2(z)a_s(z)}{E^3}\,\sin\Psi_s(z),\quad V(z)=\frac{W(z)}{W(0)}.$$

The limits of the integral

$$I(z)=\int_{\xi(0)}^{\xi(z)}\frac{\sin^2\theta\, d\theta}{(1-m\sin^2\theta)^{3/2}}$$

are found as follows:

$$\xi(z)=\arcsin\left[\mathrm{sn}\left(\sqrt{e_1-e_3}\,\sigma Ez+z_0|m\right)\right].$$

We denote the coefficients proportional to δa and $\delta\varphi$ by x_1 and y_1 which appear when we expand the roots (A3) of Eq. (A1):

$$x_1=-\frac{1}{3}\,\frac{F^a}{(-B_s/3)^{1/2}}\cos(\beta_s/3)-\left[\frac{1}{3}\,\frac{P^a}{(-B_s/3)}+\right.$$

$$\left.+\frac{1}{6}\,\frac{D_sF^a}{(-B_s/3)^2}\right]\sin(\beta_s/3)/\sqrt{1-D_s^2/4(-\beta_s/3)^3}\,.$$

$$x_{2,3}=\frac{1}{3}\,\frac{F^a}{(-B_s/3)^{1/2}}\cos(\beta_s/3\pm\pi/3)+\left[\frac{1}{3}\,\frac{P^a}{(-B_s/3)}+\right.$$

$$\left.+\frac{1}{6}\,\frac{D_sF^a}{(-B_s/3)^2}\right]\sin(-\beta_s/3\pm\pi/3)/\sqrt{1-D_s^2/4(-\beta_s/3)^3}. \qquad \text{(C5)}$$

The coefficients B_s and D_s in expression (C5) are the same as the coefficients in expression (A2) corresponding to Eq. (A1) and β_s is determined by expression (A5). Finally,

$$F^a=2\Delta\left[a_1^2(0)\,\frac{a_1^2(0)-2a_s^2(0)}{E^5}\cos\Psi(0)+\Delta a_s(0)\times\right.$$

$$\left.\times\frac{2a_1^2(0)-a_s^2(0)}{E^4}\right]+\frac{a_s(0)}{E^2}\left[\frac{4}{3}\,\Delta^2(2+\Delta^2)-2\Delta C\right],$$

$$P^a=\left[\frac{2}{3}(2+\Delta^2)\Delta-2C\right]\left[a_1^2(0)\,\frac{a_1^2(0)-2a_s^2(0)}{E^5}\times\right.$$

$$\left.\times\cos\Psi(0)+\Delta a_s(0)\,\frac{2a_1^2(0)-a_s^2(0)}{E^4}\right]+\frac{a_s(0)}{E^2}\times$$

$$\times\left[-\frac{2}{3}(2+\Delta^2)\Delta C-\frac{2}{3}\Delta^2(1+2\Delta C)+\frac{12}{27}\Delta^2(2+\Delta^2)^2\right].$$

The coefficients y_1 are obtained by substituting F^a in expression (C5) by F^φ and P^a by P^φ, where

$$F^{\varphi} = 2\Delta \frac{a_1^2(0)\ a_s(0)}{E^3} \sin\Psi(0),$$

$$P^{\varphi} = \left[\frac{2}{3}\Delta(2 + \Delta^2) - 2C\right] \frac{a_1^2(0)\ a_s(0)}{E^3} \sin\Psi(0).$$

REFERENCES

1. P.D. Drummond, K.J. McNeil, and D.F. Walls, Opt. Commun. **28**, No. 2, 255–258 (1979).
2. K.J. McNeil, P.D. Drummond, and D.F. Walls, Opt. Commun. **27**, No. 2, 292–294 (1978).
3. V. Zehnle and P. Mandel, Opt. Commun. **66**, No. 4, 216–218 (1988).
4. N. Blombergen, *Nonlinear Optics* [Russian translation] (Mir, Moscow, 1966).
5. A.V. Gainer, N.P. Konoplyova, and G.I. Surdutovich, "*General properties of the process of second–harmonic generation with arbitrary initial conditions*," Preprint No. 448, Institute of Automatics and Electrometry, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk.
6. S.A. Akhmanov and R.V. Khokhlov, "*Problems of non–linear optics*," VINITI, Moscow (1964).
7. G.A. Korn and T.M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw–Hill, New–York, 1961).