ANALYSIS OF DYNAMIC CORRECTION OF ATMOSPHERIC ABERRATIONS WITH THE HELP OF FLEXIBLE MIRRORS

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Received June 1, 1990

A correction of nonstationary atmospheric aberrations of the optical system with phase conjugation taking into account the dynamic characteristics of flexible mirrors has been examined. Suboptimal laws of control with the help of mirrors in correcting the phase distortions on the basis of the first ten Zernike polynomials have been analyzed.

An increase in "the speed of adaptive optical systems operating under conditions of the turbulent atmosphere is the urgent problem.1 One of the methods of solving this problem consists in the improvement of the characteristics of control circuit for flexible mirrors.

"Quasistatical" and dynamic correction systems[2] are distinguished depending on characteristic time of change in atmospheric aberrations. In the first case an investigation of static deformations of mirrors is enough, but in the second it is necessary to regard the wavefront corrector as an inertial system with distributed parameters.[4,5] Owing to the complexity of the latter problem, the algorithms for control with the help of flexible mirrors and dynamic characteristics of the adaptive optical systems are often studied independently. The dynamic behavior of atmospheric aberrations is usually modelled with the help of the Zernike polynomials,[6–8] and the control system is constructed for the problem of damping the eigenmodes of oscillations of a flexible mirror.[5]

This paper is concerned with-the analysis of the dynamic correction of atmospheric aberrations with the help of flexible mirrors in adaptive systems withrphase conjugation. The following questions are studied: 1) numerical calculation of the linear forming filter for the description of dynamics of phase distortions; 2) choice of a wavefront corrector model; 3) synthesis of the suboptimal control with the help of adaptive mirrors.

1. Let us consider the first problem. The correlation functions of the expansion coefficients $\xi_1(t)$ of the wavefront $\varphi(r, t)$ in the Zernike polynomials $Z_1(r)$ for the Kolmogorcv model of atmospheric turbulence are defined by the following well–known expressions:[8]

$$K_1(\tau) = \langle\xi_1(t)\ \xi_1(t + t_0)\rangle = c_1\left[\frac{D}{r_0}\right]^{5/3} \times$$

$$\times \int_0^{\infty}\left[J_0(\nu\tau) \pm (-1)^m\cdot J_{2m}(\nu\tau)\ \cos 2m\psi\right] J^2_{n+1}(\nu)\nu^{-14/3}d\nu,\quad(1)$$

where $\tau = 2Vt_0/D$, D is the aperture diameter, (V, Ψ) are the polar coordinates of an average velocity of inhomogeneities transfer, r_0 is Fried's correlation radius, J_k are the kth order Bessel functions, C_1 are the normalization constants, n is the order of the polynomial, m is its angular frequency, the sign "+" corresponds to even i, and the sign "–" – to odd.

The expressions (1) makes it impossible to use the methods for design of the optimal linear tracking systems.[9] Therefore with an account of the typical profile of the functions $K_1(\tau)$ we shall approximate them by the following expressions:

$$\tilde{K}_1(\tau) = \langle\xi_1^2\rangle \exp(-\alpha_1\tau),\ i \le 3, \quad (2)$$

$$\tilde{K}_1(\tau) = \langle\xi_1^2\rangle\exp(-\alpha_1\tau)\left[\cos\gamma_1\tau + \frac{\alpha_1}{\gamma_1}\sin\gamma_1\tau\right], i > \xi \quad (3)$$

that provide for the adequate accuracy of approximation and are relevant for the linear forming filters of the lowest orders[10]

$$\dot{\xi}_1 + \alpha_1\xi_1 = \xi_1\ i \le 3,$$

$$\ddot{\xi}_1 + 2\alpha_1\dot{\xi}_1 + (\alpha_1^2 + \gamma_1^2)\xi_1 = \zeta_1, \qquad i > 3. \quad (4)$$

Here $\dot{\xi}_1 = \frac{d}{dt}\xi_1$ and ξ_1 is the white noise with the spectral functions

$$S_1^{\zeta} = 2\langle\xi_1^2\rangle\alpha_1(\alpha_1^2 + \gamma_1^2)/\pi, \qquad i > 3, \quad (5)$$

We shall determine the constant coefficients α_1 in Eq. (2) by minimizing the errors ε_1 for some time intervals $[0, T_1]$.

$$\varepsilon_1 = \int_0^T K_1(\tau) - \tilde{K}_1(\tau)]^2 d\tau / \int_0^T K_1^2(\tau)d\tau. \quad (6)$$

In addition we shall assume that the difference between the functions K_1 and $\tilde{K}_1$ for $\tau > T_1$ affects only the lowest frequencies of atmospheric aberrations, which do not make significant contribution to the dynamic error of an adaptive system. We shall calculate the coefficients α_1 and γ_1 in the Eq. (3) according to the well–known method[10]: by setting $K_1(T_{11}) = \tilde{K}(T_{11}) = 0$ for the first root T_{11}, we shall derive $\alpha_1 = -\gamma_1 \operatorname{ctg} \gamma_1 T_{11}$; the remaining parameter γ_1 is determined by solving the problem of minimizing the errors (6).

The calculated values of the coefficients α_1 and γ_1 in the case in which the direction of the velocity of inhomogeneities transfer is aligned with the axis $\Psi = 0$ are presented in Table I, where the designation $\sigma_1 = \langle\xi_1^2\rangle(r_0/D)^{5/3}$ is used.

0235-6880/91/02 131–4 $02.00 © 1991 Institute of Atmospheric Optics

TABLE I.

i	2	3	4	5	6	7	8	9	10
σ_1	0. 449	0.449	0.023	0.023	0.023	0.006	0.006	0.006	0.006
α_1	0. 13	0.08	1.64	1.02	2.62	1.94	0.99	2.49	0.99
γ_1	–	–	1.88	2.04	0	1.80	3.79	3.24	3.28
T_{11}	–	–	1.2	1.0	–	1.3	0.5	0.7	0.7
T_1	20	20	4	4	4	2	2	2	2
ε_1	0.09	0.02	0.003	0.02	0.02	0.002	0.05	0.09	0.03

2. Let us turn to the problem of description of the adaptive system model with phase conjugation. We shall study the system in which the measuring device performs the expansion of the residual spatial error $\varphi - \tilde{W}$ ($\tilde{W}$ is the correcting function) in a series of the first ten Zernike polynomials Z_1 :

$$\varphi(r,\ t) - \tilde{W}(r,\ t) \simeq \sum_{1=1}^{10} a_1(t) Z_1(r),$$

where a_1 are the measured signals. We shall assume that the correction of the average phase and the average wavefront tilts to be performed with an adequate accuracy with the help of an additional mirror which has higher more high resonance frequencies in comparison with the frequencies of the flexible mirror. In this case we shall represent the correcting function $\tilde{W}$ in the form $\tilde{W} \simeq \sum_{i=1}^{3} a_1(t) Z_1(r) + W(r,t) + W(r,\ t)$, where W is the surface of the circular flexible mirror described by the well-known equation[2]

$$\left[\sigma_m \frac{\partial^2}{\partial t^2} + 2\mu \frac{\partial}{\partial t} + L\right] W(r,\ t) = q(r,\ t). \tag{7}$$

Here σ_m is the mass of a unit surface, μ is the damping coefficient, q is the controlling load, and L is the operator of statical deformation of the mirror. For plate mirrors $L = G_m \nabla^4$, where G_m is the cylindrical stiffness. In addition for $i \le 10$ the conditions $Lz_i = 0$ are satisfied; therefore it is expedient to take the controlling actuators out of the limits of the working aperture Ω when the problems of the optimal quasistatical correction of the lowest–order aberrations are solved.[3] Such a position of the actuators makes it impossible to realize the effective suppression of high spatial frequencies of the mirror oscillations[5]; therefore it is expedient to use the passive methods of damping the highest–order modes of the oscillations in systems of correction of the lowest–order aberrations. In addition we shall assume that the band of working frequencies of the system $\Delta\nu$ does not exceed the lowest eigenfrequency of the mirror λ_1

$$\Delta\nu < \lambda_1 \tag{8}$$

In this case we shall represent the controlling load in the form $q(r,\ t) = \sum_{i=1}^{10} U_1(t) q_1(r)$, where U_1 are the controlling signals, $q_1(r)$ is the statistical distribution of the forces exerted by the actuators in the process of approximating the polynomials $Z_1(r)$($U_1^{st} = 1$, $LZ_1 \simeq q_1$, and $r \in \Omega$). Since the circular polynomials $Z(i)$ ($I = 4$, 10) are independent harmonic functions of the angle θ (Ref. 1), the correction channels corresponding to them may be studied separately taking into account only oscillations modes $V_j(r)$ with the same angular dependence in the expansions of the functions $W(r,\ t)$ and $q_1(r)$ ($i \le 10$). In so doing, Eq. (7) for every control channel may be rewritten in the form (we shall omit the index i everywhere)

$$\sigma_m \bar{f}_j + 2\mu \dot{f}_j + \lambda_j^2 f_j = u(t) C_j \lambda_j^2, \qquad j = \overline{1,\ l}; \qquad l \gg 1, \tag{9}$$

where $\dot{f}_j = \frac{d}{dt} f_j$, $w \simeq \sum_{j=1}^{l} f_j(t) v_j(r)$, $q \simeq \sum_{j=1}^{l} C_j \lambda_j^2 v_j(r)$,

$$Lv_j = \lambda_j^2 v_j.$$

It is quite clear that in the quasistatical approximation the coefficients C_j are equal to the values f_j^{st} ($U_{st} = 1$). Therefore we shall determine them by minimizing the error of the static approximation $\|Z - \sum_{m=1}^{3} K_m Z_m - \sum_{j=1}^{l} C_j V_j\|^2$ on the working aperture (here the coefficients K_1 are introduced in order to distinguish the aberrations Z_m, $m \le 3$). The simplest expressions for C_j correspond to the case in which the mirror zone coincides with Ω (for example, in the plate mirrors with free edge). With an account of orthonormality of the oscillation modes[2] $(V_1,\ V_j) = \delta_{ij}$ we can derive

$$C_j = (z,\ V_j) + + \frac{1}{S_0} \sum_{m=1}^{3} (Z_m, V_j) \frac{\sum_{i=1}^{l} (Z, V_i)(Z_m, V_i)}{1 - \frac{1}{S_0} \sum_{i=1}^{l} (Z_m, V_i)^2}, \tag{10}$$

where $(Z_i,\ Z_j) = S_0\, \delta_{ij}$ and $S_0 = \pi D^2/4$.

Thus, equation (9) and the measurable error of correction

$$a(t) = \xi(t) - f(t), \quad f(t) = \frac{1}{S_0} \sum_{1=1}^{l} f_1(Z,\ v_1), \tag{11}$$

define the linear model system for compensation of the function $\xi(t) \cdot Z(r)$.

3. Let us study the problem of synthesis regulator $U(a)$. According to the methods of the linear system theory for automatic control we shall change over from the equations (9) and (11) to their spectral representation

$$f(s) = \tilde{u}(s)/A_1(s), \quad \tilde{u}(s) = u(s)\left[1 + \sum_{j=2}^{l} A_1(s)/A_j(s)\right], \tag{12}$$

where $A_1(s) = (\sigma_m s^2 + 2\,\mu s + \lambda^2_j)\, S_0/C_j \lambda_j^2\, (z,\ V_j)$, s designates the Laplacian operator. It is expedient to distinguish the transfer function $A_1(s)$ owing to the foregoing assumption (8). If this assumption is correct, the resonance factor may be taken into account only in this term. Meanwhile, the synthesized system will appear to be optimal only with respect to the lowest-order modes of every channel.

Then the equations of the tracking system (4) and (11) will assume the form

$$A_1(s)\, a \simeq -\tilde{u} + A_1(s)\xi, \quad B(s)\xi = \zeta, \tag{13}$$

where $B(s) = (\kappa s)^2 + 2\alpha(\kappa s) + \alpha^2 + \gamma^2$, $\kappa = 2V/D$, and ξ is the white noise with the spectral density S_ξ = const.

It is well known[9] that the regulator which minimize the functional

$$J = \lim_{T \to \infty} \frac{1}{T} \int_0^T (M^2 a^2 + \tilde{u}^2) dt = M^2 \langle a^2 \rangle + \langle \tilde{u}^2 \rangle, \tag{14}$$

with the weighting factor M^2 is defined by the expression

$$\tilde{u}^{\bullet} = -D^{\bullet}(s)a, \quad D^{\bullet}(s) = A_1(s)\left[1 - \frac{S_\zeta^{1/2}\,G(s)}{B(s)\,M_+(s)}\right], \qquad (15)$$

Here the polynomial $G(s)$ and the fractional function $M_+(s)$ are to be found with the use of conditions

$$M^2 + A_1(s)A_1(-s) = G(s)G(-s),$$

$$S_\zeta^{1/2}\frac{A_1(s)A_1(-s)}{B(s)\,G(-s)} = M_+(s) + M_-(s), \qquad (16)$$

where $M(s)$ is the proper fraction with poles in the right half-plane.

The regulator (15) was synthesized with an account of the well known spectral function of the correctable random process. In read situations such an information may be absent. In this case it is usually assumed that $B(s) = A_1(s)$. This leads us to the simpler expression for

$$D^{\bullet}(s) = A_1(s) - G(s). \qquad (17)$$

It should be noted that such a regulator corresponds to the stabilizer synthesis problem for the perturbations of the white noise type[9].

Expressions (15) and (16), that have been obtained here, satisfy the well-known conditions of stability for tracking systems with respect to the small perturbations of the parameters[9]. In addition, the variances of the values

$$\frac{\langle a^2\rangle}{S_\zeta} = \int_0^\infty \frac{d\omega}{|W_a(i\omega)|^2}, \quad \frac{\langle \tilde{u}^2\rangle}{S_\zeta} = \int_0^\infty \frac{|D^{\bullet}(i\omega)|^2}{|W_a(i\omega)|^2}d\omega, \qquad (18)$$

where $W_a(s) = [A_1(s) - D^*(s)]B(s)/A_1(s)$, can be easily represented in the analytic form[11] here.

We shall present the calculational results on the characteristics of the adaptive system with the flexible plate mirrors with free edge. The eigenfrequencies λ_j^2 and eigenforms $V_j(r)$ of the circular plate with the Poisson coefficient being equal to 0.33 are presented in the referencebook.[12] Table II includes the calculated values of $\frac{1}{S_0}(Z_i, V_j)\ (i = \overline{4, 10})$. Here, we have obtained the inequality $|1 - C_j/(Z, V_j)| \le 0.01$, which indicates that there exists a practical possibility of neglecting the second term in Eq. (10). It is evident from the Table that we may set $\tilde{U}(s) \simeq U(s)$ in Eq. (12) for operating frequencies of an adaptive system, which satisfy the condition by virtue of smallness of the values $C_j/(Z, V_j)\,S_0$ $(j \ge 2)$.

The calculated dependences of the relative correction error $\langle a^2\rangle/\langle\xi^2\rangle$ on the energy consumptions for the control $\langle\tilde{U}^2\rangle/\langle\xi^2\rangle$ at $\mu = 0$ are presented in Fig. 1. The solid curves correspond to the regulator (15), the dashed curves — to the regulator (17), and the parameter $K = k/[4\sigma_m^{1/2}/D^2G_z^{1/2}] = 0.5\,DV(\sigma_m/G_3)^{1/2}$. Hence we may draw two conclusions. First, taking into account the spectral functions of atmospheric correctable aberrations we can decrease the correction error approximately by a factor of 1.5–2.0. Second in these systems we may use optimal regulators of stabilization systems with some additional consumptions for a control process.

In conclusion we shall make some remarks about the synthesis of a control in correction systems with operating frequencies which exceed the lowest frequencies of oscillations of flexible mirrors. In this case for effective suppression of free resonance oscillations of a mirror it is necessary to place control actuators along the entire surface of a mirror and to synthesize a regulator for the starting model of system (9) and (11). The method for synthesis of optimal regulators according to the criterion of the generalized performance enables us to write the law of the control directly in the explicit form omitting all the intermediate calculations performed on the basis of standard techniques. It can be shown that for the problem of stabilizer synthesis the minimization of the functional of the generalized performance

$$J = \int_0^\infty \left\{\sum_{j=1}^{1}\left(K_{1j}^2 f_j^2 + 2\mu K_{2j}^2 \dot{f}_j^2\right) + u^2 + u^{\bullet 2}\right\}dt \qquad (19)$$

is achieved with the help of control in the following form:

$$u^{\bullet} = -\frac{1}{2}\sum_{j=1}^{1} c_j\left\{K_{1j}^2 f_j + K_{2j}^2(1 + \lambda_j^2)f_j\right\}, \qquad (20)$$

where k_{1j}^2 and k_{2j}^2 are the weighting coefficients.

TABLE II.

	$(D/2)\,(\lambda_j\,G_z^{1/2})^{1/2}$					$C_j(Z_i, V_j)/S_0$				
i/j	1	2	3	4	5	1	2	3	4	5
4	3,0	6,2	9,4	12,5	15,7	0,986	0.013	10^{-3}	$2{,}3\cdot10^{-4}$	$7{,}2\cdot10^{-5}$
5,6	2,3	5,9	9,2			0,998	0.0018	$8\cdot10^{-6}$		
7,8	4,6	7,7	10,9	14,1	17,2	0,939	0.047	$6{,}5\cdot10^{-3}$	$1{,}8\cdot10^{-3}$	$5{,}6\cdot10^{-4}$
9,10	3,5	7,3	10,6			0,992	0.0069	10^{-4}		

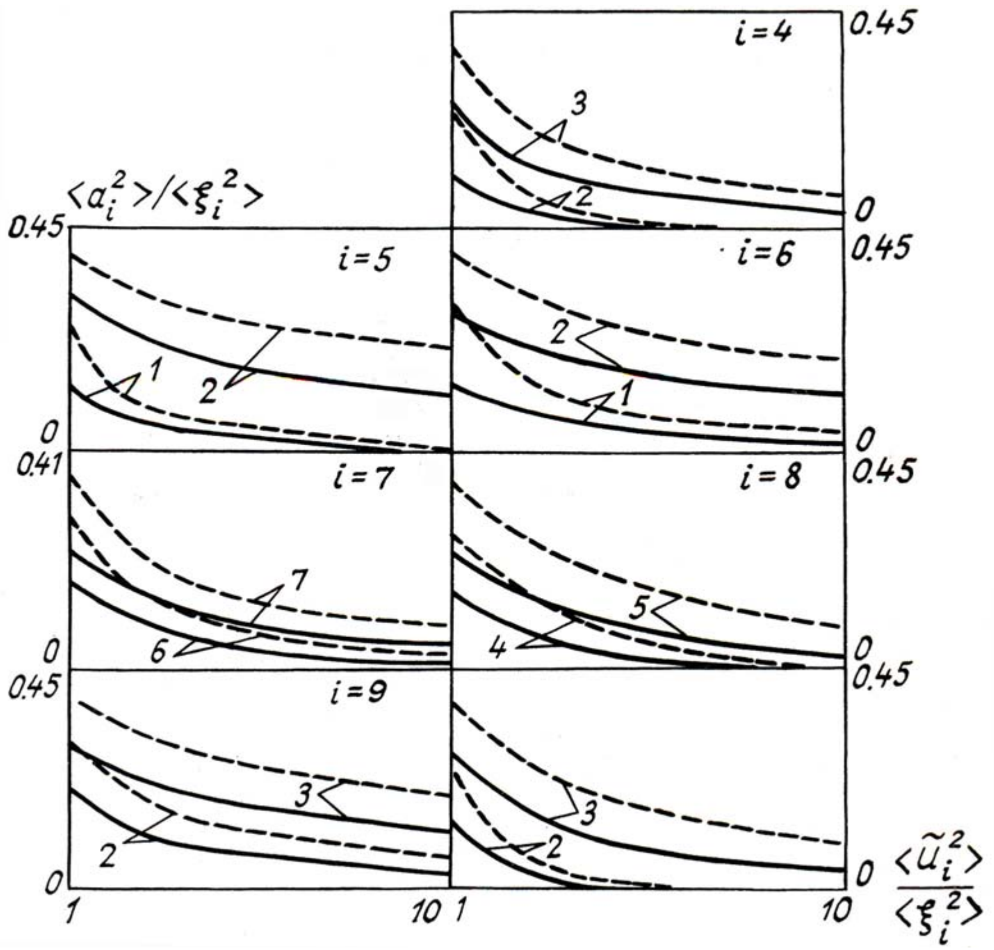

FIG. 1. Dependence of the relative correction error on the energy consumptions for the control process: 1) k = 3; 2) k = 5; 3) k = 7; 4) k = 10, 5) k = 12; 6) k = 15; 7) k = 18.

The methods of solving the more general problems of synthesis of optimal control with the help of minimization of functional (14) performed by linear tracking systems of the form (4) and (9) for the conditions of deficient observational information are described in Ref. 14. All these methods require solving the nonlinear matrix Riccati equation. This fact significantly increases the length of computations.

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