APPROXIMATION OF THE WAVEFRONT-PHASE STRUCTURE FUNCTION

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The real phase structure function for a light wave distorted by a turbulent atmosphere is approximated by a class of polynomials in ρ^2 with and without account of the outer turbulence scale. The approximations derived are used to evaluate the wavefront cored ion errors for membrane mirrors and assess the contribution of the Zernike polynomials and the correlation correction errors.

The statistical properties of the wavefront-phase light wave propagating through an isotropic, locally homogeneous turbulent atmosphere are expressed in terms of the real phase structure function $D_\varphi(\rho)$ (Ref. 1). The form of the structure function adopted appears to influence the computational complexity of the calculational scheme which is used in the solution of many problems of calculational atmospheric optics. To simplify the solution of these kinds of problems, use is often made of different approximations of D_φ (Refs. 2, 3). When describing large-scale, purely phase fluctuations, the approximations should provide for the possibility of estimating the effect of the outer turbulence scale on the results obtained.

It is the object of this paper to approximate $D_\varphi(\rho)$ by a set of polynomials in ρ^2 with and without account of the outer turbulence scale and to illustrate the approximations as applied to the solution of certain model problems.

The expression for the real phase fluctuation structure function accounting for the outer turbulence scale L_0 can be cast in the following form[1]

$$D_\varphi(r)=6.16\left[\frac{r}{r_0}\right]^{5/3}\int\limits_0^\infty\left[1-J_0(\kappa))(\kappa^2+\alpha^2r^2\right]^{-11/6}\kappa\,d\kappa \qquad (1)$$

where $r_0 = 1.68(c_n^2 zk^2)^{-3/5}$ is the correlation radius as given by Fried[4]; c_n^2 is the refractive index structure constant; z is the atmospheric path length; k is the wave number; J_0 is the zeroth-order Bessel function; $\alpha = 2\pi D/L_0$; D is the circular aperture diameter, $r = \rho/D$, and $r \in (0, 1)$.

Function (1) was examined in Refs. 1–5. For $\alpha = 0$ it has the form[4]:

$$D_{\varphi_0}(r) = 6.88\left[\frac{D}{r_0}\right]^{5/3} r^{5/3}.$$

The dependence of D_φ on a is plotted in Figure 1. To estimate the error due to the substitution of D_{φ_0} for expression (1) at $\alpha \neq 0$ we introduce the approximation $D'_\varphi = c\,D_{\varphi_0}$ (where the coefficient c is found by minimization of the normalized mean square error of the approximation of D_φ (1) by D'_φ on a unitary circular aperture) to yield

$$\delta = \int\limits_0^1\left[D_\varphi - D'_\varphi\right]^2 r\,dr \Big/ \int\limits_0^1 D_\varphi^2\, r\,dr \qquad (2)$$

Numerical computations indicate that the value α = 0.01, 01, 0.5 correspond to c = 0.923, 0.713, 0.443 and $\delta = 3.5\cdot10^{-4}$, $4.4\cdot10^{-3}$, 0.18. It leads us to suppose that, starting from $\alpha > 0.1$ the outer turbulence scale will have an effect on the solution of a number of atmospheric optics problems pertaining to the large-scale, purely phase fluctuations, especially over near-ground atmospheric paths.

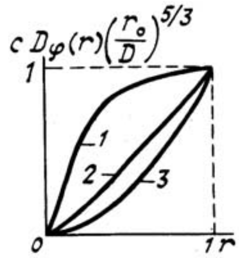

FIG. 1. The structure function.

The choice of the relevant approximation of the structure function is determined by its advantages in handling a definite class of stet problems. Problems involving phase fluctuation expansions over a chosen set of functions $f_1(r)$ (e.g., in adaptive optics[6]) often require the evaluation of quadruple Integrals of the form $\iint D_\varphi(r_1 - r_2) f_i(r_1) f_j(r_2) d^2r_1 d^2r_2$. The change-over to double integrals can be made through the following approximation

$$D_\varphi(r) = \left[\frac{D}{r_0}\right]^{5/3}\sum_{l=1}^{M} a_l\, r^{2l} \qquad (3)$$

where M is the number of expansion terms, and a_i are the expansion coefficients determined by minimization of error (2). The computational results obtained for $M \leq 5$ are summarized in Table 1 where ε_0 stands for the ensemble average of the mean-square wavefront error calculated by Fried's formulae[4] with $D_\varphi(r)$ from Eq. (1).

Table 1

α		0	0.1	1	6.28
$\langle\varepsilon\rangle(r_0/D)^{5/3}$		1.0299	0.7807	0.3598	0.0704
M=2	a_1	8.868	6.918	3.356	0.641
	a_2	-2.112	-2.317	-1.770	-0.508
	δ	$3.2\ 10^{-4}$	$7.5\ 10^{-4}$	$3.3\ 10^{-3}$	$4.4\ 10^{-2}$
M=3	a_1	9.942	8.010	4.350	1.094
	a_2	-5.683	-6.245	-5.058	-2.019
	a_3	2,676	2.946	2.455	1.133
	δ	$6.4\ 10^{-5}$	$1.5\ 10^{-4}$	$8.0\ 10^{-4}$	$1.6\ 10^{-2}$
M=4	a_1	10.788	9.024	5.159	1.576
	a_2	-10.795	-11.811	-9.911	-4.907
	a_3	11.620	12.687	10.948	6.187
	a_4	-4.768	-5.200	-4.530	-2.700
	δ	$1.9\ 10^{-5}$	$4.7\ 10^{-5}$	$2.7\ 10^{-4}$	$6.5\ 10^{-3}$
M=5	a_1	11.496	9.839	5.872	2.042
	a_2	-17.406	-19.421	-16.563	-9.262
	a_3	31.483	35.518	30.905	19.254
	a_4	-28.614	-32.583	-28.479	-18.376
	a_5	9.935	11.416	9.979	6.534
	δ	$6.7\ 10^{-6}$	$1.6\ 10^{-5}$	$1.2\ 10^{-4}$	$2.9\ 10^{-3}$

Let us consider specific examples of model problems whose solutions can be significantly simplified by the use of approximation (3). For instance, the relations[4] for the variances of the coefficients of expansion of the wavefront $\langle b_i^2\rangle$ in the Zernike polynomials on a circular aperture are now integrated by elementary functions. Calculated results for $\alpha = 0$ are displayed in Table 2. The case of $M = \infty$ is taken from Refs. 4, and 7. Changes in the coefficients $\langle b_i\rangle (r_0/D)$ with variation of a for approximation (3) with $M = 5$ are presented in Table 3.

Table 2

M		2	3	4	5	∞
$\langle\varepsilon\rangle(r_0/D)^{5/3}$		0.9985	1.0197	1.0263	1.0286	1.0299
$\langle b_i^2\rangle \times \left(\frac{r_0}{D}\right)^{5/3}$	i=2,3	0.466	0.458	0.453	0.451	0.449
	i=4,6	0.022	0.028	0.026	0.025	0.023
	i=7-10	0	0.0052	0.0078	0.0076	0.0062
	i=11-15	0	0	0.0019	0.0031	0.0024
	i=16-21	0	0	0	0.0008	0.0012

Table 3

	α \ i	2.3	4-6	7,8	9,10
z_i	0.1	0.919	0.0276	0.0084	0.0084
	1.	0.122	0.0214	0.0071	0.0071
	6.28	0.00114	0.00617	0.00358	0.00353
G_i	0.1	0.322	0.0305	0.0078	0.0088
	1.	0.126	0.0239	0.0059	0.0075
	6.28	0.0141	0.0080	0.00236	0.00880

Let us now consider the α-dependence in the analogous coefficients of the Karhunen-Loeve functions G_1 evaluated for Eq. (3) at $M = 4$ according to formulae from Ref. 7. Let G_1 be numbered just as the Zernike polynomials. Numerical computations show that approximation (3) enables us to derive the functions G_1 on a circular aperture in the form of cyclotomic polynomials with weakly correlated coefficients b_1. For $\alpha = 0$ the formulae from Ref. 7, yield $\langle\tilde{b}_i^2\rangle \approx \langle b_i^2\rangle$, $i = 2, \ldots, 10$, $\langle\tilde{b}_2\tilde{b}_8\rangle = \langle\tilde{b}_3\tilde{b}_7\rangle = 0.33$ $\langle\tilde{b}_8^2\rangle = 0.33$ $\langle\tilde{b}_7^2\rangle = -0.15$ $\langle b_2 b_8\rangle = -0.15$ $\langle b_3 b_7\rangle$ for structure Function (1). Changes in $\langle\tilde{b}_i^2\rangle (r_0/D)^{5/3}$ with variation of a for calculations based on structure function (3) are shown in Table 3.

We will now consider the calculation of the errors of the wave phase correction for a membrane mirror for $M = 2$ and $\alpha = 0$ (Ref. 6). The computations are carried out for two mirrors employing three actuators with $0.51R_a$ radii and four actuators with $0.42R_a$ radii, respectively. The actuator centers are uniformly distributed on a circle with a $0.6R_a$ radius, where $R_a = D/2$. The radius of the mirror is $1.5R_a$. The mirror response to the actuator action is derived via the well-known Green's function[8]. Replacing the quadruple integrals by their double counterparts reduces the computing time by a factor of ~ 70 for a 600-point numerical integration. The phase correction error is estimated to be $0.22(D/r_0)^{5/3}$ and $0.14(D/r_0)^{5/3}$, respectively, for the multiple-actuator mirror assemblies under consideration. In summary, applications of the real phase structure function approximations have been discussed. By virtue of the fact that the complex-phase structure function has a similar form[1], approximation (3) may also prove to be useful in this case.

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