ULTRASHORT PULSE PROPAGATION IN NONLINEAR DISPERSIVE MEDIA WITH ABSORPTION

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We present here an equation derived in the geometric optics approximation that describes the evolution of ultrashort pulses in an absorbing medium. This is the Burgers–Korteweg–de Vries (BKV) equation where the amplitude of the Poynting vector is taken as the unknown function. We have numerically simulated the evolution of picosecond pulses in a quartz optical wave guide where no absorption is present (the Korteweg–de Vries equation). During their evolution picosecond pulses are decomposed into femtosecond solitons (≈ 200 fs)*. Duration of the solitons is inversely proportional to the soliton amplitude to the power of* $1/2$*. In the case when no dispersion is present the width of the arising shock wave is proportional to the absorption coefficient and to the electromagnetic wave period, and inversely proportional to the radiation intensity. At the intensity of* 100–1000 W/cm^2*, the width of the shock wave front equals 100–1000 periods of the electromagnetic radiation wave.*

Development of some new leads in non-linear wave theory is stimulated by the present-day high level of investigations into the ultrashort light pulses (USLP) by the methods of non-linear optics. Theory of optical solitons and soliton lasers, generalization of the method of slow varying amplitudes to problems of femtosecond non-linear optics, search for new mathematical models of non-linear processes, as well as the development of mathematical techniques for solving problems of laser radiation self-action in a non-linear dispersive medium are the tasks of current concern.

The progress in experiments on the USLP formation in a non-linear dispersive medium has stimulated studies in the theory of non-linear wave equations. An overview of such studies may be found, for instance, in Refs. 1–4. In the first approximation, the propagation of short pulses is described by non-linear equation (NSE). In the quasi-optics approximation, the process can be described by the system of equations for the amplitude and phase of the field (Ref. 4, p. 470). Similar approach (the Poynting vector is more convenient for use in the vector-form approach) was used in Ref. 2, where the Korteweg–de Vries equation (KdV) was obtained for the amplitude of the Poynting vector from the vector wave equations for the electric **E** and magnetic **H** fields in the case of a non-linear absorption-free dispersive medium. In this paper we present similar equation derived for the absorbing medium. The equation is the Burgers–Korteweg–de Vries equation (BKV). Based on KdV, we simulate propagation of ultrashort pulses in an optical wave guide.

From the Maxwell equations for a non-absorbing dispersive medium, it is easy to obtain[2] wave equations in a vector form

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{E} = k^2\,\mathbf{E} + \left[i\,\frac{\partial k^2}{\partial\omega}\,\frac{\partial \mathbf{E}_0}{\partial t} - \frac{1}{2}\,\frac{\partial^2 k^2}{\partial\omega^2}\,\frac{\partial^2 \mathbf{E}_0}{\partial t^2} - \right.$$

$$\left. - \frac{i}{6}\,\frac{\partial^3 k^2}{\partial\omega^3}\,\frac{\partial^3 \mathbf{E}_0}{\partial t^3} \right] \exp\,(-i\omega t + i\Psi)\,, \quad (1)$$

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{H} = k^2\,\mathbf{H} + \left[i\,\frac{\partial k^2}{\partial\omega}\,\frac{\partial \mathbf{H}_0}{\partial t} - \frac{1}{2}\,\frac{\partial^2 k^2}{\partial\omega^2}\,\frac{\partial^2 \mathbf{H}_0}{\partial t^2} - \right.$$

$$\left. - \frac{i}{6}\,\frac{\partial^3 k^2}{\partial\omega^3}\,\frac{\partial^3 \mathbf{H}_0}{\partial t^3} \right] \exp\,(-i\omega t + i\Psi)\,,$$

where $k = n\omega/c$; $\omega = 2\pi/\lambda$, λ is the light wavelength in vacuum; Ψ is the wave phase; $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t + i\Psi)$; $\mathbf{H} = \mathbf{H}_0 \exp(-i\omega t + i\Psi)$, $\mathbf{E}_0$ and $\mathbf{H}_0$ are the amplitudes of the electric and magnetic fields slowly varying on the wavelength scale; n is the refractive index of the medium.

When deriving the BKV, let us first neglect the summands $\sim \partial^3(k^2)/\partial\omega^3$ in Eq. (1). In the case of an absorbing medium, **E** and **H** also satisfy Eq. (1), but the refractive index is a complex number $n = n' + in''$. In view of all the above reasonings, the first equation in Eq. (1) can be written as follows:

$$2ik'\left[\frac{\partial \mathbf{E}_0}{\partial s} + \frac{1}{2k'}\frac{\partial (k')^2}{\partial \omega}\frac{\partial \mathbf{E}_0}{\partial t} + \frac{\omega n''}{c}\mathbf{E}_0\right] +$$

$$+\left[\frac{\partial^2 \mathbf{E}_0}{\partial s^2} - \frac{1}{2}\frac{\partial^2 (k')^2}{\partial \omega^2}\frac{\partial^2 \mathbf{E}_0}{\partial t^2} - 4k'\frac{n''}{c}\frac{\partial \mathbf{E}_0}{\partial t}\right] -$$

$$- 2i\frac{k'n''}{\omega c}\frac{\partial^2 \mathbf{E}_0}{\partial t^2} = 0, \qquad (2)$$

$$2ik'\left[\frac{\partial \mathbf{H}_0}{\partial s} + \frac{1}{2k'}\frac{\partial (k')^2}{\partial \omega}\frac{\partial \mathbf{H}_0}{\partial t} + \frac{\omega n''}{c}\mathbf{H}_0\right] +$$

$$+\left[\frac{\partial^2 \mathbf{H}_0}{\partial s^2} - \frac{1}{2}\frac{\partial^2 (k')^2}{\partial \omega^2}\frac{\partial^2 \mathbf{H}_0}{\partial t^2} - 4k'\frac{n''}{c}\frac{\partial \mathbf{H}_0}{\partial t}\right] -$$

$$- 2i\frac{k'n''}{\omega c}\frac{\partial^2 \mathbf{H}_0}{\partial t^2} = 0, \qquad (3)$$

where $k = k' + ik''$; $k' = n'\omega/c$; $k'' = n''\omega/c$.

As seen from Eqs. (2) and (3), the solutions $\mathbf{E}_0$ and $\mathbf{H}_0$, as a first approximation, can be represented in the form

$$\mathbf{E}_0 = \tilde{\mathbf{E}}_0\left(t - \frac{1}{2k'}\frac{\partial (k')^2}{\partial \omega}s\right)\exp\left(-\frac{\omega}{c}n''s\right);$$

$$\mathbf{H}_0 = \tilde{\mathbf{H}}_0\left(t - \frac{1}{2k'}\frac{\partial (k')^2}{\partial \omega}s\right)\exp\left(-\frac{\omega}{c}n''s\right). \qquad (4)$$

Taking into account Eq. (4), the expression in the second square brackets of Eq. (2) can be written in the form

$$\frac{\partial^2 \mathbf{E}_0}{\partial s^2} - \frac{1}{2}\frac{\partial^2 (k')^2}{\partial \omega^2}\frac{\partial^2 \mathbf{E}_0}{\partial t^2} - 4k'\frac{n''}{c}\frac{\partial \mathbf{E}_0}{\partial t} =$$

$$= -k'\frac{\partial^2 k'}{\partial \omega^2}\frac{\partial^2 \mathbf{E}_0}{\partial t^2} - 2k'\frac{n''}{c}\frac{\partial \mathbf{E}_0}{\partial t} +$$

$$+\left(\frac{\omega}{c}n''\right)^2\mathbf{E}_0 - \mathbf{E}_0\delta(k')^2, \qquad (5)$$

where $\delta(k')^2$ is the correction to the square of the wave number $(k')^2$ whose value will be determined below; in deriving Eq. (5), we suppose that $\omega\frac{\partial k'}{\partial \omega} = k'$.

In Eq. (2), let us equate the terms containing real and complex coefficients to zero. Taking into account Eq. (5), we obtain

$$\frac{\partial \mathbf{E}_0}{\partial s} + \frac{1}{2k'}\frac{\partial (k')^2}{\partial \omega}\frac{\partial \mathbf{E}_0}{\partial t} + \frac{\omega n''}{c}\mathbf{E}_0 - \frac{n''}{\omega c}\frac{\partial^2 \mathbf{E}_0}{\partial t^2} = 0, \qquad (6)$$

$$\mathbf{E}_0\delta(k')^2 = -k'\frac{\partial^2 k'}{\partial \omega^2}\frac{\partial^2 \mathbf{E}_0}{\partial t^2} - 2k'\frac{n''}{c}\frac{\partial \mathbf{E}_0}{\partial t},$$

where the summands $\sim(n'')^2$ in Eq. (6) are omitted.

Similar calculations for Eq. (3) yield the system of equations for $\mathbf{H}_0$

$$\frac{\partial \mathbf{H}_0}{\partial s} + \frac{1}{2k'}\frac{\partial (k')^2}{\partial \omega}\frac{\partial \mathbf{H}_0}{\partial t} + \frac{\omega n''}{c}\mathbf{H}_0 - \frac{n''}{\omega c}\frac{\partial^2 \mathbf{H}_0}{\partial t^2} = 0, \qquad (7)$$

$$\mathbf{H}_0\delta(k')^2 = -k'\frac{\partial^2 k'}{\partial \omega^2}\frac{\partial^2 \mathbf{H}_0}{\partial t^2} - 2k'\frac{n''}{c}\frac{\partial \mathbf{H}_0}{\partial t}.$$

The equations (6) and (7) are obtained from Eqs. (2) and (3) as the first approximation which supposes that the wave number k' is determined only by the medium parameters. The solution $\mathbf{E}_0$, in the form of the equality (4), makes it possible to determine the correction to the value $\delta(k')^2$. This means that, in the second approximation, the wave number k' depends both on the medium parameters and on the profile of the initial signal. According to the perturbation theory, the expression $\frac{\partial[(k')^2 + \delta(k')^2]}{\partial \omega}$ should be replaced by $\frac{\partial (k')^2}{\partial \omega}$ in the second equation of the system (6) (the correction to other terms is small). Taking into account the above said, it follows from Eqs. (6) and (7) that

$$\mathbf{H}_0^* \times \frac{\partial \mathbf{E}_0}{\partial s} + \frac{\partial k'}{\partial \omega}\mathbf{H}_0^* \times \frac{\partial \mathbf{E}_0}{\partial t} + \frac{\omega n''}{c}\mathbf{H}_0^* \times \mathbf{E}_0 -$$

$$-\frac{n''}{\omega c}\left[\mathbf{H}_0^* \times \frac{\partial^2 \mathbf{E}_0}{\partial t^2} + \frac{\partial \mathbf{H}_0^*}{\partial t} \times \frac{\partial \mathbf{E}_0}{\partial t}\right] - \qquad (8)$$

$$-\left[\frac{1}{2k'}\frac{\partial}{\partial \omega}\left(k'\frac{\partial^2 k'}{\partial \omega^2}\right)\frac{\partial^2 \mathbf{H}_0^*}{\partial t^2} \times \frac{\partial \mathbf{E}_0}{\partial t}\right] = 0,$$

where Eq. (8) is derived under the assumption that

$$\frac{\partial k'}{\partial \omega} = \frac{k'}{\omega}.$$

Similar calculations for $\mathbf{H}_0$ with the allowance for the last summands in Eq. (1) yield the equation

$$\frac{\partial}{\partial s}\mathbf{S} + \frac{\partial k'}{\partial \omega}\frac{\partial}{\partial t}\mathbf{S} + \frac{\omega n''}{c}\mathbf{S} - \frac{n''}{\omega c}\frac{\partial^2}{\partial t^2}\mathbf{S} - \frac{1}{6}\frac{\partial^3 k'}{\partial \omega^3}\frac{\partial^3}{\partial t^3}\mathbf{S} = 0, \qquad (9)$$

where $\mathbf{S}$ is the Poynting vector:

$$\mathbf{S} = \frac{c}{16\pi}\left[\mathbf{E}_0 \times \mathbf{H}_0^* + \text{complex conjugated}\right].$$

When deriving the Eq. (9) we took into account that the inequality

$$\frac{\partial k'}{\partial \omega}\frac{\partial^2 k'}{\partial \omega^2} \ll k'\frac{\partial^3 k'}{\partial \omega^3}$$

holds in an optical wave guide.

Now, let us take into account the nonlinear contribution to the vector of electric induction (Ref. 7, p. 517):

$$\mathbf{D} = \varepsilon\mathbf{E} + \alpha S\mathbf{E}. \qquad (10)$$

It is easy to show[5] that Eq. (9) has an additional nonlinear summand in this case. Thus we obtain the BKV equation

$$\frac{\partial S}{\partial s}+\frac{\omega\mu\alpha}{2kc^2}S\frac{\partial S}{\partial t}-\frac{1}{6}\frac{\partial^3 k'}{\partial\omega^3}\frac{\partial^3 S}{\partial\tilde{t}^3}+2\frac{\omega n''}{c}S=\frac{n''}{\omega c}\frac{\partial^2 S}{\partial\tilde{t}^2}, \quad (11)$$

where the following designations are introduced:

$$\tilde{s} = s,\ \tilde{t} = t - \frac{\partial k'}{\partial\omega}s.$$

If $n'' = 0$, the BKV equation (11) reduces to KdV. In this case, there exists a stationary (soliton) solution of the equation

$$S = S_1 ch^{-2}\left[\left(\tilde{t} - \frac{\omega\mu\alpha S_1}{6k'c^2}\tilde{s}\right)\left(\frac{\omega\mu\alpha S_1}{4k'c^2\varkappa}\right)^{1/2}\right], \quad (12)$$

where $\varkappa = -\frac{\partial^3 k'}{\partial\omega^3}$; S_1 is the soliton amplitude.

As seen from Eq. (12), the velocity of radiation propagation satisfies the equality

$$\frac{1}{v} = \frac{1}{v_0} + \frac{1}{6}S_1\frac{\omega\mu\alpha}{k'c^2}, \quad (13)$$

where v_0 is the group velocity of radiation. Correspondingly, the characteristic length of soliton equals

$$\Delta s = v_0\,[4c^2k'\varkappa/(\omega\mu\alpha S_1)]^{1/2}. \quad (14)$$

The soliton has certain energy, velocity, and it behaves as a particle. If the power and the length of an input signal exceeds those of a soliton, the initial signal, as shown below, is decomposed into separate solitons.

For $n \neq 0$, as follows from Eq.(12), the amplitude of a Poynting vector falls off as an exponent

$$S = S_1\exp(-(2\omega n''/c)\,\tilde{s}). \quad (15)$$

If the last summand of Eq. (11) is neglected, then, with the allowance for Eq. (13) up to $\sim n''$, we obtain the non-linear Burgers equation

$$\frac{\partial S_1}{\partial\tilde{s}} + \frac{\omega\mu\alpha}{2kc^2}S_1\frac{\partial S_1}{\partial\tilde{t}} = \frac{n''}{\omega c}\frac{\partial^2 S_1}{\partial\tilde{t}^2}. \quad (16)$$

The equation (16) with the boundary conditions $S_1(0, \tilde{s}) = A$, $S_1(t_0, \tilde{s}) = 0$ has a stationary soluton[8] which is a shock wave with the width of the leading edge

$$\Delta\tilde{t} \sim n''/(\omega\alpha A). \quad (17)$$

Let us estimate the width of the leading edge of the shock wave Δt in pure water. For water,[9] at the wavelength $\lambda = 546$ nm we have $n'' \approx 6\cdot 10^{-9}$, $\alpha \approx \approx 10^{-16}$ cm^2/W. At the intensity $S = 100$–1000 W/cm^2, we have $\Delta t \approx (10^3$–$10^4)T$, where T is the period of the electromagnetic wave. Let for nitrogen[9] the value n'' be equal $n'' = 10^{-11}$ and $\alpha \approx 10^{-20}$ cm^2/W at the same wavelength. Then, for $S = 1$ W/cm^2, we obtain $\Delta t \approx 10^9 T$. These results show that, in some cases, the width of the leading edge of the shock wave is comparable with the length of the emitted signal what, in its turn, can distort the received signal.

In this paper, we simulate propagation of ultrashort pulses in an optical wave guide on the basis of the Korteweg–de Vries equation ($n'' = 0$ in Eq. (9)). Let $\mu = 1$, $\varepsilon^{1/2} = n_0 = 1.5$, $n'' = 0$ in Eq. (9), all other medium and radiation parameters being taken from Ref. 3. In this case, assuming that the efficient section of a one-mode optical wave guide equals ~ 10 μm^2, we obtain $n_2S_0 \approx 3\cdot 10^{-6}$ for $\delta n = n_2S_0$, $n_2 = a/(2n_0) = 3.2\cdot 10^{-16}$ cm^2/W, $S_0 = 1.1\cdot 10^{10}$ W/cm^2. Assuming that $t_0 = 200$ fs, we obtain $L_d = \tau_0^2/k'' \approx 30$ m for the dispersive length. Here we take $|k''| = 1.4\cdot 10^{-29}$ s^2/cm (see Ref. 1, p. 20), $k'' = \partial^2 k/\partial\omega^2 < 0$ for quartz glass.

By introducing new variables

$$\zeta = s/L_d, \quad \eta = \tilde{t}/(1.75\tau_0), \quad S = uS_0 \quad (18)$$

we write Eq. (11) for $n'' = 0$ in the form

$$\frac{\partial u}{\partial\zeta} + u\frac{\partial u}{\partial\eta} + 0.01\frac{\partial^3 u}{\partial\eta^3} = 0, \quad (19)$$

where we take into account that, according to Ref. 1, $k'''k \gg k''k'$, $\frac{\partial^3 k^2}{\partial\omega^3} \approx 2kk'''$, $|k'''| = 0.9\cdot 10^{-42}$ s^3/cm, $k''' = \frac{\partial^3 k}{\partial\omega^3} < 0$.

In numerical simulation, the input signal was

$$u(\zeta = 0, \eta) = \exp\{-[(\eta - 15)/a]^2\}, \quad (20)$$

where a equals 2.5 and 5. In the case when $a = 5$, the pulse duration $\Delta\eta \approx 10$ or, in correspondence with Eq. (18), $\Delta t = 1.75\Delta\eta\tau_0 = 3.5$ fs what is approximately 8–9 times less than that in Ref. 6. In both cases, the signal was translated to $\zeta_{max} = 30$ (this corresponds to the optical wave guide length $s \approx 900$ m). Figure 1 presents the value u as a function of η for $a = 5$ when $\zeta = 5$ and 8. As seen from the figure, first, a shock wave of the envelope is formed from the pulse (20). Then the generation of ultrashort pulses begins for $5 \le \zeta \le 8$ ($\zeta = s/L_d$, $L_d = 30$ m). It should be noted that, for the pulse considered ($\tau_0 = 3$ ps), the nonlinear length

$$L_{nl} = \tau_0[n_0/(k_2k_0n_2S_0)]^{1/2} \approx 25 \text{ m},$$

i.e., $L_d \approx L_{nl}$. If the process of radiation propagation is described by NSE,[4] the dispersion pulse blurring must be exactly compensated for by the contraction, i.e., the pulse must keep its shape (a soliton is formed).

Even at $\zeta = 8$ (an optical wave guide of that length was used in the experiment[6]), there appeared already four solitons. Figure 2 presents the value u as a function of η for $\zeta = 30$ ($a = 5$). As seen from the figure, 15 solitons appear in this case.

Figures 3 and 4 present the value u as a function of η for the signal (20) when $\zeta = 0$, 2, 8, and 30 ($a = 2.5$). As seen from Figs. 1 and 3, the front self-steeping occurs more rapidly at $a = 2.5$ as compared to that at $a = 5$. Generation of ultrashort pulses begins at $2 \le \zeta \le 8$. From Figure 4 one may see that at $\zeta = 30$ the tenth soliton appears. In contrast to the case of $a = 5$ mm, the energy of the input signal is completely redistributed among the solitons.

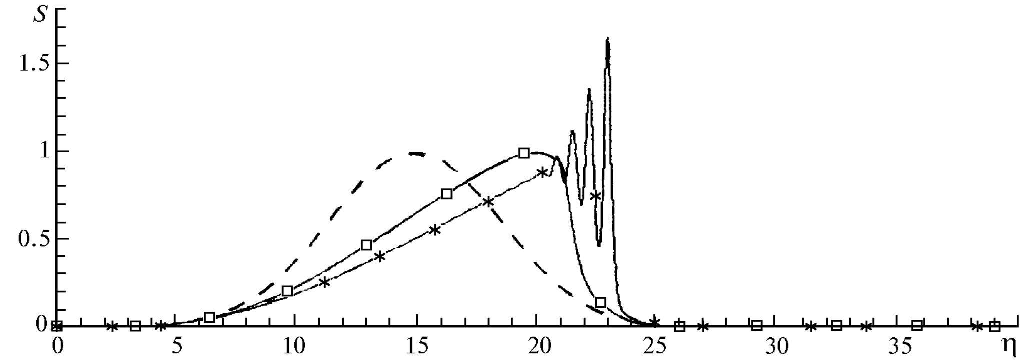

FIG. 1. Amplitude of the Poynting vector **S** *as a function of* η *(*$a = 5$*). The dashed line corresponds to* $\zeta = 0$*, the labels □ and ∗ correspond to* $\zeta = 5$ *and* $\zeta = 8$*.*

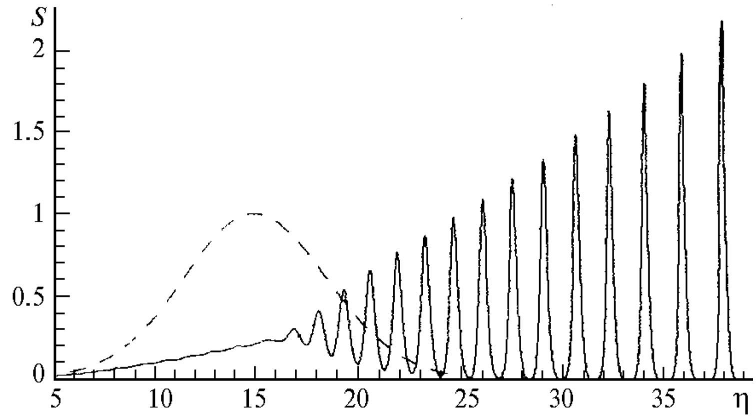

FIG. 2. Amplitude of the Poynting vector **S** *as a function of* η *(*$a = 5$*):* $\zeta = 0$ *(dashed line);* $\zeta = 30$ *(solid line).*

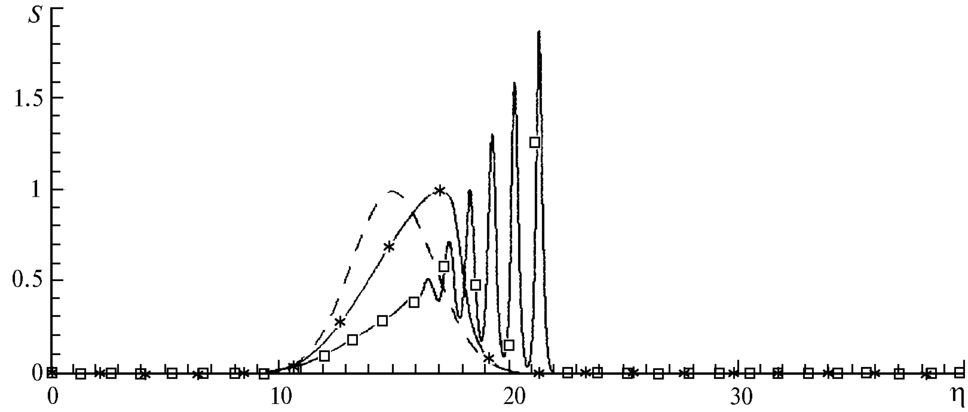

FIG. 3. Amplitude of the Poynting vector **S** *as a function of* η *(*$a = 2.5$*):* $\zeta = 0$ *(dashed line);* $\zeta = 8$ *(□);* $\zeta = 2$ *(∗).*

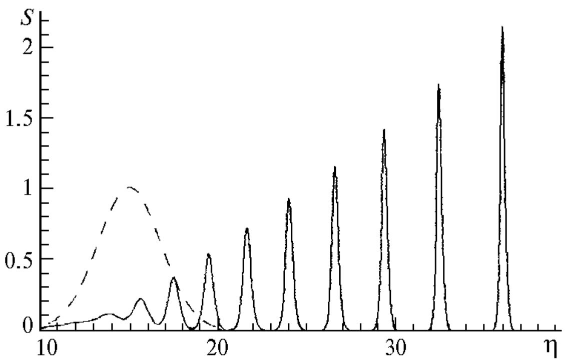

FIG. 4. Amplitude of the Poynting vector **S** *as a function of* η *(a = 2.5): $\zeta = 0$ (dashed line); $\zeta = 30$ (solid line).*

This regularity can be explained by the fact that the steeping process of the trailing edge occurs more rapidly for a shorter input pulse (see Fig. 1). Now let us consider the spectral radiation. Let us assume that the amplitude of the electric (or magnetic) field approximately corresponds to the S profile, provided that ζ is the same. Then, for the signal (20) ($\zeta = 0$), the spectral width of the signal is equal to $\Delta\omega_1 \approx 2/(7\tau_0)$. For $\zeta = 30$, let us approximate S by a product of the exponent and some periodic function

$$S \sim \exp(-\lambda_2\tau)\, f(\lambda_1\tau) ,$$

where $f(\lambda_1, \tau + \pi n) = f(\lambda_1\tau)$, $n = 0, 1, 2, 3,\dots$.

As follows from Fig. 2, $\lambda_2 \approx 0.0057/\tau_0$ and $\lambda_1 \approx \pi/\tau_0$. Finally, it follows that the spectral width of the pulse (20) equals $\Delta\omega_2 \approx \pi/\tau_0$ for $\zeta = 30$, i.e., if $\zeta = 30$, the spectrum broadens 10 times as compared with the initial one.

Thus, the results presented in this paper show that in a single-mode optical wave guide picosecond pulses are decomposed into subpicosecond ($\approx$ 200 fs) ones at the distance of 2–8 dispersion lengths (60–240 m).

The pulse energy and duration are proportional to $(u_0S_0)^{1/2}$ and $(u_0S_0)^{-1/2}$, respectively. The velocity of a soliton depends on its peak value in the accompanying coordinate system. For a peak value of the soliton, $\approx$ 1800 W, its energy is $\approx 2.5\cdot10^{-10}$ J, the characteristic duration being about 160 fs.

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