CLOUD-RADIATION INTERACTIONS: THE TITOV AND OTHER MODELS

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Dedicated to my friend and colleague Georgii Titov

We discuss the treatment of radiative transfer through a two-component discrete stochastic mixture. A broken cloud field can be modeled as such a stochastic mixture, with the clouds and clear sky representing the two components. The integral equation approach of Titov for a Markovian mixture is shown to be equivalent to a differential model introduced in the kinetic theory literature. Simplifications and extensions of this model are also discussed. The simplifications include a renormalized equation of transfer, as well as various diffusive models. The extensions involve more accurate descriptions utilizing additional radiative transfer equations, as well as allowance for non–Markovian mixing statistics.

PREFACE

This special issue of JAOO is dedicated to the memory of Georgii Titov, and prior to my technical paper I would like to make a few personal remarks. I didn't know Georgii long, but I considered him a dear and special friend. I first met Georgii in 1993 at the Third ARM Science Team Meeting in Norman, Oklahoma. At that time, his command of English was somewhat limited, and Sasha Marshak kindly acted as a translator for us. In spite of the language difficulties, it was clear that Georgii and I shared a special bond, both scientifically and personally. We would often sneak out of the meeting to share a smoke, a bad habit we both had developed many years earlier. Some time after this meeting, Georgii visited me for a few days in Los Angeles. By this time, his English had improved substantially so that we could carry on spirited discussions concerning the treatment of radiative transfer through a broken cloud field. We also found time for social activities. At one point, we were discussing what to do for dinner that evening. Georgii said he would like to have a real American steak, and Georgii, I, and my friend Lucie Rodriguez went to a fine steak house. The three of us shared good steak, good wine, good conversation, and, of course, a good smoke. We also took Georgii to the floral gardens at the Huntington Library in Pasadena. This is a beautiful place, and Georgii seemed to find real peace and contentment on this Sunday afternoon. The third and last time I saw Georgii was in Alaska at the 1996 International Radiation Symposium. We hugged each other, shared a smoke of course, and picked up our friendship as if no time had passed since our meeting in Los Angeles.

Lucie and I had planned to meet Georgii and his family, together with Norm McCormick (a faculty member at the University of Washington) and his wife, in Seattle during the summer of 1998. Unfortunately, Georgii's illness prevented this meeting and, sad to say, I never saw my dear friend again. Upon hearing of Georgii's passing, Lucie and I returned to the gardens of the Huntington Library and help a private memorial to mark the loss of this special man. We spread rose petals in front of the tea house, where the three of us had shared so much love and friendship. Lucie wrote a beautiful poem in honor of Georgii, which we sent to Georgii's wife. We also included photographs of Georgii, Lucie and I sharing happy times at the Huntington.

Georgii, we will miss you. We will miss the scientific contributions you would have continued to make, but more importantly we will miss you the person. You were the best this world has to offer. Rest in peace, my dear friend. The paper which follows is dedicated to you, your family, and my memories of you.

1. INTRODUCTION

Cloud-radiation interactions are an important ingredient in the determination of the global climate. An accurate treatment of these interactions is essential to our understanding of long term climate change. A promising approach to this problem is to treat a broken cloud field as a two-component stochastic mixture, with the clouds and clear sky representing the two components. Georgii Titov,[1] together with researchers in the kinetic theory community, have successfully pursued this line of inquiry. In this paper, we review the Titov and kinetic theory models, and show the equivalence of the Titov integral equation model to one of the differential kinetic theory models. We also discuss simplifications and extensions of this model, including a renormalized equation of transfer, diffusive approximations, higher order models, and non-Markovian mixing effects. Because of space

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limitations, our discussion is necessarily brief, but more detailed accounts are available in two recent review articles[2,3] and the references therein.

The radiative transfer equation we consider describes time-independent, monochromatic photon transport and is written

$$\mathbf{\Omega}\cdot\nabla I + \sigma I = \sigma_s \int_{4\pi} d\mathbf{\Omega}' \, f(\mathbf{\Omega}\cdot\mathbf{\Omega}') \, I(\mathbf{\Omega}') + S. \quad (1)$$

The dependent variable in Eq. (1) is the specific intensity of radiation $I(\mathbf{r}, \mathbf{\Omega})$, with $\mathbf{r}$ and $\mathbf{\Omega}$ denoting the spatial and angular (photon flight direction) variables, respectively. The quantity $\sigma(\mathbf{r})$ is the macroscopic total cross section (extinction coefficient), $\sigma_s(\mathbf{r})$ is the macroscopic scattering cross section, $f(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ is the single scatter angular redistribution function normalized according to

$$\int_{4\pi} d\mathbf{\Omega} \, f(\mathbf{\Omega}\cdot\mathbf{\Omega}') = 2\pi \int_{-1}^{1} d\xi \, f(\xi) = 1, \quad (2)$$

and $S(\mathbf{r})$ denotes any emission source of photons. Under the assumption of local thermodynamic equilibrium for the matter, $S = \sigma_a B$, where B is the local Planck function, and $\sigma_a = \sigma - \sigma_s$ is the macroscopic absorption cross section corrected for induced emission. We take the boundary condition on Eq. (1) to be

$$I(\mathbf{r}_s, \mathbf{\Omega}) = \Gamma(\mathbf{r}_s, \mathbf{\Omega}), \qquad \mathbf{n}\cdot\mathbf{\Omega} < 0, \quad (3)$$

where $\mathbf{n}$ is normal outward pointing unit vector at a surface point $\mathbf{r}_s$, and Γ is the specified boundary data. This boundary condition corresponds to specifying the radiation field incident upon each surface point of the system.

To treat the case of a binary statistical mixture, the quantities σ, σ_s, f, and S in Eq. (1) are considered as discrete random variables, each of which assumes at any $\mathbf{r}$ one of two sets of values characteristic of the two components constituting the mixture, namely the clouds and the clear sky. We denote these two sets by σ_i, σ_{si}, f_i, and S_i with $i = 0$ and 1. That is, as a photon traverses the broken cloud field along any path, it encounters alternating segments of clouds and clear sky, each of which has known deterministic values of σ, σ_s, f, and S. The stochastic nature of the problem enters through the statistics of the cloud field, i.e., through the statistical knowledge as to whether a cloud or clear sky is present at point $\mathbf{r}$. Since σ, σ_s, f, and S in Eq. (1) are (two-state, discrete) random variables, the solution of Eq. (1) is also a (continuous) random variable, and we let $\langle I \rangle$ denote the ensemble-averaged intensity. The primary goal in any statistical model of cloud-radiation interactions is to obtain a relatively simple and accurate set of equations for $\langle I \rangle$. It is also of interest to have a model for the higher moments of the stochastic radiation field, in particular the variance. We assume that the boundary data Γ in Eq. (3) is nonstochastic, i.e., it is the same for each physical realization of the statistics.

The outline of the remainder of this paper is as follows. In the next section, we describe the simplest of the kinetic theory models, based upon the assumption of Markovian mixing and the use of the Liouville master equation. Section 3 describes the Titov model, and shows its equivalence to the kinetic theory model of Sec. 2. Section 4 consists of two parts. We first discuss simplifications of the Titov model, including various diffusive approximations and a renormalized equation of transfer. This renormalized equation is of the classic radiative transfer equation form, but contains effective material properties and an effective emission source to account for the stochasticity of the problem. The second part of Sec. 4 makes reference to improved (but more complex) Markovian models, and discusses possible treatments of non-Markovian statistics. The final section is devoted to a few concluding remarks.

2. THE LIOUVILLE MASTER EQUATION TREATMENT

We assume that the statistics of the broken cloud field are Markovian, described entirely by the equation (4)

$$\text{Prob}\,(i \to j) = \frac{ds}{\lambda_i(s)}, \qquad j \neq i, \quad (4)$$

where s is a spatial coordinate along the direction $\mathbf{\Omega}$, and Prob $(i \to j)$ is the differential probability that point $s + ds$ is in component j, given that point s is in component i. Here the two $\lambda_i(s)$ are the prescribed Markovian transition lengths associated with the cloud field, and they completely describe the statistics of the clouds – clear sky mixture. The probabilities $p_i(s)$ of finding component i at position s are related to the $\lambda_i(s)$ by the Chapman–Kolmogorov equations[4]

$$\frac{dp_i}{ds} = \frac{p_j}{\lambda_j} - \frac{p_i}{\lambda_i}, \qquad j \neq i. \quad (5)$$

The $p_i(s)$ have the simple interpretation of being the volume fractions of the two components of the mixture at position s. For homogeneous (λ_i independent of s) statistics, they are related to the λ_i according to

$$p_i = \frac{\lambda_i}{\lambda_0 + \lambda_1}. \quad (6)$$

In this homogeneous case, λ_i has the interpretation of being the mean chord length in component i. A quantity $\lambda_c(s)$, which has the interpretation of a correlation length, is related to the $\lambda_i(s)$ by[4]

$$\frac{2}{\lambda_c(s)} = \frac{1}{\lambda_0(s)\, p_1(s)} + \frac{1}{\lambda_1(s)\, p_0(s)}, \quad (7)$$

which, for homogeneous statistics, reduces to

$$\frac{1}{\lambda_c} = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} . \qquad (8)$$

Finally, in the case of homogeneous statistics, the chord lengths in each component of the mixture are exponentially distributed with mean λ_i.[4]

To incorporate this statistical description into the radiative transfer problem, we first consider a purely absorbing ($\sigma_s = 0$) system, in which case Eq. (1) reduces to

$$\frac{dI(s)}{ds} + \sigma(s)\, I(s) = S(s). \qquad (9)$$

Denoting the surface point $\mathbf{r}_s$ by $s = 0$, the boundary condition on Eq. (9) as given in general by Eq. (3) becomes

$$I(0) = \Gamma. \qquad (10)$$

Vanderhaegen[5] pointed out that the transport problem defined by Eqs. (9) and (10) can be treated exactly under the assumption of Markovian statistics as defined by Eq. (4). He observed that Eqs. (9) and (10) describe an initial value problem, with the space coordinate s playing the role of time. If the mixing is taken as Markovian, the stochastic transport problem is then a joint Markov process, and thus the Liouville master equation applies. For the binary discrete statistics under consideration, this master equation is given by

$$\frac{\partial P_i}{\partial s} - \frac{\partial}{\partial I} [(\sigma_i I - S_i)\, P_i] = \frac{P_j}{\lambda_j} - \frac{P_i}{\lambda_i} , \quad j \neq i. \qquad (11)$$

Here $P_i(I, s)$ is defined such that $P_i\, dI$ is the probability of finding component i at position s, and having the stochastic solution lie between I and $I + dI$. The boundary conditions on Eq. (11) are

$$P_i[I, 0] = p_i(0)\, \delta(I - \Gamma), \qquad (12)$$

which expresses the certainty of the solution at $s = 0$. The volume fraction $p_i(s)$ is related to $P_i(I, s)$ according to

$$p_i(s) = \int_0^\infty dI\, P_i(I, s), \qquad (13)$$

which follows from the definitions of $p_i(s)$ and $P_i(I, s)$.

If we define $I_i(s)$ as the conditional ensemble average of I, conditioned upon position s being in component i, the definition of $P_i(I, s)$ gives

$$p_i(s)\, I_i(s) = \int_0^\infty dI\, I\, P_i(I, s). \qquad (14)$$

It then follows that

$$\langle I(s)\rangle = p_0(s)\, I_0(s) + p_1(s)\, I_1(s), \qquad (15)$$

where $\langle I(s)\rangle$ is the overall, unconditional, ensemble average of the intensity. Multiplication of Eq. (11) by I and integration over $0 \leq I < \infty$ yields, upon integrating by parts,

$$\frac{d(p_i I_i)}{ds} + \sigma_i\, p_i\, I_i = p_i\, S_i + \frac{p_j I_j}{\lambda_j} - \frac{p_i I_i}{\lambda_i} , \quad j \neq i. \qquad (16)$$

These two coupled equations for I_0 and I_1 are subject to the boundary conditions

$$I_0(0) = I_1(0) = \Gamma. \qquad (17)$$

Equations (15) through (17) give a complete and exact description for $\langle I(s)\rangle$ for the purely absorbing, binary Markovian mixture under consideration. Similar results can be obtained for higher stochastic moments of the intensity, such as the variance.[6]

Although the Liouville master equation is not strictly valid as a treatment of stochastic radiative transfer in a Markovian mixture when scattering is present, it has been suggested[7] that the use of this master equation, while not rigorous, might produce a useful and simple, albeit approximate, model of particle transport in Markovian stochastic mixtures. If Eq. (1) is the underlying equation of transfer, this model arises from treating the integral scattering term in Eq. (1) on the same basis as the emission source S. This leads to the two equations

$$\mathbf{\Omega}\cdot\nabla(p_i\, I_i) + \sigma_i\, p_i\, I_i =$$

$$= p_i\, S_i + \sigma_{si} \int_{4\pi} d\Omega'\, f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, p_i\, I_i(\mathbf{\Omega}') + \frac{p_j I_j}{\lambda_j} - \frac{p_i I_i}{\lambda_i} ,$$

$$j \neq i, \qquad (18)$$

with $\langle I(\mathbf{r}, \mathbf{\Omega})\rangle$ given by

$$\langle I(\mathbf{r}, \mathbf{\Omega})\rangle = p_0(\mathbf{r})\, I_0(\mathbf{r}, \mathbf{\Omega}) + p_1(\mathbf{r})\, I_1(\mathbf{r}, \mathbf{\Omega}). \qquad (19)$$

Several other derivations of the approximate model given by Eqs. (18) and (19) have been given. One is based upon stochastic balance methods, introducing an approximate closure to relate an interface ensemble average to the I_i.[8] Sahni has given two alternate derivations, based upon utilizing ideas from nuclear reactor noise analysis,[9] and invoking the approximation that each photon track is independent of prior track.[10] Lastly, there is the approach of Titov,[1] which we consider in the next section.

3. THE TITOV TREATMENT

Titov[1] presented an integral equation formalism for the stochastic transport problem under consideration, in the special case of negligible emission ($S = 0$) and a completely transparent clear sky ($\sigma_0 = \sigma_{s0} = 0$). Here we have let $i = 0$ denote the clear sky and $i = 1$ denote the clouds. The Titov model is given by

$$<I(s)> + \int_0^s \mathrm{d}s'\, C_1\, p_1(s')\, I_1(s') = \Gamma, \qquad (20)$$

$$p_1(s)\, I_1(s) + \int_0^s \mathrm{d}s'\, P_{11}(s', s)\, C_1\, p_1(s')\, I_1(s') = p_1\, \Gamma, \quad (21)$$

where s is the spatial coordinate along the direction $\mathbf{\Omega}$ (so that $\mathbf{\Omega}\cdot\nabla = \mathrm{d}/\mathrm{d}s$), and C_1 is the collision operator defined by

$$C_1(p_1\, I_1) = \sigma_1\, p_1\, I_1 - \sigma_{s1} \int_{4\pi} \mathrm{d}\mathbf{\Omega}'\, f_1(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, p_1\, I_1(\mathbf{\Omega}'). \quad (22)$$

Titov obtained Eqs. (20) and (21) under a Markovian assumption for both the mixing and the transport process. The surface of the system is taken as $s = 0$, and Γ denotes the incoming intensity at this point. The quantity $P_{ij}(s', s)$ occurring in Eq. (21) with $i = j = 1$ is defined as the conditional probability that position s is in mixture component j, given that position s' is in component i. For Markovian statistics, the $P_{ij}(s', s)$ satisfy the Chapman–Kolmogorov equations given by[4]

$$\frac{\partial P_{ii}}{\partial s} = \frac{P_{ij}}{\lambda_j} - \frac{P_{ii}}{\lambda_i}, \quad \frac{\partial P_{ij}}{\partial s} = \frac{P_{ii}}{\lambda_i} - \frac{P_{ij}}{\lambda_j}, \quad j \neq i, \qquad (23)$$

with boundary conditions

$$P_{ii}(s', s') = 1; \quad P_{ij}(s', s') = 0, \quad j \neq i. \qquad (24)$$

It is clear from their definition that the $P_{ij}(s', s)$ satisfy

$$P_{ii} + P_{ij} = 1, \quad j \neq i. \qquad (25)$$

We now cast the integral Titov model given by Eqs. (20) and (21) into an equivalent differential form, and show that this differential form is identical to Eqs. (17) and (18) (in the case that $S_i = \sigma_0 = \sigma_{s0} = 0$), the kinetic theory model discussed in the last section. We first subtract Eq. (21) from Eq. (20) and make use of Eq. (19) to obtain

$$p_0(s)\, I_0(s) + \int_0^s \mathrm{d}s'\, P_{10}(s', s)\, C_1\, p_1(s')\, I_1(s') = p_0(s)\, \Gamma, \qquad (26)$$

where we have used $p_0 + p_1 = P_{10} + P_{11} = 1$.

Differentiating Eqs. (20) and (21), making use of Eq. (24), given

$$\frac{\mathrm{d}<(s)>}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) = 0, \qquad (27)$$

$$\frac{\mathrm{d}[p_1(s)\, I_1(s)]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) + \int_0^s \mathrm{d}s'\, \frac{\partial P_{11}(s', s)}{\partial s}\, C_1\, p_1(s')\, I_1(s') = \frac{\mathrm{d}p_1(s)}{\mathrm{d}s}\, \Gamma. \qquad (28)$$

We now use the first equation in Eq. (23) with $i = 1$ and $j = 0$ to rewrite Eq. (28) as

$$\frac{\mathrm{d}[p_1(s)\, I_1(s)]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) - \frac{1}{\lambda_1(s)} \int_0^s \mathrm{d}s'\, P_{11}(s', s)\, C_1\, p_1(s')\, I_1(s') + \frac{1}{\lambda_0(s)} \int_0^s \mathrm{d}s'\, P_{10}(s', s)\, C_1\, p_1(s')\, I_1(s') = \frac{\mathrm{d}p_1(s)}{\mathrm{d}s}\, \Gamma. \qquad (29)$$

Equations (21) and (26) can be used to eliminate the integral terms in Eq. (29), resulting in

$$\frac{\mathrm{d}[p_1(s)\, I_1(s)]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) = \frac{p_0(s)}{\lambda_0(s)}\, I_0(s) - \frac{p_1(s)}{\lambda_1(s)}\, I_1(s) + \left[\frac{\mathrm{d}p_1(s)}{\mathrm{d}s} + \frac{p_1(s)}{\lambda_1(s)} - \frac{p_0(s)}{\lambda_0(s)} \right] \Gamma. \qquad (30)$$

Making use of Eq. (15) with $i = 1$ and $j = 0$, Eq. (30) simplifies to

$$\frac{\mathrm{d}[p_1(s)\, I_1(s)]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) = \frac{p_0(s)}{\lambda_0(s)}\, I_0(s) - \frac{p_1(s)}{\lambda_1(s)}\, I_1(s). \qquad (31)$$

Finally, subtracting Eq. (31) from Eq. (27) yields

$$\frac{\mathrm{d}[p_0(s)\, I_0(s)]}{\mathrm{d}s} = \frac{p_1(s)}{\lambda_1(s)}\, I_1(s) - \frac{p_0(s)}{\lambda_0(s)}\, I_0(s). \qquad (32)$$

Equations (31) and (32) are the final result of our algebraic manipulations, and are entirely equivalent in content to the integral equations of Titov given by Eqs. (20) and (21), once they are supplemented with the identity

$$<I(s)> = p_0(s)\, I_0(s) + p_1(s)\, I_1(s), \qquad (33)$$

and the initial conditions

$$I_0(0) = I_1(0) = \Gamma. \qquad (34)$$

Using $d/ds = \mathbf{\Omega}\cdot\nabla$ and the definition of C_1 as given by Eq. (22) in Eqs. (31) and (32), it is clear that the Titov model is identical to the Markovian kinetic theory model given by Eq. (18), recognizing that Titov considered the special case $S_i = \sigma_0 = \sigma_{s0} = 0$ (no emission and a transparent clear sky).

4. SIMPLIFICATIONS AND EXTENSIONS

The approximate model given by Eq. (18) can be simplified by introducing further approximations. Most of the work reported in the literature in this regard is restricted to isotropic statistics $[\lambda_i \neq \lambda_i(\mathbf{\Omega})]$ and isotropic scattering ($4\pi f_i = 1$). For isotropic scattering, Eq. (18) reduced to

$$\mathbf{\Omega}\cdot\nabla(p_i I_1) + \sigma_i p_i I_i = p_i S_i + \frac{\sigma_{si}}{4\pi}\int_{4\pi} d\mathbf{\Omega}'\, p_i I_i(\mathbf{\Omega}') + \frac{p_j I_j}{\lambda_j} - \frac{p_i I_i}{\lambda_i}, \quad j \neq i. \tag{35}$$

If it assumed in Eq. (35) that one or both of the λ_i are small (so that λ_c is small), simple asymptotic analysis[11] gives the expected result, referred to as the atomic mix limit,

$$\mathbf{\Omega}\cdot\nabla \langle I\rangle + \langle\sigma\rangle \langle I\rangle = \langle S\rangle + \frac{\langle\sigma_s\rangle}{4\pi}\int_{4\pi} d\mathbf{\Omega}' \langle I(\mathbf{\Omega}')\rangle, \tag{36}$$

where

$$\langle\sigma\rangle = p_0\,\sigma_0 + p_1\,\sigma_1 , \tag{37}$$

with similar expressions for $\langle\sigma_s\rangle$ and $\langle S\rangle$. A second asymptotic limit which has been examined[11] corresponds to a small amount of large cross section material admixed with a large amount of small cross section material. In this case, one finds a renormalized equation of transfer given by

$$\mathbf{\Omega}\cdot\nabla \langle I\rangle + \sigma_{\text{eff}} \langle I\rangle = S_{\text{eff}} + \frac{\sigma_{s,\text{eff}}}{4\pi}\int_{4\pi} d\mathbf{\Omega}' \langle I(\mathbf{\Omega}')\rangle, \tag{38}$$

where S_{eff}, σ_{eff}, and $\sigma_{s,\text{eff}}$ represent an effective source and effective cross sections which approximately account for the stochasticity of the problem. These effective quantities are explicit algebraic expressions in terms of the S_i, σ_i, σ_{si}, and λ_i which, in the interest of conserving space, we shall not give here. We do point out, however, that all three of these effective quantities are always nonnegative, even far from the asymptotic limit under consideration, indicating the robustness of this asymptotic treatment.

Several papers are available describing still another asymptotic limit, that reduces Eq. (35) to a diffusive description of stochastic radiative transfer. The first of these[11] scales $\sigma_{ai} = \sigma_i - \sigma_{si}$ and S_i in Eq. (35) as $O(\ ^2)$, where is a formal smallness parameter, and scales the gradient term as $O(\ )$. The λ_i are unscaled, and hence taken as $O(1)$. One obtains two coupled diffusion equations for the energy densities E_i (the angular integral of I_i divided by the speed of light). Subsequent work[12] generalized this analysis to include various scalings of the λ_i. Depending upon the assumed scaling, one obtains either two coupled diffusion equations for the E_i, or a single diffusion equation for E , the ensemble-averaged energy density. In a further generalization,[13] asymptotic diffusive descriptions have been obtained in the presence of anisotropic statistics, and independent scalings for the two λ_i.

Other diffusive descriptions corresponding to Eq. (35) have been reported. These are the P_1 and P_2 spherical harmonic approximations,[14] a diffusion description based upon the Case discrete modes of transport theory,[14] and flux-limited diffusion theories.[15,16] Flux-limiting means that the diffusion coefficient in Fick's law of diffusion is nonlinear, in just such a way that the magnitude of the radiative flux can never exceed the product of the radiation energy density and the speed of light, no matter how steep the spatial gradients.

All of the above discussion deals with simplifications that have been suggested, taking Eq. (18) or its isotropic scattering version Eq. (35), which in themselves are approximate, as the underlying transport model. Several models, which are more accurate but more complex than Eqs. (18) and (35), have also been suggested for treating this Markovian stochastic transport problem. They are all based upon the exact stochastic balance equation given by, for isotropic scattering,[8]

$$\mathbf{\Omega}\cdot\nabla(p_i I_i) + \sigma_i p_i I_i = p_i S_i + \frac{\sigma_{si}}{4\pi}\int_{4\pi} d\mathbf{\Omega}'\, p_i I_i(\mathbf{\Omega}') + \frac{p_j \bar{I}_j}{\lambda_j} - \frac{p_i \bar{I}_i}{\lambda_i}, \quad j \neq i. \tag{39}$$

These two equations (for $i = 0$ and $i = 1$) are not closed in that they contain four unknowns, namely the I_i as previously defined and the $\bar{I}_i$, defined as the ensemble average of the specific intensity conditioned upon the position $\mathbf{r}$ lying at an interface between materials, with material i to the left (the vector $\mathbf{\Omega}$ points from left to right). Thus a closure is needed to make Eq. (39) useful. The simplest suggested closure[8] is to use

$$\bar{I}_i = I_i. \tag{40}$$

With this (approximate) closure, Eq. (39) becomes identical to the master equation model given by Eq. (35). We note that this closure is exact for purely absorbing ($\sigma_{si} = 0$) Markovian problems.

To improve upon the accuracy of Eq. (35), several other closures have been suggested. Two of these are algebraic, in which Eq. (40) is replaced[17,18]

$$\bar{I}_i = \alpha I_i, \quad (41)$$

or

$$\bar{I}_i = \beta I_i + \gamma E_i, \quad (42)$$

with relatively simple and explicit expressions given for α, β, and γ. The qualitative difference between these two closures is that Eq. (42) mixes the various directions, through the energy density term E_i, with no such mixing present in Eq. (41). For purely absorbing problems ($\sigma_{si} = 0$), one has $\alpha = \beta = 1$ and $\gamma = 0$, thus assuring that both closures reduce to Eq. (40) for this class problems, and hence are exact closures in this case. Two additional closures have been suggested, which involve additional equations of transfer for the $\bar{I}_i$.[19,20] Thus these two models involve four coupled equations of transfer for the four unknowns I_0, I_1, $\bar{I}_0$, and $\bar{I}_1$. As such, these models are quite complex, but they are also the most accurate. Space limitations prohibits any detailed discussion of these models. We only point out that both are exact for purely absorbing ($\sigma_{si} = 0$) problems.

We close this section by briefly discussing problems involving non-Markovian mixing statistics. All of the models of stochastic transport discussed thus far in this paper have assumed that the statistics of the mixing is described by a Markov process. As pointed out in Sec. 2, homogeneous Markovian statistics correspond to exponentially distributed cloud sizes and intercloud spacings. If more realistic, non-Markovian, distributions are available, one can use renewal theory to obtain an exact description for the purely absorbing problem.[4,21,22] This analysis uses the integral, rather than the differential, equation of transfer as the underlying description of the transport process. The renewal theory description consists of four coupled integral equations, whose solution yields $\langle I \rangle$, the ensemble-averaged intensity. It can be shown[22] that for Markovian statistics these four integral equations are equivalent to the two differential equations given by Eq. (16). If the scattering interaction is present, an approximate model for non-Markovian statistics can be constructed by treating the integral scattering term on the same basis as the emission source S in the renewal equations. This is the same philosophy as was used to incorporate scattering into the purely absorbing Liouville master equation result, Eq. (16), which led to Eq. (18) as a Markovian model with scattering.

A second approach to dealing with non-Markovian mixing statistics in stochastic transport analysis has also been proposed.[23] This model uses the Markovian model given by Eq (18) as the relevant equations, but replaces the Markov transition length λ_i in Eq. (18) with an effective quantity $\lambda_{i,\mathrm{eff}}$ given by

$$\lambda_{i,\mathrm{eff}} = q\, \lambda_i. \quad (43)$$

The factor q, which is independent of the index i, and which lies in the range $0 < q \le 1$ with $q = 1$ corresponding to Markovian statistics, was determined by solving the renewal equations in the purely absorbing case. It was found that with a certain choice for q, Eq. (18), with $\sigma_{si} = 0$ and λ_i replaced by $q\lambda_i$, preserves both the correct photon mean free path and the deep-in solution. This model for non-Markovian statistics is particularly appealing in that it retains the simplicity of two coupled equations of transfer in the form given by Eq. (18), in contrast to the more complex four coupled integral equations of renewal theory.

5. CONCLUDING REMARKS

In this paper we have briefly reviewed various kinetic theory models describing particle transport and radiative transfer through a two-state, discrete, stochastic mixture. These models provide one way of approaching the cloud-radiation interaction problem, by treating the clouds and clear sky as the two components of a stochastic mixture. We have shown in detail that the Titov integral equation model for describing radiative transfer through a broken cloud field is equivalent to one of the differential kinetic theory models. Georgii Titov will no longer be contributing to this scientific literature, but his prior contributions and the man himself will long be remembered. We will all miss Georgii, and the special warmth that was so much a part of him.

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REFERENCES

1. G.A. Titov, J. Atmos. Sci. **47**, 24–38 (1990).
2. G.C. Pomraning, Advances in Nuclear Science and Technology **24**, 47–93 (1996).
3. G.C. Pomraning, Trans. Th. Statist. Phys. in press. (1998).
4. G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **42**, 279–293 (1989).
5. D. Vanderhaegen, J. Quant. Spectrosc. Radiat. Transfer **36**, 557–561 (1986).
6. G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **56**, 629–646 (1996).
7. G.C. Pomraning, C.D. Levermore, and J. Wong, Lecture Notes in Pure and Applied Mathematics **115**, 1–35 (1989).
8. M.L. Adams, E.W. Larsen, and G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **42**, 253–266 (1989).
9. D.C. Sahni, Ann. Nucl. Energy **16**, 397–408 (1989).
10. D.C. Sahni, J. Math. Phys. **30**, 1554–1559 (1989).
11. F. Malvagi, C.D. Levermore, and G.C. Pomraning, Trans. Th. Statist. Phys. **18**, 287–312 (1989).

12. M. Sammartino, F. Malvagi, and G.C. Pomraning, J. Math. Phys. **33**, 1480–1501 (1992).
13. G.C. Pomraning, Ann. Nucl. Energy **19**, 737–763 (1992).
14. S. Su and G.C. Pomraning, Nucl. Sci. Eng **120**, 75–90 (1995).
15. M. Sammartino and G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **46**, 237–249 (1991).
16. G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **54**, 637–649 (1995).
17. M. Su and G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **51**, 893–912 (1994).
18. G.C. Pomraning and B. Su, in: *Proc. International Conf. On Reactor Physics and Reactor Computations,* Ben–Gurion University of the Negev Press, Israel (1994), pp. 672–679.
19. G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **46**, 221–236 (1991).
20. B. Su and G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **54**, 779–801 (1995).
21. D. Vanderhaegen, J. Quant. Spectrosc. Radiat. Transfer **39**, 333–337 (1988).
22. G.C. Pomraning, J. Quant. Spectrosc. Radiat. Transfer **57**, 295–307 (1997).
23. C.D. Levermore, G.C. Pomraning, and J. Wong, J. Math. Phys. **29**, 995–1004 (1988).