ESTIMATION OF THE PARAMETERS OF A PARTIALLY OBSERVED ALTERNATING FLOW OF EVENTS

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The estimations of the parameters of a partially observed alternating flow of events have been obtained by the method of moments. Properties of the parameters were considered. The theoretical results obtained were verified by means of a simulation model.

Many physical processes and phenomena are often described using a mathematical model of events. But the majority of publications deal only with the case when the events in a flow may be observed. But in practice an event in the flow can obscure observation of the subsequent events. As an example can be taken the processes in a Geiger counter. The peculiarity of this device is that a particle penetrating inside a counter induces a discharge. The discharge continues some time during which other particles cannot be recorded. So a problem arises of recording real flow of particles by means of such devices. Similar problem was considered in Refs. 1 and 2 for Poisson flow of events. In this paper the results obtained are generalized for the case of alternating flow of events. In atmospheric optics one faces similar problems when considering propagation of radiation through broken clouds and also in the problems of determining optical parameters of the turbulent atmosphere.

1. STATEMENT OF THE PROBLEM

Let us consider a stationary Poisson flux with the intensity λ, which can be observed only during some intervals. The process of observations is defined as a discrete Markovian process with continuous time which has two (0 and 1) states. When the controlling process is in the state 1 the Poisson flux can be observed, if it is in state 0 observations are impossible. The intensity of control process transition from state 1 to state 0 and back are α and β. Stationary intervals of the control processes are distributed according to exponential law. So control process results in an alternating flow of events which is partially obscured because after an occurrence of an event in this flow there exists some finite interval T during which other events may not be observed. Let us call this interval "dead time". The events in the dead time do not result in its prolongation, but after termination of the first event in the alternating flow it causes again the next period T when observations are impossible, etc. This situation is depicted in Fig. 1, where the intervals of dead time are hatched, 0 and 1 are states of the controlling process, t is the current time, and $\{t_1, \ldots, t_n\}$ are moments when the events occur in the flow. The assessment of Poisson flux intensity λ should be obtained from these results and also assessments of dead time intervals and intensities of the transitions α and β as well.

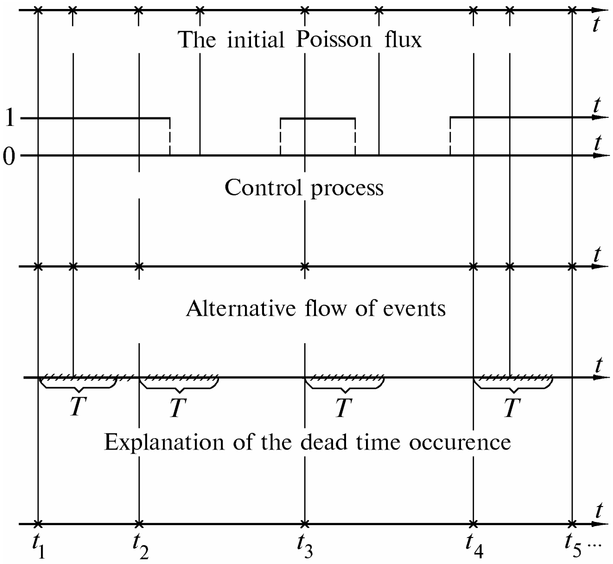

FIG. 1.

2. DISTRIBUTION OF THE OBSERVED VALUES PROBABILITIES

Let us use time intervals $\tau_i = t_{i+1} - t_i$, $i = \overline{1, n}$ instead of $\{t_1, \ldots, t_n\}$. Let us take $\{t_1, \ldots, t_n\}$ as the initial parameters for processing. It can easily be shown that the observed flux is recurrent, i.e., values $\{t_1, \ldots, t_n\}$ are independent and distributed according to the same law. So the probability density $p(\tau)$ of time intervals between successive events in the flow can be taken as a basis for further analysis.

Let us assume that an event is observed in the flow. Also we assume that this event occurs at the moment $t = 0$. So the controlling process is in state 1. For the probability to find the controlling process in state j ($j = 0, 1$) let us introduce the designation $\pi_j(t)$. Then the following system of equations is valid for $\pi_j(t)$

$$\begin{cases} \pi_0'(t) = -\beta\, \pi_0(t) + \alpha\, \pi_1(t), \\ \pi_1'(t) = \beta\, \pi_0(t) - \alpha\, \pi_1(t). \end{cases} \quad (1)$$

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Because the event occurs at the moment $t = 0$, the initial conditions for $t = 0$ are as follows $\pi_1(0) = 1$ and $\pi_0(0) = 0$. So the solution of system (1) can be written in the form

$$\begin{cases} \pi_0(t) = \dfrac{\alpha}{\alpha+\beta} - \dfrac{\alpha}{\alpha+\beta}\exp(-(\alpha+\beta)t), \\ \pi_1(t) = \dfrac{\alpha}{\alpha+\beta} + \dfrac{\alpha}{\alpha+\beta}\exp(-(\alpha+\beta)t). \end{cases} \quad (2)$$

In what follows we will need the solution of system (2) for $t = T$. So, by substituting $t = T$ into Eqs. (2) we obtain

$$\begin{cases} \pi_0(T) = \dfrac{\alpha}{\alpha+\beta} - \dfrac{\alpha}{\alpha+\beta}\exp(-(\alpha+\beta)T), \\ \pi_1(T) = \dfrac{\alpha}{\alpha+\beta} + \dfrac{\alpha}{\alpha+\beta}\exp(-(\alpha+\beta)T). \end{cases} \quad (3)$$

Let $P_j(t)$ be the probability of the process being in state j at the moment $T + t$. This means that events do not occur in the interval $(T, T + t)$. So $P_j(t)$ is a solution to the following system

$$\begin{cases} P_1'(t) = -(\lambda+\alpha)\,P_1(t) + \beta\,P_0(t), \\ P_0'(t) = \alpha\,P_1(t) - \beta\,P_0(t), \end{cases} \quad (4)$$

In this case the initial conditions are

$P_1(0) = \pi_1(T)$, $P_0(0) = \pi_0(T)$,

where $\pi_0(T)$ and $\pi_1(T)$ are defined by Eqs. (3).

Then the solution of the system (4) can be written as

$$\begin{cases} P_1(t) = \dfrac{\beta - z_1}{\alpha} B_1 \exp(-z_1 t) + \dfrac{\beta - z_2}{\alpha} B_2 \exp(-z_2 t), \\ P_0(t) = B_1 \exp(-z_1 t) + B_2 \exp(-z_2 t), \end{cases}$$

where

$$z_1 = \frac{\lambda + \alpha + \beta - \sqrt{(\lambda+\alpha+\beta)^2 - 4\lambda\beta}}{2},$$

$$z_2 = \frac{\lambda + \alpha + \beta + \sqrt{(\lambda+\alpha+\beta)^2 - 4\lambda\beta}}{2},$$

$$B_1 = \frac{\alpha}{(z_2 - z_1)(\alpha+\beta)}\{z_2 + (z_1 - \lambda)\exp(-(\alpha+\beta)T)\},$$

$$B_2 = -\frac{\alpha}{(z_2 - z_1)(\alpha+\beta)}\{z_1 + (z_2 - \lambda)\exp(-(\alpha+\beta)T)\}. \quad (5)$$

Evidently, the probability density $p(t)$ for a time interval between the end of the dead time and the beginning of the first event in an alternating flow is

$p(t) = \lambda\,P_1(t)$.

The interval between two successive events in the alternating flow is equal $\tau = T + t$, so allowing for the last formula we obtain

$p(\tau) = \lambda\,P_1(\tau - T)$.

Then taking into account Eq. (5) and after simple mathematical manipulations we obtain the following equation for $p(\tau)$

$$p(\tau) = \gamma z_1 \exp(-z_1(\tau - T)) + (1-\gamma) z_2 \exp(-z_2(\tau - T)), \quad (6)$$

where

$$\gamma = \frac{z_2 - 1}{(z_2 - z_1)(\alpha+\beta)}\,[z_2 + (z_1 - \lambda)\exp(-(\alpha+\beta)T]\,.$$

Variables z_1 and z_2 are defined in Eq. (5).

Let us note that Eq. (6) is valid only at $\tau \ge T$. For $0 \le \tau < T$ $p(\tau) \equiv 0$.

3. CONSTRUCTING ESTIMATES

Estimations of the parameters were obtained by the method of moments[3] using four statistics

$$C_k = \frac{1}{k!}\frac{1}{n}\sum_{i=1}^{n} \tau_i^k\,, \quad k = \overline{1,4}\,. \quad (7)$$

True moments are calculated by the formula

$$M\{\tau^k\} = \int_0^\infty \tau^k\,p(\tau)\,d\tau\,.$$

By substituting $p(\tau)$ into this formula from Eq. (6) we obtain

$$M\{\tau^k\} = k!\left[\frac{\gamma\exp(z_1 T)}{z_1^k} + \frac{(1-\gamma)\exp(z_2 T)}{z_2^k}\right]. \quad (8)$$

According to the method of moments the estimations for $\hat{z}_1$, $\hat{z}_2$, $\hat{T}$, $\hat{\gamma}$ can be derived from the system of equations

$M\{\tau^k\} = C_k$, $k = \overline{1,4}$.

Allowing for Eq. (8) this system changes to the form

$$\frac{\hat{\gamma}\exp(\hat{z}_1\hat{T})}{\hat{z}_1} + \frac{(1-\hat{\gamma})\exp(\hat{z}_2\hat{T})}{\hat{z}_2} = C_1\,,$$

$$\frac{\hat{\gamma}\exp(\hat{z}_1\hat{T})}{\hat{z}_1^2} + \frac{(1-\hat{\gamma})\exp(\hat{z}_2\hat{T})}{\hat{z}_2^2} = C_2\,,$$

$$\frac{\hat{\gamma}\exp(\hat{z}_1\hat{T})}{\hat{z}_1^3} + \frac{(1-\hat{\gamma})\exp(\hat{z}_2\hat{T})}{\hat{z}_2^3} = C_3\,,$$

$$\frac{\hat{\gamma}\exp(\hat{z}_1\hat{T})}{\hat{z}_1^4} + \frac{(1-\hat{\gamma})\exp(\hat{z}_2\hat{T})}{\hat{z}_2^4} = C_4\,.$$

Let us write a solution for this system as

$$\hat{T} = \frac{\ln\left(\dfrac{\hat{z}_1^2(C_2\,\hat{z}_2 - C_1)}{\hat{\gamma}(\hat{z}_2 - \hat{z}_1)}\right)}{\hat{z}_1}\,,$$

where $\hat{z}_1$ and $\hat{z}_2$ are the roots of the equation

$$(C_3^2 - C_2 C_4)\hat{z}^2 - (C_2 C_3 - C_1 C_4)\hat{z} + (C_2^2 - C_1 C_3) = 0,$$

$\hat{\gamma}$ is a root of the equation

$$(1-\gamma) = \hat{z}_2^2 \frac{C_1 - \hat{z}_1 C_2}{\hat{z}_2 - \hat{z}_1}\left[\frac{\hat{z}_2 - \hat{z}_1}{\hat{z}_1^2 (C_2 \hat{z}_2 - C_1)}\right]^{\hat{z}_2/\hat{z}_1} \gamma^{\hat{z}_2/\hat{z}_1} . \quad (9)$$

Estimations $\hat{z}_1$, $\hat{z}_2$, $\hat{T}$, $\hat{\gamma}$ explicitly define a form of the estimations

$$\hat{\lambda} = \hat{z}_1 + \hat{z}_2 - x^* \hat{z}_2 ,$$

$$\hat{\alpha} = x^* \hat{z}_2 - \frac{\hat{z}_1 \hat{z}_2}{\hat{z}_1 + \hat{z}_2 - x^* \hat{z}_2} ,$$

$$\hat{\beta} = \frac{\hat{z}_1 \hat{z}_2}{\hat{z}_1 + \hat{z}_2 - x^* \hat{z}_2} ,$$

where x^* is a root of the following equation:

$$\hat{\gamma}\,[1 - \hat{z}_1/\hat{z}_2]\frac{x}{x - \hat{z}_1/\hat{z}_2} = 1 + (x-1)\exp(-x\,\hat{z}_2\,\hat{T}) . \quad (10)$$

4. PROPERTIES OF ESTIMATIONS

Let us consider the fidelity of the estimations obtained. Using Kolmogorov theorem[4] we can write

$$C_k = \frac{1}{k!}\frac{1}{n}\sum_{i=1}^{n} \tau_i^k \to \frac{1}{k!} M\{\tau_i^k\}$$

which is almost certainly true for $n \to \infty$. Allowing for Eq. (8) at $n \to \infty$ the following formula is almost certainly valid:

$$C_k \to \frac{\gamma \exp(z_1 T)}{z_1^k} + \frac{(1-\gamma)\exp(z_2 T)}{z_2^k} . \quad (11)$$

Using Eq. (11) it is easy to show that the following equation:

$$(C_3^2 - C_2 C_4)\hat{z}^2 - (C_2 C_3 - C_1 C_4)\hat{z} + (C_2^2 - C_1 C_3) = 0$$

at $n \to \infty$ almost certainly reduces to the from

$$\hat{z}^2 - (z_1 + z_2)\hat{z} + z_1 z_2 = 0 .$$

From this equation it follows that with high probability at $n \to \infty$ $\hat{z}_1 \to z_1$ and $\hat{z}_2 \to z_2$.

So we obtain the equations

$$\hat{\gamma}\exp(\hat{z}_1 \hat{T}) = \hat{z}_1^2 \frac{\hat{z}_2 C_2 - C_1}{\hat{z}_2 - \hat{z}_1} ,$$

$$(1-\hat{\gamma})\exp(\hat{z}_2 \hat{T}) = \hat{z}_2^2 \frac{C_1 - \hat{z}_1 C_2}{\hat{z}_2 - \hat{z}_1} . \quad (12)$$

Allowing for Eq. (11) and taking into account convergence of estimations $\hat{z}_1$ and $\hat{z}_2$, to the true values z_1 and z_2, at $n \to \infty$ Eq. (12) reduces to the equations

$$\hat{\gamma}\exp(\hat{z}_1 \hat{T}) = \gamma \exp(z_1 T) ,$$

$$(1-\hat{\gamma})\exp(\hat{z}_2 \hat{T}) = (1-\gamma)\exp(z_2 T) . \quad (13)$$

From the first of these equations we obtain

$$\hat{T} = \ln \gamma^{1/\hat{z}_1} + T - \ln \hat{\gamma}^{1/\hat{z}_1} .$$

Substituting this equation into the second of Eqs. (13) we obtain

$$(1-\hat{\gamma}) = \left(\frac{1-\hat{\gamma}}{\hat{\gamma}^{\hat{z}_2/\hat{z}_1}}\right)\hat{\gamma}^{\hat{z}_2/\hat{z}_1} .$$

From this it follows that at $n \to \infty$ with high probability we have $\hat{\gamma} \to \gamma$.

If taken into consideration together with Eq. (13) at $n \to \infty$ this formula almost certainly leads to the equation $\hat{T} \to T$.

Since estimations $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\beta}$ are related to estimations $\hat{z}_1$, $\hat{z}_2$, $\hat{T}$, and $\hat{\gamma}$, the second estimation is almost certainly convergent, so the convergence of the first estimations to true values $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\beta}$ is almost certain too. So the estimations $\hat{\lambda}$, $\hat{\alpha}$, and $\hat{\beta}$ obtained are valid.

Because the estimations are valid, using the method of linearization we may derive asymptotic standard deviations for the estimations

$$D(\hat{\lambda}) = \frac{1}{n} B_\lambda^{\mathrm{T}} E B_\lambda , \; D(\hat{\alpha}) = \frac{1}{n} B_\alpha^{\mathrm{T}} E B_\alpha ,$$

$$D(\hat{\beta}) = \frac{1}{n} B_\beta^{\mathrm{T}} E B_\beta , \; D(\hat{T}) = \frac{1}{n} B_T^{\mathrm{T}} E B_T ,$$

where

$$B_\lambda^{\mathrm{T}} = (B_{11}, B_{12}, B_{13}, B_{14}),\; B_\alpha^{\mathrm{T}} = (B_{21}, B_{22}, B_{23}, B_{24}),$$

$$B_\beta^{\mathrm{T}} = (B_{31}, B_{32}, B_{33}, B_{34}),\; B_T^{\mathrm{T}} = (b_{41}, b_{42}, b_{43}, b_{44}) .$$

Here we have introduced the following designations: E is a matrix with the elements

$$E_{\mathrm{sm}} = (z_2)^{-(s+m)}\left\{\frac{(s+m)!}{s!m!}\left[\frac{\gamma\exp(z_1 T)}{(z_1/z_2)^{s+m}} + (1-\gamma)\exp(z_2 T)\right] - \right.$$

$$- \left[\frac{\gamma\exp(z_1 T)}{(z_1/z_2)^s} + (1-\gamma)\exp(z_2 T)\right] \times$$

$$\left. \times \left[\frac{\gamma\exp(z_1 T)}{(z_1/z_2)^m} + (1-\gamma)\exp(z_2 T)\right]\right\},$$

$$B_{1i} = \Delta_{11}\sum_{j=1}^{4}\omega_j b_{ji} + 2\Delta_{12} b_{1i} + 2\Delta_{13} b_{2i} ,$$

$$B_{2i} = \Delta_{21}\sum_{j=1}^{4}\omega_j b_{ji} + 2\Delta_{22} b_{1i} + 2\Delta_{23} b_{2i} ,$$

$$B_{3i} = \Delta_{31}\sum_{j=1}^{4}\omega_j b_{ji} + 2\Delta_{32} b_{1i} + 2\Delta_{33} b_{2i} , \quad i = \overline{1, 4} ,$$

where

$$\omega_1 = -\frac{\alpha+\beta-z_2}{(\alpha+\beta)(z_2-z_1)^2}[z_2+(\alpha+\beta-z_2)\exp(-(\alpha+\beta)T] ,$$

$$\omega_2 = -\frac{\alpha+\beta-z_1}{(\alpha+\beta)(z_2-z_1)^2}[z_1+(\alpha+\beta-z_1)\exp(-(\alpha+\beta)T],$$

$$\omega_3 = -1 ; \quad \omega_4 = -\frac{\mathrm{la}}{z_2-z_1}\exp(-(\alpha+\beta)T) ,$$

$\Delta_{ij} = A_{ji}/(\det \Delta)$ $(i, j = \overline{1, 3})$, A_{ji} is an algebraic supplement of the element $\Delta^{(j,i)}$ of the matrix Δ. Elements of this matrix are given by the formulas

$$\Delta^{(1,1)} = 0 ,$$

$$\Delta^{(1,2)} = \frac{1}{(\alpha+\beta)^2(z_2-z_1)}\{\lambda\beta + [(\alpha+\beta)^2 - \lambda\beta] \times$$

$$\times \exp(-(\alpha+\beta)T) + \lambda\alpha(\alpha+\beta)T\exp(-(\alpha+\beta)T)\} ,$$

$$\Delta^{(1,3)} = \frac{1}{(\alpha+\beta)^2(z_2-z_1)}\{\lambda\beta + [(\alpha+\beta)^2 - \lambda\beta] \times$$

$$\times \exp(-(\alpha+\beta)T) + \lambda\alpha(\alpha+\beta)T\exp(-(\alpha+\beta)T)\} ,$$

$$\Delta^{(2,1)} = -1 + \frac{\lambda+\alpha-\beta}{\sqrt{(\lambda+\alpha+\beta)^2-4\lambda\beta}} ,$$

$$\Delta^{(3,1)} = -1 - \frac{\lambda+\alpha-\beta}{\sqrt{(\lambda+\alpha+\beta)^2-4\lambda\beta}} ,$$

$$\Delta^{(2,2)} = -1 + \frac{\lambda+\alpha+\beta}{\sqrt{(\lambda+\alpha+\beta)^2-4\lambda\beta}} ,$$

$$\Delta^{(3,2)} = -1 - \frac{\lambda+\alpha+b}{\sqrt{(\lambda+\alpha+\beta)^2-4\lambda\beta}} ,$$

$$\Delta^{(2,3)} = -1 + \frac{\alpha+\beta-\lambda}{\sqrt{(\lambda+\alpha+\beta)^2-4\lambda\beta}} ,$$

$$\Delta^{(3,3)} = -1 - \frac{\alpha+\beta-\lambda}{\sqrt{(\lambda+\alpha+\beta)^2-4\lambda\beta}} ,$$

$b_{ji} = A_{ji}/(\det B)$ $(i, j = \overline{1, 4})$; A_{ji} is an algebraic supplement of the element $b^{(j,i)}$ of the matrix B. Elements of this matrix are given by the following formulas:

$$b^{(1,1)} = \frac{T z_1 - 1}{z_1^2}\gamma\exp(z_1 T) ,$$

$$b^{(1,2)} = \frac{T z_2 - 1}{z_2^2}\gamma\exp(z_2 T) ,$$

$$b^{(1,3)} = \frac{\exp(z_1 T)}{z_1} - \frac{\exp(z_2 T)}{z_2} ,$$

$$b^{(2,1)} = \frac{T z_1 - 2}{z_1^3}\gamma\exp(z_1 T) ,$$

$$b^{(2,2)} = \frac{T z_2 - 2}{z_2^3}\gamma\exp(z_2 T) ,$$

$$b^{(2,3)} = \frac{\exp(z_1 T)}{z_1^2} - \frac{\exp(z_2 T)}{z_2^2} ,$$

$$b^{(3,1)} = \frac{T z_1 - 3}{z_1^4}\gamma\exp(z_1 T) ,$$

$$b^{(3,2)} = \frac{T z_2 - 3}{z_2^4}\gamma\exp(z_2 T) ,$$

$$b^{(3,3)} = \frac{\exp(z_1 T)}{z_1^3} - \frac{\exp(z_2 T)}{z_2^3} ,$$

$$b^{(4,1)} = \frac{T z_1 - 4}{z_1^5}\gamma\exp(z_1 T) ,$$

$$b^{(4,2)} = \frac{T z_2 - 4}{z_2^5}\gamma\exp(z_2 T) ,$$

$$b^{(4,3)} = \frac{\exp(z_1 T)}{z_1^4} - \frac{\exp(z_2 T)}{z_2^4} ,$$

$$b^{(1,4)} = \gamma\exp(z_1 T) + (1-\gamma)\exp(z_2 T) ,$$

$$b^{(2,4)} = \frac{\gamma\exp(z_1 T)}{z_1} + \frac{\gamma\exp(z_2 T)}{z_2} ,$$

$$b^{(3,4)} = \frac{\gamma\exp(z_1 T)}{z_1^2} + \frac{\gamma\exp(z_2 T)}{z_2^2} ,$$

$$b^{(4,4)} = \frac{\gamma\exp(z_1 T)}{z_1^3} + \frac{\gamma\exp(z_2 T)}{z_2^3} .$$

5. RESULTS OF SIMULATIONS

A statistic experiment has been performed to verify the obtained theoretical result. A flow of observable events which have been simulated. The values $\{\tau_1, \ldots, \tau_n\}$ were obtained using which the statistics C_k were constructed according to Eq. (7). Based on the statistics C_k the estimations $\hat{\lambda}$, $\hat{T}$, $\hat{\alpha}$, and $\hat{\beta}$ are calculated by Eqs. 9 and 10. A series of 100 experiments allowed us to calculate the mean values in samples and standard deviations which are essential for obtaining of interval estimations. At the 95% level of confidence ($p = 0.95$) intervals of confidence were obtained for estimations of the true values of parameters. In Fig. 2 the results of simulations are presented for the following set of parameters: $\lambda = 10$, $T = 0.001$, $\alpha = 0.5$, $\beta = 5$. So we can conclude that the results of statistical experiment obtained by simulation of the system behavior confirmed the obtained theoretical results.

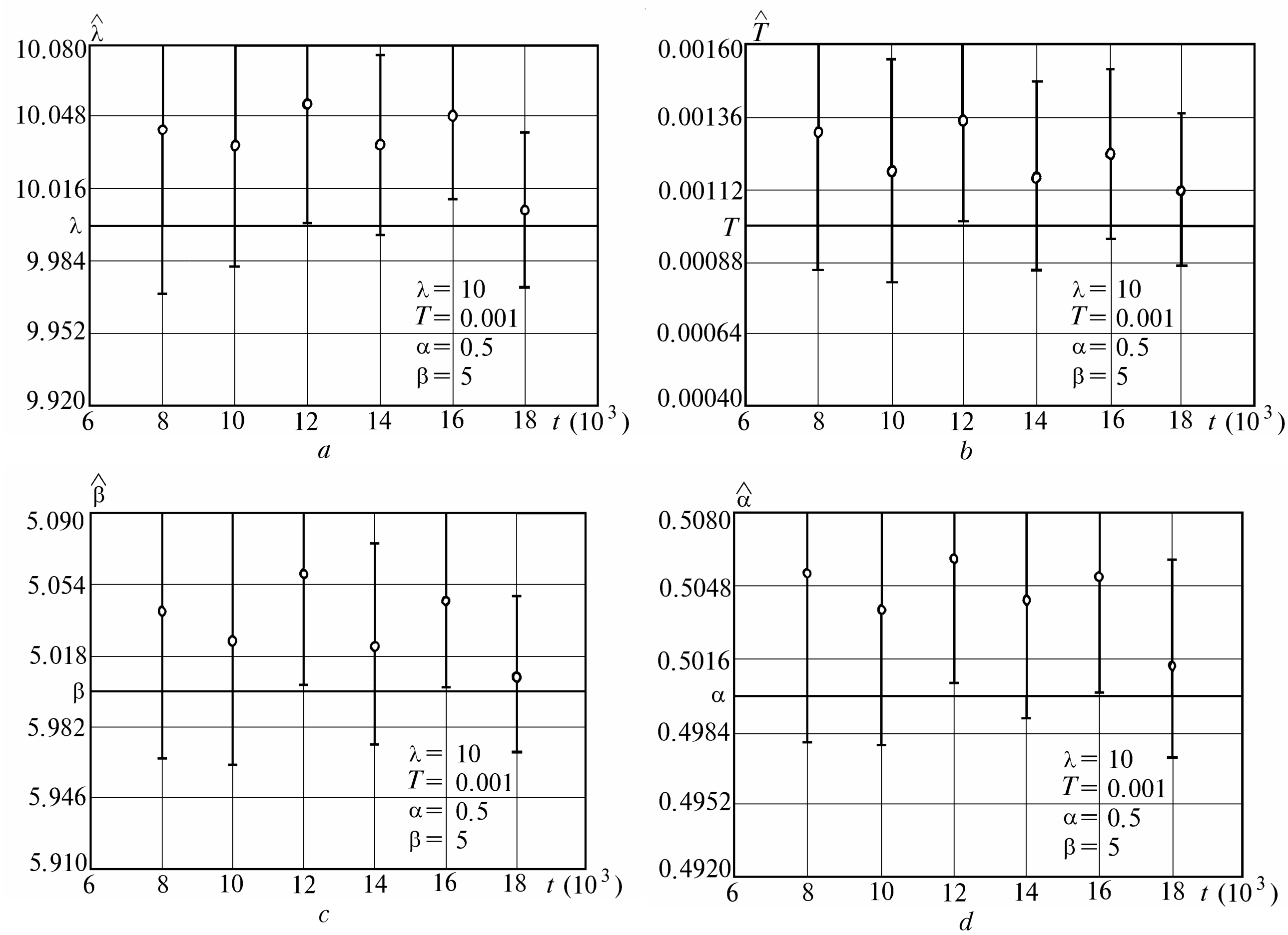

FIG. 2.

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