STATISTICAL MODEL OF A PHASE CONJUGATE ADAPTIVE OPTICAL SYSTEM WHICH ACCOUNTS FOR THE MEASUREMENT NOISE

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Algorithms for reconstructing the wave front and for compensating the phase distortions of the radiation field by means of a phase conjugate adaptive optical system which account for the measurement noise and errors in control with a phase corrector are developed in the framework of the statistical approach on the basis of the methods of calculus of variations and matrices. The phase distributions being reconstructed and compensated are calculated using the a priory *information about the statistical characteristics of the wave front being measured, measurement noise, and errors of control. Equations have been obtained and studied for the Karhunen-Loeve orthogonal modes which guarantee the least error in compensating the phase distortions of the radiation field for the fixed number of modes realized by the phase corrector with the prescribed response functions without errors in their approximation. Results of calculations of the errors of reconstruction and compensation for the number of the measuring channels of a Hartmann sensor and of the actuators being equal to 3, 7, and 19 are given and discussed depending on the intensity of measurement noise and errors of control.*

INTRODUCTION

Algorithm for control with an executing device (phase corrector) of an adaptive optical system (AOS) determines the efficiency of compensation for the phase distortions (PD) of the radiation field and depends on the purpose and the performance figures of the AOS,[1,2] on the character of the PD,[1,3] on the methods of measuring, on the algorithms for data processing,[1] and on the type of the employed phase corrector. The random (noise) and/or deterministic errors in estimating the magnitude of the PD being compensated and errors in realization of the controlling actions by the phase distortion corrector strongly affect the quality of compensation for the PD. The effect of noise can be decreased by optimizing the algorithm for control.[1,4,5] In this case the magnitudes of controlling phase actions will depend on variance, correlation length, and other statistical characteristics of the PD and noise of the radiation field of the AOS. Therefore, implementing the algorithms of this type, it is necessary to carry out either direct measurements of these characteristics or to rely on the *a priori* known information about them.

This classification distinguishes between the zonal and modal algorithms for control of the AOS.[1] Controlling action for the algorithms of the first type are found and realized for each executing channel (actuator) of the phase corrector separately, while in the case of algorithms of the second type –– for each group of actuators corresponding to the definite line combinations (modes) of the response functions of the actuators. The modal algorithms are preferable for the slow AOS, which have no time to organize the control of each actuator separately, or for the search AOS, in the case of independent responses of the goal function of the AOS on the actions of different modes (i.e., the orthogonality of modes is employed).

In connection with this, the problem of search for the Karhunen––Loeve modes[6,7] realized by the phase corrector without the error in their approximation, i.e., which can be represented in the basis of the response functions of the actuator and ensures the fastest decrease of the error in compensating the PD of the radiation field depending on the number of modes employed in the AOS, is actual.

In this paper we study the algorithms for control with the phase corrector (based on the deformable mirror) of the phase conjugate AOS, ensuring the minimum error in compensating the PD with an account of the noise of the PD measurements and of the errors in the realization of the actions of control exerted by the phase corrector. We also consider the problem of reconstructing the PD from the measurements made by means of the wave front detector in the presence of measurement noise with the minimum residual error.

FORMULATION OF THE PROBLEM AND STARTING RELATIONS

Let the PD of the radiation field of the AOS being compensated be described by the random function $\varphi(\boldsymbol{p}, t)$ with the well–known statistics. As a result of measurement with the help of the wave front sensor, the set of readings is entered into the AOS, we represent them in the form

$$M_m = S_m + N_{am} + N_{pm} , \qquad (1)$$

where $M_m = M(\xi_m)$, $\xi_m = (\boldsymbol{p}_i, t_j, k)$; $\boldsymbol{p}_i$ are the radius–vectors of the points of measurements, $i = 1, 2. . .$; t_j is the time of measurement, $j = 1, 2. . .$; k is the serial number of the characteristic of the wave front being measured (for example, when $k = 1$, we measure the local tilts of the wave front along the x axis, while when $k = 2$ – along the y axis); the vectors

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ξ_m are enumerated from $m = 1$ to L according to a definite rule; L is the total number of spatio–temporal readings; the quantity S_m corresponds to the function φ, and the quantity N_{pm} corresponds to the function $\varphi_p(\boldsymbol{p}, t)$ describing the random PD of the radiation field in the receiving optical channel of the AOS, which are absent in its transmitting channel and cannot be eliminated from M_m by means of the preliminary measurements; and the vector N_{am} takes into account the noise of the sensor of the PD of the radiation field (for example, the electronic noise of the radiation detector). The character of the noise N_{am} and the dependence of the readings M_m on the functions φ and φ_p are specified taking into account the type of the wave front sensor.

In the phase conjugate AOS controlled by the zonal algorithm we represent the phase distribution compensating the PD at the moment t and being realized with the help of the phase corrector, in the form of the linear combination of the response functions R_n of the phase corrector

$$\varphi_c(\boldsymbol{p}, t) = \sum_1^L (R_n(\boldsymbol{p}) + \delta R_n(\boldsymbol{p}))\, u_n(t) \ ; \tag{2}$$

$$u_n = (1 + \nu_n) \sum_1^L H_{nm} M_m \ , \tag{3}$$

where $u_n(t)$ are the controlling actions applied at the actuators of the phase corrector with the serial numbers $n = 1, 2. . . N$, ν_n is the error in realization of the controlling action due to the noise in the electron unit intended for control with the phase corrector, H_{nm} are the deterministic weighting coefficients ($n = 1, . . ., N$, $m = 1, . . ., L$), which can be used to realize $u_n(t)$ with an account of the spatio–temporal statistical characteristics of the PD, R_n is the response function of the phase corrector on the action of the nth actuator, and δR_n is the error in realization of the response function by the phase corrector of the AOS.

Thus, the problem of finding the algorithm for control of the AOS is reduced to the determination of the coefficients H_{nm}, which guarantee extremum of the goal function of the AOS (which is understood, for example, as obtaining the nearly diffraction–limited beam or maximization of axial intensity of the beam and so on), assuming that we have already known the mean statistical values of the quantities ν_n, δR_n, S_m, N_{pm}, N_{am}, and $\varphi(\boldsymbol{p}, t)$ and their products.

When implementing the modal algorithm for compensating the PD, each mode $F_n(\boldsymbol{p}, t)$ must be realized by the phase corrector without approximation error, therefore

$$F_n(\boldsymbol{p}, t) = \sum A_{nm}(t)\, (R_m(\boldsymbol{p}) + \delta R_m)\, (1 + \nu_m) \ , \tag{4}$$

where the sought–after weighting coefficients A_{nm}, forming the matrix $A = |A_{nm}|$, determine the contribution of the response function of the phase corrector to the action of the mth actuator when forming mode number n. The weighting coefficients A_{nm} are found from the conditions of orthogonality of the modes and of the fastest decrease of the error in the compensating the PD depending on the number of modes being used for compensation. The compensating phase distribution realized by the corrector in the course of modal control, is represented as a linear combination of the modes $F_m(\boldsymbol{p}, t)$

$$\varphi_c(\boldsymbol{p}, t) = \sum_1^L \upsilon_m F_m(\boldsymbol{p}, t) \ , \tag{5}$$

where the weighting coefficients υ_m form the matrix $V = |\upsilon_n|$ of the controlling modal actions. Since to control the AOS by the modal and zonal algorithms, the functions $\varphi_c(\boldsymbol{p}, t)$ compensating the PD are formed as a superposition of the same response functions of the phase corrector, the matrices of controlling actions in these algorithms are directly related

$$V = A^{-T} H M \ , \tag{6}$$

where $|. . .|^{-\mathrm{T}} = (|. . .|^{-1})^{\mathrm{T}}$, $|. . .|^{\mathrm{T}}$ is the transposed matrix, $|. . .|^{-1}$ is the inverse matrix, $M = |M_n|$ is the column matrix of the readings of the wave front sensor defined by Eq. (1), and H is the matrix of the weighting coefficients, related with the magnitude of the controlling actions of the phase corrector when implementing the zonal control algorithm given by Eqs. (2) and (3).

We disregard the errors of control with the AOS phase corrector when solving the problem of reconstructing the PD. Let us represent the PD being reconstructed as a sum

$$\varphi_c(\boldsymbol{p}, t) = \sum_1^L M_m R_m(\boldsymbol{p}, t) \ , \tag{7}$$

where the weighting coefficients, i.e., the functions $R_m(\boldsymbol{p}, t)$ must be determined from the condition of extremum of the goal function of the AOS. The functions $R_m(\boldsymbol{p}, t)$ can be considered to be the best response functions for the corrector of the AOS in the sense that in the case in which the response functions of the corrector coincide with the functions $R_m(\boldsymbol{p}, t)$ (or their linear combinations), the theoretical extremum of the goal function of the AOS can be obtained in the absence of the error of control with the phase corrector.

For the performance figures of the AOS, we take the quantities of the mean–square residiual errors in compensating for the PD δ^2_φ and its gradient δ^2_g. They define the Strehl factor and divergence of the beam formed by the AOS[1,2] and are calculated according to the formulas

$$\delta^2_\varphi = \overline{<\varphi - \varphi_c - <\varphi - \varphi_c>>^2} \ ; \tag{8}$$

$$\delta^2_g = \overline{<\nabla\varphi - \nabla\varphi_c>^2} \ , \tag{9}$$

where a bar above the formula denotes statistical averaging over the realizations, the angular brackets denote spatial averaging according to the rule:

$$<U> = \int A(\boldsymbol{p})\, U(\boldsymbol{p}) \mathrm{d}\boldsymbol{p} / \int A(\boldsymbol{p}) \mathrm{d}\boldsymbol{p},$$

in the case of Eq. (8) and according to the rule $<\nabla U> = \int A^2(\boldsymbol{p})\, (\nabla U(\boldsymbol{p})) \mathrm{d}\boldsymbol{p} / \int A^2(\boldsymbol{p}) \mathrm{d}\boldsymbol{p}$ in the case of Eq. (9) ($U(\boldsymbol{p})$ is an arbitrary function), $A(\boldsymbol{p})$ is the amplitude distribution of the field in the beam, and ∇ is the gradient operator, which acts along the transverse coordinates of the beam. Algorithms for reconstructing and compensating the PD must minimize expressions (8) and (9).

Relations (1) – (9) are the starting ones for finding the zonal and modal algorithms for control of the phase conjugate AOS and algorithm for reconstructing the PD.

ALGORITHM FOR RECONSTRUCTING THE PD

The problem of finding the algorithm for reconstructing the PD incorporates the determination of the weighting functions $R_m(\boldsymbol{p}, t)$, providing the reconstruction of the PD of the radiation field with minimum errors given by Eq. (8) or (9). Substituting Eq. (7) into Eq. (8) or (9) and varying either δ_φ^2 or δ_g^2 as functional of R_n by the methods of calculus of variations,[8] we find the optimal weighting functions R_n

$$|R_n(\boldsymbol{p}, t)| = |\overline{M_n M_m}|^{-1} |K_m| ; \quad n, m = 1,\ldots,L ; \quad (10)$$

$$K_m = K_m(\boldsymbol{p}, t; \xi) = \overline{\varphi(\boldsymbol{p}, t)\, M_m} ,$$

where $|R_n(\boldsymbol{p}, t)|$ and $|K_m|$ are the column matrices. Relation (10) assumes linear independence of the functions K_m.

Relations (7) and (10) describe the algorithm for optimal reconstructing the PD and the PD gradients from the discrete readings of the wave front sensor. Residiual errors of reconstructing the PD are minimized for both performance figures given by Eqs. (8) and (9) simultaneously and are calculated according to the formulas

$$\delta_\varphi^2 = \sigma_\varphi^2 - \mathrm{Sp}[\,|\langle K_m K_n\rangle - \langle K_m\rangle\langle K_n\rangle|\, |\overline{M_n M_k}|^{-1}] , \quad (11)$$

$$\delta_g^2 = \sigma_g^2 - \mathrm{Sp}[\,|\langle\nabla K_m \nabla K_n\rangle|\, |\overline{M_n M_k}|^{-1}] , \quad (12)$$

where $\delta_\varphi^2 = \langle \overline{\varphi^2} \rangle - \overline{\langle\varphi\rangle}^{\,2}$ and $\sigma_\varphi^2 = \langle \overline{(\nabla\varphi(\rho))^2} \rangle$ are the mean squares of the PD and of the PD gradient, respectively; the symbol Sp denotes the operation of finding of the spur of matrix.

Let us find the Karhunen–Loeve functions, which can be used to minimize the volume of calculations in reconstructing the PD with a fixed error. Let us use the matrix method of construction of the Karhunen–Loeve modes. Since they must be spatially orthogonal and be represented in the basis $R_m - \langle R_m\rangle$, let us construct the system of the functions $F_n - \langle F_n\rangle = \sum A_{nm}(R_m - \langle R_m\rangle)$, orthonormalized with the weight $A(\boldsymbol{p})$, where the coefficients A_{nm} are the sought–after values. In a matrix form, the condition of orthonormalization

$$A^{-T} |\langle R_n R_m\rangle - \langle R_n\rangle\langle R_m\rangle|^{-1} A^{-1} = E , \quad (13)$$

where E is the unit matrix. Equation (13) corresponds to the performance figure given by Eq. (8). When determining the AOS modes with the performance figure of the correction of the PD in the form of Eq. (9), the functions $\nabla F_n(\boldsymbol{p})$ are orthonormalized with the weight $A^2(\boldsymbol{p})$, and the matrix of overlap $|\langle R_n R_m\rangle - \langle R_n\rangle \langle R_m\rangle|$ is replaced by $|\langle\nabla R_n(\boldsymbol{p})\, \nabla R_m(\boldsymbol{p})\rangle|$. Since Eq. (13) has an infinite number of solutions, let us assume that the matrix product entering into Eq. (11) is diagonal in the basis of functions F_n, i.e.,

$$A^{-T} |M_n M_m|\, A^{-1} = |\delta_{nm}\lambda_n^2| , \quad (14)$$

where λ_n^2 are the characteristic numbers[9] and δ_{nm} is the Kronecker–Capelli symbol. The system of equations (13) and (14) corresponds to the problem of reduction of two symmetrical quadratic forms to their canonical form. The modes F_n being determined are characterized by the following property of extremum:[10] let the chracteristic numbers be enumerated in such a way that they form a nonincreasing sequence $\lambda_{n+1}^2 \le \lambda_n^2$. Let us assume that there exists a system of the orthogonal modes W_n different from F_n, which, analogous to F_n, is constructed in the basis of the functions $R_m - \langle R_m\rangle$; then the first $m \le N$ modes $(F_1, \ldots F_m)$ always ensure the error in reconstructing of the PD which is smaller than that for modes $(W_1, \ldots W_m)$; in addition, the mean square of the reconstructed function of the PD is equal to $\sum_1^m \lambda_n^2$, and the mean square of the error of reconstruction is calculated according to the formula:

$$\delta_{\varphi,g}^2 = \sigma_{\varphi,g}^2 - \sum_1^m \lambda_n^2 . \quad (15)$$

Thus, the system of matrix equations (13) and (14) determines the Karhunen–Loeve modes when reconstructing the PD from the discrete spatio–temporal readings with an account of the measurement noise.

The Karhunen–Loeve modes can be found also by the method described in Ref. 6. In this case the integral equations

$$\lambda_i^2 (F_i(\boldsymbol{p}) - \langle F_i\rangle) = \int A(\boldsymbol{p}_1)(F_i(\boldsymbol{p}) - \langle F_i\rangle) \times$$

$$\times \sum_{n,}^{L} \sum_{m=1}^{L} (R_n(\boldsymbol{p}_1) - \langle R_n\rangle)\,(R_m(\boldsymbol{p}) - \langle R_m\rangle)\, \overline{M_n M_m}\, \mathrm{d}\boldsymbol{p}_1 ; \quad (16)$$

$$\lambda_i^2\, F_i(\boldsymbol{p}) = \int A^2(\boldsymbol{p}_1)(\nabla F_i(\boldsymbol{p}_1)) \times$$

$$\times \sum_{n,}^{L} \sum_{m=1}^{L} (\nabla R_n(\boldsymbol{p}_1)\, R_m(\boldsymbol{p}))\, \overline{M_n M_m}\, \mathrm{d}\boldsymbol{p}_1 , \quad (17)$$

$i = 1, \ldots$, M coinciding by their content with relations (13) and (14) were derived. The first equation corresponds to the figure of performance given by Eq. (8), and the kernel of this equation is proportional to the correlation function $\overline{(\varphi_c(\boldsymbol{p}) - \langle\varphi_c\rangle)\,(\varphi_c(\boldsymbol{p}_1) - \langle\varphi_c\rangle)}$. The second equation corresponds to the figure of performance given by Eq. (9), and its kernel is proportional to $\overline{(\varphi_c(\boldsymbol{p})\, \nabla\varphi_c(\boldsymbol{p}_1))}$. The equivalence of the system of equations (13) and (14) with Eq. (16) can be established by means of substitution of the relation $F_n = \sum A_{nm} R_m$ into formula (16).

Thus, relations (7) and (10) determine the algorithm for optimal reconstructing the PD from the discrete spatio–temporal readings of the wave front sensor in the presence of the measurement noise, and equations (13) and (14) or (16) and (17) determine the Karhunen–Loeve modes which can be used for modal reconstructing the PD. In this case, the residual errors in reconstructing the PD are determined according to formulas (11), (12), and (15).

For the AOS in which the spatio–temporal measurements of the PD are carried out continuously (for example, the interference wave front sensor is used as the PD sensor in the AOS) Eq. (7) takes the form of the integral

$$\varphi_c(\boldsymbol{p},\ t) = \int R(\boldsymbol{p},\ t;\ \mathbf{x})\, M(\xi)\mathrm{d}\xi \qquad (18)$$

for the PD function being reconstructed, and the problem of reconstruction in this statement (in contrast to the problem of reconstructing the PD from discrete readings) is, in general case, to separate out the statistical function of the PD from the measured function and the function minimizing the error in reconstructing the function $R(\boldsymbol{p},\ t,\ \xi)$. In formula (18), $M(\xi)$ is the function of the measured quantities, and ξ is the vector defined earlier by relation (1). We note that in the case of discrete measurements, the equation determining the optimal weighting functions R_n is obtained by means of representing readings given by Eq. (1) in the form of the function $\sum \delta(\xi - \xi_m)M_m$.

From relations (8) and (9) taking into account Eq. (8) we find the integral equation for determining the optimal weighting function $R(\boldsymbol{p},\ t;\ \xi)$

$$\overline{\varphi(\boldsymbol{p},\ t)\, M(\xi)} = \int R(\boldsymbol{p},\ t;\ \mathbf{x}_1)\ \overline{M(\xi)\, M(\xi_1)}\ \mathrm{d}\xi_1 \qquad (19)$$

by the variational method. Equation (19) is the spatio—temporal Wiener—Hopf equation. Starting from Eq. (19), we find the equation for $R(\boldsymbol{p},\ t;\ \xi)$. Let the solutions of the equation

$$\gamma_n Q_n(\xi) = \int Q_n(\xi_1)\ \overline{M(\xi)\, M(\xi_1)}\ \mathrm{d}\xi_1 \qquad (20)$$

be well known, where Q_n is the complete system of the orthonormalized functions, $n = 1, 2, \ldots$, γ_n are the corresponding eigenvalues, and $|\gamma_{n+1}| \le |\gamma_n|$. Expanding $R(\boldsymbol{p},\ t;\ \xi)$ in terms of the functions $Q_n(\xi)$ and substituting them into Eq. (19), we derive

$$R(\boldsymbol{p},\ t;\ \xi) = \int \overline{\varphi(\boldsymbol{p},\ t)\, M(\xi_1)}\ Q(\xi,\ \xi_1)\mathrm{d}\xi_1\ ; \qquad (21)$$

$$Q(\xi,\ \xi_1) = \sum Q_n(\xi)\, Q_n(\xi_1)/\gamma_n\ .$$

Thus, relation (18) is the optimal solution of the problem of reconstructing the PD and its gradient from continious measurements. The weighting function $R(\boldsymbol{p},\ t;\ \xi)$, entering into Eq. (18) is given by Eq. (21). Having determined the weighting function $R(\boldsymbol{p},\ t;\ \xi)$ we can calculate the residual error in reconstructing the PD and its gradients. Equations for their determination are obtained after substitution of Eqs. (18) and (19) into Eqs. (8) and (9), respectively:

$$\delta_\varphi^2 = \sigma_\varphi^2 - \int \mathrm{A}(\boldsymbol{p})\, \mathrm{R}(\boldsymbol{p},\ t,\ \xi) \times$$

$$\times\ \overline{(\varphi(\boldsymbol{p},\ t) - \langle\varphi(\boldsymbol{p},\ t)\rangle)\, M(\xi)}\ \mathrm{d}\boldsymbol{p}\mathrm{d}\xi \Big/ \int \mathrm{A}(\boldsymbol{p})\mathrm{d}\boldsymbol{p}\ ;$$

$$\delta_g^2 = \sigma_g^2 - \int \mathrm{A}^2(\boldsymbol{p})\, \nabla\mathrm{R}(\boldsymbol{p},\ t,\ \xi) \times$$

$$\times\ \overline{(\nabla\varphi(\boldsymbol{p},\ t))\, M(\xi)}\ \mathrm{d}\boldsymbol{p}\mathrm{d}\xi \Big/ \int \mathrm{A}^2(\boldsymbol{p})\mathrm{d}\boldsymbol{p}\ . \qquad (22)$$

Let us write out the integral equations for determining the Karhunen—Loeve modes in the case of continious measurements of the PD in the presence of the measurement noise. Taking into account Wiener—Hopf equation (19) and the algorithm for reconstructing given by Eq. (18), the sought—after integral equations for the performance figures in the form of Eqs. (8) and (9), respectively, take the form

$$\lambda_i^2\, (F_i(\boldsymbol{p},\ t) - \langle F_i\rangle) = \int A(\boldsymbol{p}_1)(F_i(\boldsymbol{p}_1,\ t) - \langle F_i\rangle) \times$$

$$\times\ R(\boldsymbol{p}_1,\ t,\ \xi)\ \overline{(\varphi(\boldsymbol{p},\ t) - \langle\varphi\rangle)\, M(\xi)}\ \mathrm{d}\boldsymbol{p}_1\mathrm{d}\xi\ , \qquad (23)$$

$$\lambda_i^2\, F_i(\boldsymbol{p},\ t) = \int A^2(\boldsymbol{p}_1)\nabla F_i(\boldsymbol{p}_1,\ t)\ [\ \overline{(\varphi(\boldsymbol{p},\ t))\, M(\xi)}\ ] \times$$

$$\times\ \nabla\mathrm{R}(\boldsymbol{p}_1,\ t,\ \xi)\mathrm{d}\boldsymbol{p}_1\mathrm{d}\xi\ . \qquad (24)$$

Error in the reconstructing the PD, similary to the case of discrete readings, is calculated according to formula (15).

Relations, given in this section, describe the algorithms for the reconstruction of the PD from discrete and continious measurements of the PD which accounts for the measurement noise. Study of the efficiency of the developed algorithms for the reconstruction of the PD in the AOS as applied to the solution of practical problems in applied optics requires precise definition of the statistical properties of the PD and noise.

ALGORITHM FOR CONTROL WITH THE PHASE CORRECTOR

Determining the algorithm for control with the AOS phase corrector incorporates an adjustment of the coefficients H_{nm} and calculation, using them (taking into account the readings of the wave front sensor M_m), the controlling actions u_n exerted by the phase corrector in the form of Eqs. (2) and (3), and minimization of the performance figures of the AOS in the form of Eqs. (8) and (9).

Equation determining the coefficients H_{nm} of the matrix H are found from the condition of minimization of the residual error in compensating the PD δ_φ^2 in the form of Eq. (8) and the PD gradients δ_g^2 in the form of Eq. (9), as functions of H_{nm}:

$$H = |RR|^{-1}\, |R\varphi M|\ |\ \overline{M_n M_m}\ |^{-1}, \qquad (25)$$

in which, if the quality of compensating the PD is determined by Eq. (8), we have

$$|RR| = |\ \overline{\langle \check{R}_n \check{R}_m\rangle} - \overline{\langle \check{R}_n\rangle\langle \check{R}_m\rangle}\ |\ \text{and}$$

$$|R\varphi M| = |\ \overline{\langle \check{R}_n}\ \ \overline{\varphi M_m\rangle} - \overline{\langle \check{R}_n}\ \rangle\langle\ \overline{\varphi M_m}\rangle\ |,$$

and if the quality of compensating the PD is given by Eq. (9), we have

$$|RR| = |\ \overline{\langle\nabla\check{R}_n(\boldsymbol{p})\, \nabla\check{R}_m(\boldsymbol{p})\rangle}\ |\ \text{and}$$

$$|R\varphi M| = |\ \overline{\langle\nabla\check{R}_n(\boldsymbol{p})}\ \ \overline{(\nabla\varphi M_m)\rangle}\ |$$

where notation $\check{R}_n(\boldsymbol{p}) = (R_n(\boldsymbol{p}) + \delta R_n) \cdot (1 + \nu_n)$ has been employed. In the derivation of relation (25), we assumed that the random variables ν_n and δR_n were statistically independent of M_n and φ.

Relations (2), (3) and (25) determine the optimal algorithm for control with the phase corrector of the AOS, which provides the correction of the PD with minimum residual error δ^2_φ in the form of Eq. (8) or its gradient δ^2_g in the form of Eq. (9). In addition, we assume that measuring of the spatial distribution of the PD is carried out by the wave front sensor in a discrete number of points and the noise component, according to Eq. (1), is present in readings, while the correcting phase actions are executed by the phase corrector with the error of control incorporating the components δR_n and ν_n, respectively.

In this case the error in compensating the PD of the radiation field in the AOS is calculated according to the formula

$$\delta^2_{\varphi,g} = \sigma^2_{\varphi,g} - \mathrm{Sp}[\,|R\varphi M|\;|M_n M_m|^{-1}\,|R\varphi M|^{\mathrm{T}}\,|RR|^{-1}], \quad (26)$$

in which the values $\sigma^2_{\varphi,g}$ are given by Eqs. (8) and (9).

The compensating phase distribution realized in optimal control of the AOS with respect to the performance figure in the form of Eq. (8) does not coincide with the distribution being optimal with respect to the performance figure in the form of Eq. (9) since optimizing the figure of performance in the form of Eq. (8) maximizes the axial beam intensity, whereas in the form of Eq. (9) minimizes the beam angular divergence. In the absence of the noise component in the case of ideal operation of the AOS thus that the ideal plane or spherical radiation wave front is reconstructed as a result of its action, the algorithm for control of the AOS minimizes the performance figures according to Eqs. (8) or (9), i.e., in the absence of noise with increase of the number of control channels of the AOS and of the measurements of the PD, the algorithm described by Eqs. (2), (3), and (25) ensures the errors in compensating for the PD which approach zero when using either of the performance figures given by Eqs. (8) or (9).

As well as for the solution of the problem of the PD reconstruction, to find the algorithm for optimal control with the corrector of the phase conjugate AOS of modal type, the Karhunen–Loeve modes are constructed. The system of equations for determining the weighting coefficients A_{nm} needed for construction of these modes $F_n(\boldsymbol{p})$, has the following form:

$$A^{-\mathrm{T}}\,|RR|^{-1}\,A^{-1} = E\;; \quad (27)$$

$$A^{-\mathrm{T}}H\;|\,\overline{M_n M_m}\,|\;H^{\mathrm{T}}A^{-1} = |\lambda^2_n \delta_{nm}|\;. \quad (28)$$

The error in compensating the PD is calculated according to formula (15), and controlling modal actions are calculated taking into account Eq. (6).

It should be noted that in the absence of errors of control, the modes $F_n(\boldsymbol{p}) - \langle F_n\rangle$ are spatially orthogonal in the sense of the requirement $\langle (F_n - \langle F_n\rangle)(F_m - \langle F_m\rangle)\rangle = \delta_{nm}$. In the presence of the errors in control of the AOS, the modes $F_n(\boldsymbol{p})$ given by Eqs. (4) with an account of Eqs. (27) and (28), are spatially orthogonal in a statistical sense, i.e.,

$$\overline{\langle (F_n - \langle F_n\rangle)(F_m - \langle F_m\rangle)\rangle} = \delta_{nm}.$$

In the phase corrector of the AOS the mean statistically orthogonal modes F_n can be realized with null error of their approximation by virtue of condition (4), therefore the implementation of algorithms (5) and (6) in the phase conjugate AOS of modal type will provide realization of the compensation for the PD with the theoretically smallest residual error.

Thus, relations (2), (3), and (25) and (5), (6), (27), and (28) determine the zonal and modal algorithms, respectively, for optimal control with the phase corrector of the AOS based on the discrete spatio–temporal measurements which accounts for the measurement noise and for the errors of realization of controlling actions. The errors of compensation are calculated according to formulas (26) and (15).

Generalization of the above–considered problem on the case of control of the AOS in which the PD are measured continuously, leads to the necessity of constructing its solution in the integral form. Thus, formula (3) for determining the controlling actions of the AOS phase corrector, will have the form

$$u_n(t) = \int H_n(t;\,\xi)\,M(\xi)\,\mathrm{d}\xi\;, \qquad n = 1,\ldots,N\;, \quad (29)$$

in which the functions H_n must be optimized. Realizing the variation of the functions H_n in Eqs. (8) and (9) with an account of Eq. (29) we arrive at the relation for their determination

$$\Big|\int H_n(t;\,\xi_1)\;\overline{M(\xi)\,M(\xi_1)\,\mathrm{d}\xi_1}\;\Big| =$$

$$= \Big|\;\overline{\langle \check{R}_n \check{R}_m\rangle} - \overline{\langle \check{R}_n\rangle\langle \check{R}_m\rangle}\;\Big|^{-1} \times$$

$$\times\;\Big|\;\overline{\langle \check{R}_m}\;\;\overline{\varphi M(\xi)\rangle} - \overline{\langle \check{R}_m}\rangle\langle\,\overline{\varphi M(\xi)\rangle}\;\Big|\;; \quad (30)$$

$$\Big|\int H_n(t;\,\xi_1)\;\overline{M(\xi)\,M(\xi_1)}\;\mathrm{d}\xi_1\Big| =$$

$$= \Big|\;\overline{\langle \nabla\check{R}_n(\boldsymbol{p})\nabla\check{R}_m(\boldsymbol{p})\rangle}\;\Big|^{-1}\Big|\langle\nabla\;\overline{\check{R}}_m(\boldsymbol{p})\;\overline{(\nabla\varphi(\boldsymbol{p})M(\xi))}\;\rangle\Big|. \quad (31)$$

Integral equations (30) and (31) are the Wiener–Hopf equations for the filter functions $H_n(t;\,\xi)$ of the phase conjugate AOS with the prescribed response function on the actions of the actuators in the presence of noise of measurement of the PD and errors of control of the AOS. Equation (30) corresponds to the figure of performance in the form of Eq. (8), and Eq. (9) corresponds to the figure of performance in the form of Eq. (9). If we know the solutions of auxiliary integral equation (20), the solutions of equations (30) and (31) can be represented in the explicit form:

$$|H_n(t;\,\xi)| = |RR|^{-1}\,|R\varphi M(\xi)|\;, \quad (32)$$

where, in the case of the figure of performance given by Eq. (8),

$$|R\varphi M(\xi)| = \Big|\int\Big\{\langle\;\overline{\check{R}_n(\boldsymbol{p})}\;\;\overline{\varphi(\boldsymbol{p},\,t)M(\xi_1)}\rangle -$$

$$- < \overline{\check{R}_n(\boldsymbol{p})} > < \overline{\varphi M(\xi_1)} > \} Q(\xi, \xi_1)\mathrm{d}\xi_1 \Big|$$

and in the case of the figure of performance given by Eq. (9)

$$|R\varphi M(\xi)| = \Big| \int \Big\{ < \overline{\check{R}}_n(\boldsymbol{p}) \times$$

$$\times \overline{(\nabla\varphi(\boldsymbol{p})M(\xi_1))} > \Big\} Q(\xi, \xi_1)\mathrm{d}\xi_1 \Big| .$$

Here the function Q is determined by relation (21). The error of compensation is calculated according to the formula

$$\delta^2_{\varphi,g} = \sigma^2_{\varphi,g} - \mathrm{Sp}[\,|RR|\ |HH|\,] ,$$

where

$$|HH| = \Big| \int \int H_n(t;\, \xi)\, H_m(t;\, \xi_2) \times$$

$$\times \overline{M(\xi_1)\, M(\xi_2)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2} \Big| .$$

Equations for determining the Karhunen–Loeve functions are analogous to the system of equations (27) and (28) and are found taking into account Eq. (29):

$$A^{-\mathrm{T}}\, |RR|^{-1} A^{-1} = E\ ; \qquad (33)$$

$$A^{-\mathrm{T}}\, |HH|\, A^{-1} = \big| \lambda_n^2\, \delta_{nm} \big| . \qquad (34)$$

Thus, relations (2) and (29)–(32) and (4), (33), and (34) describe the zonal and modal algorithms for control of the phase conjugate AOS, based on the continuous measurements of the PD which guarantee the compensation for the PD with a minimum residual error in the presence of noise of the measurement of the PD and the errors of control. In addition, in the case of the modal algorithm for control, the modes formed by the phase corrector, are statistically mean orthogonal and provide the fastest decrease of the error of the compensation with increase of the number of the modes used in the AOS.

The approach to the development of the algorithm for control with the phase corrector of the AOS incorporating the determination of controlling phase actions, which optimize the goal function of the AOS (being the function of performance figures in the form of Eqs. (8) and (9)), directly recorded by the system without the intermediate stage of measuring and processing of the measurements of the PD is well–known.[1] Search for the extremum of the goal function can be implemented with the help of the search – for modal algorithm for control with the corrector by means of a successive search optimization of the goal function of the AOS for each mode whose number and shape must be determined in accordance with the application of the AOS. In this case the requirements to the modes of the AOS practically reproduce the requirements to the Karhunen–Loeve modes for the phase conjugate AOS, namely, the modes must be spatially orthogonal for independence of the control channels of the AOS and for reaching the global extremum of the goal function; the error of compensation in the case of a successive realization of modes must decrease rapidly with increase of the serial number of mode, for the fast AOS with fixed quality of compensation; the modes must be formed by the corrector of the AOS without the error in their approximation, which will lead to deterioration of the convergence of the algorithm and to the increase of the error of compensation. To apply relations for determining the above–obtained optimal modes to the AOS with search for an algorithm of control, it is sufficient to set the noise of the measurement of the PD equal to zero in formulas (29)–(34). Corresponding modification of Eqs. (29)–(34) leads to the following system of equations:

$$A^{-\mathrm{T}}\, |RR|^{-1} A^{-1} = E\ ; \qquad (35)$$

$$A^{-\mathrm{T}}\, G A^{-1} = \big| \mathrm{l}_n^2\, \mathrm{d}_{nm} \big| , \qquad (36)$$

where

$$G = |RR|^{-1} \times \Big| \int \int A(\boldsymbol{p}_1) A(\boldsymbol{p})\, \overline{(\varphi(\boldsymbol{p}_1, t) - <\varphi(\boldsymbol{p}_1, t)>)(\varphi(\boldsymbol{p}, t) - <\varphi(\boldsymbol{p}, t)>)}\ \overline{\check{R}_n}(\boldsymbol{p}_1)\ \overline{\check{R}_m}(\boldsymbol{p}) \mathrm{d}\boldsymbol{p}_1 \mathrm{d}\boldsymbol{p} \Big/ \Big(\int A(p)\mathrm{d}p\Big)^2 \Big|\ |RR|^{-1}$$

$$G = |RR|^{-1} \Big| \int \int A^2(\boldsymbol{p}_1) A^2(\boldsymbol{p}) \times \overline{(\nabla\overline{\check{R}_n}(\boldsymbol{p}_1)\nabla\varphi(\boldsymbol{p}_1, t))(\nabla\overline{\check{R}_m}(\boldsymbol{p})\nabla\varphi(\boldsymbol{p}, t))}\ \Big(\int A^2(p)\mathrm{d}p\Big)^{-2} \mathrm{d}p_1\ \mathrm{d}p \Big|\ |RR|^{-1}$$

in the cases of the performance figures in the form of Eqs. (8) and (9), respectively. Thus, relations (35) and (36) determine the Karhunen–Loeve modes for the search modal algorithm for compensating the PD, in addition, the residual error of compensation is calculated according to formula (15).

The spatially orthogonal modes minimizing either performance figures in the form of Eqs. (8) and (9) simultaneously, are of interest for the AOS with modal control. In this case the first equation of the system of equations (28) and (29) (or (24), (25), (35), and (36)) is substituted by two equations describing the condition of orthogonality of the functions ($F_n - <F_n>$) and (∇F_n), i.e., the problem of finding the algorithm for control of the AOS is reduced to the solution of the system of three equations. The modes ($F_n - <F_n>$), constructed for this problem, will be mean–statistically spatially–orthogonal in the process of control of the AOS with respect to either energy performance figures, i.e., such an algorithm for control will guarantee the formation by the AOS of the radiation fluxes having a minimum divergence and a maximum pick intensity. However, proof of existence of solutions of the above–described system of equations requires a special study.

STUDY OF THE ALGORITHMS FOR RECONSTRUCTING AND COMPENSATING THE PD

A developed theory was used to study the efficiency of the zonal and modal algorithms for control with the phase corrector on the basis of the deformable mirror (DM) of the phase conjugate AOS with deterministic response functions in the presence of the noise component in the measured PD. To this end, we determined the error in compensating the PD as a function of the number of control channels of the phase corrector of the AOS, of the signal–to–noise ratio when taking each reading of the PD by the Hartmann wave front sensor, and of the correlation length of the PD and compared this error with its minimum value obtained when realizing the developed procedure of optimal reconstructing the PD.

Numerical calculations of the Karhunen–Loeve modes were carried out on the basis of matrix equations (13) and (14) and (27) and (28), since they required much shorter computation time compared to the calculations on the basis of the integral equations.

When carrying out the calculations, we assumed that the round actuators of the DM were located in the nodes of the hexagonal grid. The response functions of the round DM were taken in the form[12] $R_n(\boldsymbol{p}) = \exp[\ln 0.15((\boldsymbol{p} - \boldsymbol{p}_n)/b)^2]$, where b is the distance between the actuators and $\boldsymbol{p}_n$ are the coordinates of the nodes of the hexahonal grid. The number of the channels of the Hartmann wave front sensor was taken equal to 3, 7, and 19 and was equal to the number of the actuators of the DM.

Vector of the measured PD in this assumption had the form

$$M_n = \partial\varphi(\boldsymbol{p}_n, t - \tau)/\partial w_n + N_{aw}(\boldsymbol{p}_n) \; ; \; n = 1, 2, ..., 2N \; ,$$

where τ is the time interval between the moments of Numerical measurement of the PD and realization of control with the DM, $w = x$ for $n = 1, 2, ..., N$ and $w = y$ for $n = N + 1, , 2N$, $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ are the slopes of the function of the FD being reconstructed or compensated, and N_{ax} and N_{ay} are the errors of their measurements (measurement noise). The statistical characteristics of the PD were given by the following relations:

$$\overline{\varphi(\boldsymbol{p}_1, t_1)\varphi(\boldsymbol{p}_2, t_2)} = \sigma^2 B_t(|t_1 - t_2|)\exp(-|\boldsymbol{p}_1 - \boldsymbol{p}_2|^2/R_c^2) \; ;$$

$$\overline{N_{ax}(\boldsymbol{p}_n) N_{ay}(\boldsymbol{p}_m)} = 0 \; ;$$

$$\overline{N_{ax,y}(\boldsymbol{p}_n)\, d\varphi(\boldsymbol{p}_m)/\mathrm{d}w} = 0 \; ;$$

$$\overline{N_{ax,y}(\boldsymbol{p}_n)\, \varphi(\boldsymbol{p}_m)} = 0 \; ;$$

$$\overline{N_{ax}(\boldsymbol{p}_n) N_{ax}(\boldsymbol{p}_m)} = \overline{N_{ay}(\boldsymbol{p}_n) N_{ay}(\boldsymbol{p}_m)} = N_a^2\, \delta_{nm}/2 \; ,$$

in which $B_t(|t_1 - t_2|)$ is the coefficient of the time correlation $\varphi(\boldsymbol{p}, t)$; σ^2 and R_c are the variance and correlation length of the PD being reconstructed, respectively, and N_a^2 is the noise variance. Calculations of the residual errors in compensating the PD were carried out according to formula (26), and errors in reconstructing – according to Eq. (11); the modes optimal for the modal algorithm for control were found from the systems of matrix equations (13) and (14) and (27) and (28).

The family of dependencies of the relative residual error in compensating the PD $\varepsilon(S) = \delta_\varphi^2/(<\overline{\varphi^2}> - \overline{<\varphi>^2})$ on the relative noise level $S = N_a^2/g^2$, where the parameter $g^2 = 4\sigma^2/c^2$ is the mean square of the PD gradient, $c = R_c/R_m$ is the relative correlation length of the PD, and R_m is the radius of the mirror, is shown in Fig. 1 for the AOS without delay of control ($\tau = 0$). Curves *1*, *2* and *3* in Fig. 1 have been obtained for the AOS with the number of the controlling channels $N = 3$, 7, and 19, respectively, with relative correlation length of the PD $c = 0.5$. Curves *4* and *5* in Fig. 1 correspond to $c = 1$ and 2 for the AOS with the number of the control channels $N = 7$. The solid curves correspond to the error in compensating the PD of the AOS, the dashed curves correspond to the errors in reconstructing the PD.

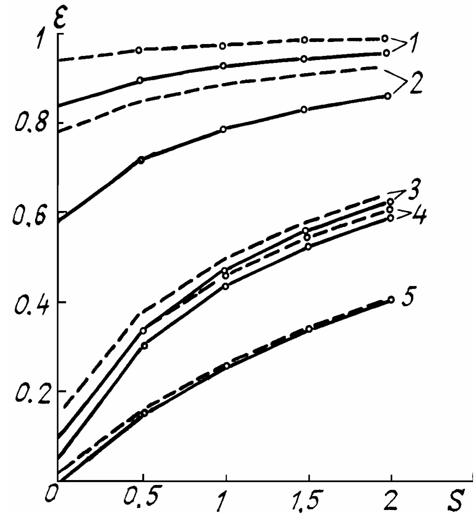

FIG. 1. Dependence of the relative residual error in compensating (solid curves) and reconstructing (dashed curves) the PD on the level of noise S. Curves 1, 2, and 3 correspond to N = 3, 7, and 19 for c = 0.5. Curves 4 and 5 correspond to N = 7 for c = 1 and 2.

Comparison of the dependencies shows that the error in optimal reconstructing the PD of the AOS is always less than the error of their compensation. Relative difference in the errors of compensation and reconstruction in the considered range of variation in the parameters increases with increase of the number of the control channels of the AOS and reaches 5–8 times for $N = 19$. Decrease of their relative difference with decrease of the control channels N of the AOS is explained by a total decrease of the efficiency of compensating and reconstructing the PD. Here the values of ε are close to 1 in both cases. Increase of the noise level in the AOS leads to the increase in the error in compensating and reconstructing the PD and, as a consequence, to the decrease of the relative difference between them, which for $S \geq 0.5...1$ is <10%. Absolute difference between the errors of compensation and reconstruction for $S = 0...0.3$ remains approximately constant and equal to its value without noise. Thus, the dependences can be used to determine the efficiency of the reconstruction and compensation of the PD for the phase conjugate AOS and to justify the choice of the efficient parameters of the AOS intended for the solution of practical problems of correction of the PD.

Figure 1 can be used for estimation of the efficiency of the AOS with the time delay τ in executing the controllable compensating phase actions. In this case the errors of compensation (and reconstruction) $\varepsilon(\tau, S)$ are related to $\varepsilon(S)$ ($\varepsilon(0, S) = \varepsilon(S)$ for $\tau = 0$) by the formula following from Eqs. (8), (9) and (17):

$$\varepsilon(\tau, S) = 1 - B_t^2(\tau) * [1 - \varepsilon(S)] \; .$$

This equation can be used to estimate the admissible time delay τ_0 in executing the controllable phase action compensating the PD. We will start from the condition that for time delays $\tau < \tau_0$ the dynamic part of the error in compensating the PD ($\varepsilon(\tau, S) - \varepsilon(S)$) does not exceed its statistical part ($\varepsilon(S)$). In the case of the near–Gaussian correlation coefficient $(B_t(\tau) = \exp(-|\tau/\tau_c|^q)$, where τ_c is correlation time and $q > 0$), the given condition is fulfilled for $\tau_0 \approx \tau_c\,[\varepsilon(S)/2]^{1/q}$. Less stringent requirements for fast adaptive control correspond to large values of q.

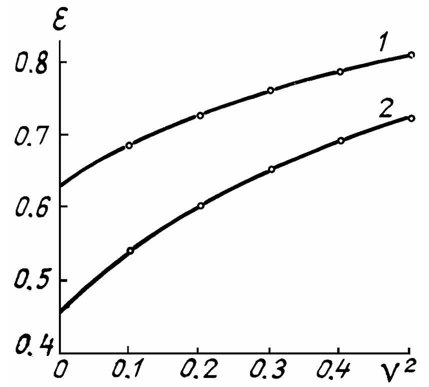

FIG. 2. Dependence of the relative residual error of compensation on the error variance of control. Curves 1 and 2 correspond to $S = 0.0$ *and* $S = 0.5$ *for* $N = 7$, $c = 0.5$, *and* $b = 0.5$.

Dependence of the error in compensating the PD on the error variance of control ν^2 is shown in Fig. 2 for $\tau = 0$. Statistical characteristics of the errors of control were taken in the form $\overline{\nu_n \nu_m} = \nu^2 \delta_{nm}$. The relative error of compensation $\varepsilon(\nu, S)$ increases with ν^2 and is $\varepsilon(\nu, S) \approx 1-(1-\varepsilon(S))/(1+\nu^2)$.

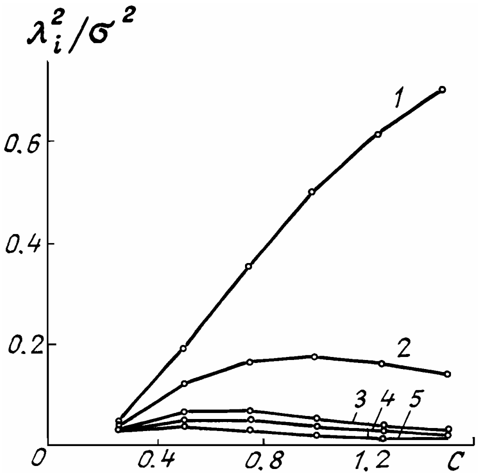

FIG. 3. Dependence of the relative contribution of the Karunen–Love modes to the mean square of the compensating phase on the correlation length c for $N = 7$, $S = 0$, *and* $b = 0.5$. *The curves correspond to the serial numbers of modes: 1) 1, 2) 2 and 3, 3) 4 and 5, 4) 6, and 5) 7.*

Dependence of the values λ_i^2/σ^2 of the relative contribution of the orthogonal Karhunen–Loeve modes to the mean square of compensating phase $\sigma_\varphi^2 = \overline{<\varphi_c - <\varphi_c>>^2}$, for $N = 7$ on the correlation length of the PD are shown in Fig. 3. Data of Fig. 3 can be used to evaluate the number of the AOS modes needed for the compensation for the PD with a fixed accuracy in the presence of noise of a given intensity. Analysis of these dependencies shows that contributions of different modes to the mean square of the compensating phase, being substantially different for long correlation lengths of the PD, become close in values when they are reduced. Each mode is characterized by the corresponding spatial frequency. When the correlation length of the PD is sufficiently reduced, the spatial frequencies of the modes turn out to be in the region of the constant power density of the frequency spectrum of the compensating phase. As a result, the contribution of different modes in the compensation is independent of the frequency and the serial number of the mode.

It follows from Fig. 3 that the contribution of each mode, depending on the correlation length, has maximum whose value and position depends on the serial number of the mode. When the serial number of the mode increases, the position of maximum is shifted toward the shorter correlation lengths. The presence of maximum is associated with the fact that the contribution of each mode vanishes as both $c \to 0$ and as $c \to \infty$. When the correlation length of the PD approaches zero, the PD power density per a mode decreases due to broadening of the PD spectrum, while when the correlation length increases, it decreases due to the concentration of the spectrum around the zeroth frequency. Maximum of the contribution of the lowest–order modes lies in the region of the correlation lengths $c = 0.4 - 1$.

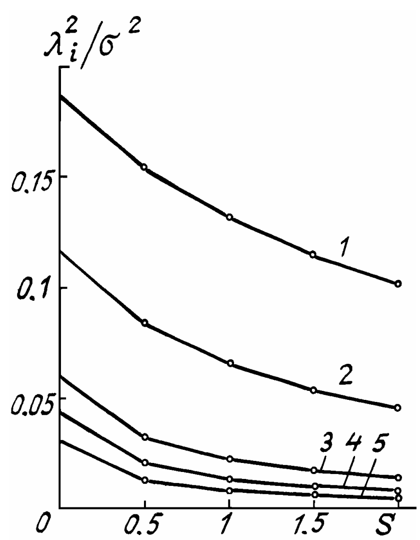

FIG. 4. Dependence of the relative contribution of the Karhunen–Loeve modes to the mean square of the compensating phase on the noise level S for $c = 0.5$, $N = 7$, *and* $b = 0.5$. *The curves correspond to the following serial numbers of modes: 1) 1, 2) 2 and 3, 3) 4 and 5, 4) 6, and 5) 7.*

The graphs of the relative contribution λ_i^2/σ^2 of the modes, employed for compensation of the PD, to the mean square of the compensating phase distribution as a function of the relative intensity of the measurement noise S for $N = 7$ and $c = 0.5$ are shown in Fig. 4. The contribution of each mode decreases with noise.

Thus, the calculated results in the framework of the developed model representation can be used to predict the efficiency and to determine the conditions of the optimal use of the phase conjugate AOS for solving the applied problems in atmospheric and laser optics.

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