OPTIMUM ESTIMATION OF PHASE–FRONT LOCAL TILTS MEASURED WITH A HARTMANN SENSOR AGAINST THE BACKGROUND OF THE POISSON NOISE

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Based on an analysis of a system of cumulants of random values, analytical expressions are derived for characteristic functions and distribution densities, which describe signals in a channel of a Hartmann sensor. Likelihood ratio equation has been solved and an expression is obtained for statistically optimum estimate of local tilts measured with a sensor of Hartmann type.

1. INTRODUCTION

The Hartmann sensor is a basic element of an adaptive optical system of phase conjugation. A quadrantal (with four sensitive elements) photodetector included in it is commonly used as a receiver of optical radiation. After processing of output electrical signals of these photodetectors, signals are obtained proportional to local phase–front tilts. In this case the reconstruction of the phase on the optical system aperture becomes possible.[1,2] Many papers devoted to synthesis of phase reconstruction algorithms from the results of measurements with Hartmann–type sensors[3,4] *a priori* assume existence of statistically optimum estimates $\hat{V} = \dfrac{\partial S(x,\, y)}{\partial x}$ and $\hat{U} = \dfrac{\partial S(x,\, y)}{\partial y}$, where $S(x, y)$ is the phase distribution on the optical system aperture. However, no consideration has been given to the approach to retrieval of such optimum estimates under conditions of the Poisson noise. It should be noted that the application of known approaches based on the Gaussian approximation for the noise is incorrect in this case, because it is *a priori* known that a light flux with the intensity I_i of the order I/M^2 (where M is the number of subapertures of a sensor) is incident on each quadrantal photodetector of the Hartmann sensor. Thus, we cannot say about "strong" signal, for which the Gaussian approximation can be applied. It is known[5] that detection of an optical field with the help of a photosensitive surface is connected with the observation of an electron flow during a certain time interval. Fundamental properties of photodetection are that probability of electron detachment from the photosensitive surface of a detector is described by the Poisson distribution:

$$P(k) = \frac{\lambda^k}{k!} \exp(-\lambda), \qquad (1)$$

where λ is the parameter of distribution.

It is known that the density of distribution of a sum of the arbitrary number L of the Poisson random variables is the Poisson random variable as well, with the parameter $L\lambda$. This is explained by the properties of the Poisson discrete process. However, taking into account the preceding, not only summation of signals from separate sensitive elements, but also their subtraction are done in a Hartmann sensor when obtaining the signals proportional to U and V. It is obvious that the difference between the Poisson random variables disobeys the Poisson law.

Thus, a problem of calculating an analytical expression for a distribution density of the Poisson random variable difference in the form convenient for analysis and synthesis of optimum decision rules for estimation of $\hat{U}$ and $\hat{V}$ is topical.

In this paper, we have obtained relations for the characteristic function of the difference between two Poisson random variables and written analytical relations for corresponding distribution densities with the use of mathematical apparatus of cumulant analysis. We have also obtained the expression for optimum estimates of U and V.

2. DERIVATION OF BASIC RELATIONS FOR PLANE PHASE FRONT

Let us consider the problem of recording of an optical field with a quadrantal photodetector of a Hartmann sensor in the following formulation. Because we say about random variables, the basic results obtained can be easily extended to random processes.

Let a light flux of low intensity focused with a lens be incident on a quadrantal photodetector. With phase–front tilt, the summation–difference signal processing is performed for calculating its value. In this case after summation and subtraction, signals can be written in the form

$$\begin{aligned} U &= (u_1 + m_1 + u_2 + m_2) - (u_3 + m_3 + u_4 + m_4), \\ V &= (u_1 + m_1 + u_3 + m_3) - (u_2 + m_2 + u_4 + m_4), \end{aligned} \qquad (2)$$

where m_i is the additive Poisson noise with the parameter λ, and u_i is the valid Poisson signal corresponding to the ith quadrant of the photodetector. Because u_i and m_i obey the Poisson law, hereafter it is expedient to consider Eq. (2) in the form

$$\begin{aligned} n_x &= (n_1 + n_2) - (n_3 + n_4), \\ n_y &= (n_1 + n_3) - (n_2 + n_4). \end{aligned} \qquad (3)$$

Here $n_i = m_i + u_i$ is the Poisson random variable with the parameter λ.

Because the sum of two Poisson variables with the parameter λ is the Poisson variable with the parameter 2λ, we take the following designations:

$$\begin{aligned} n_1 + n_2 &= N_1, \qquad n_1 + n_3 = N_3, \\ n_3 + n_4 &= N_2, \qquad n_2 + n_4 = N_4, \end{aligned} \qquad (4)$$

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where N_j is the Poisson random variable with the parameter 2λ.

Expectations of n_x and n_y are equal to

$$M[n_x] = M[N_1 - N_2] = M[N_1] - M[N_2] = 0 , \quad (5)$$

$$M[n_y] = M[N_3 - N_4] = M[N_3] - M[N_4] = 0 , \quad (6)$$

where $M[.]$ is the symbol of expectation.

The first moments after simple transformations in view of Eqs. (5) and (6) are written as

$$M[n_x n_y] = M[(N_1 - N_2)(N_3 - N_4)] = 0 , \quad (7)$$

$$M[n_x n_x] = M[(N_1 - N_2)(N_1 - N_2)] = 4\lambda - k_1 , \quad (8)$$

where k_1 is the correlation coefficient of random variables N_i.

Considering the nature of the Poisson noise, the correlation coefficient should be set equal to zero.

Moments of higher order for random variables n_x and n_y are written in the form

$$M\left[n_x^n\right] = M\left[(N_1 - N_2)^n\right] = M\left[\sum_{k=0}^{n} c_n^k (-1)^k N_1^{n-k} N_2^k\right] = \sum_{k=0}^{n} c_n^k (-1)^k m_{n-k,k}^{1,2} , \quad (9)$$

where $m_{n-k,k}^{1,2}$ are the joint moments of random variables N_1 and N_2 of the orders $n - k$ and k.

Hereafter to designate the order of moments and cumulants of variables, subscripts correspond to the moment order, whereas superscripts denote the corresponding variable.

It is seen from Eq. (9) that all odd moments m_{2n+1}^x of the random variable n_x are equal to zero, whereas the joint moments $m_{n-k,k}^{1,2}$ of random variables N_1 and N_2 involved in Eq. (10) cannot be set equal to zero in general for even moments m_{2n}^x.

It is known from the theory of cumulant analysis[6] that in the case of two independent random variables N_1 and N_2 all their joint cumulants are equal to zero, whereas this cannot be unambigously asserted for corresponding moments. So we take an advantage of this fact and consider a system of cumulants of the random variable n_x. Unknown cumulants can be found on the basis of linearity and invariance properties[6] from Eq. (9):

$$\chi_n^x = \sum_{k=0}^{n} c_n^k (-1)^k \chi_{n-k,k}^{1,2} , \qquad n = 0, 2 \ldots . \quad (10)$$

Here χ_n^x are the nth order cumulants of the random variable n_x, and $\chi_{n-k,k}^{1,2}$ are the joint cumulants of random variables N_1 and N_2.

Analysis of Eq. (10) shows that random variables n_x are described solely by a system of even cumulants that are equal to

$$\chi_n^x = 4\lambda , \qquad n = 0, 2, 4, \ldots . \quad (11)$$

With the use of Eq. (10) we write down the relation for the characteristic function of the sought–after distribution of the probability density taking into account that only even cumulants exist

$$\theta(i\nu) = \exp\left[\sum_{k=1}^{\infty} \frac{4\lambda}{(2k)!}(i\nu)^{2k}\right] . \quad (12)$$

By summing a series in brackets, we obtain

$$\theta(i\nu) = \exp\{4\lambda(\cosh(i\nu) - 1)\} = \exp\{4\lambda(\cos(\nu) - 1)\} . \quad (13)$$

Let us take the Fourier transform of Eq. (13)

$$P(x_n) = \int_{-\infty}^{\infty} \exp\{4\lambda(\cos(\nu) - 1) - i\nu x_n\}\, d\nu . \quad (14)$$

Considering a well–known series expansion of exponential function[7]

$$\exp\{\pm i z \sin(\nu)\} = \sum_{n=-\infty}^{\infty} I_n(z) \exp\{\pm i n \nu\}, \quad (15)$$

where $I_n(z)$ is the modified Bessel function of the nth order, after simple transformation we obtain

$$P(x_n) = \exp\{-4\lambda\} \sum_{n=-\infty}^{\infty} I_n(4\lambda)\, \delta[x_n - n], \quad (16)$$

where $\delta(x)$ is the delta function.

Because the random variable n_x takes only discrete values, final expression for the sought–after density is written as

$$P(n) = \exp\{-4\lambda\} I_n(4\lambda). \quad (17)$$

It is seen that density (17) is normalized with the weight of unity.

Let us consider physical meaning of the obtained results. First, zero odd distribution moments, as it follows from Eq. (11), indicate that the *a priori* density of signal distribution at the output from a quadrantal photodetector of the Hartmann sensor is symmetric. Existence of higher cumulants for fixed values of the parameter 4λ allows us to conclude that the obtained distribution differs from the Gaussian one.

3. CASE OF A PHASE–FRONT TILT (*A POSTERIORI* DISTRIBUTION DENSITY)

It is obvious that in the presence of the phase–front tilt, the Airy circle on the quadrantal photodetector is shifted; as it takes place, the parameters λ and μ of the Poisson distributions of the random variables N_1 and N_2 are no longer equal:

$$\lambda \neq \mu . \quad (18)$$

In this case

$$M[n_x] = M[N_1 - N_2] = M[N_1] - M[N_2] = \lambda - \mu . \quad (19)$$

Equation (10) is valid for a system of cumulants of the random variable n_x in the general case. All joint cumulants are equal to zero as well. The odd cumulants χ_{2n+1}^x are equal to $\lambda - \mu$, and the even cumulants χ_{2n}^x are equal to $\lambda + \mu$. It is connected with the fact that $(-1)^k$ in Eq. (10) gives only positive factors for even k and alternating ones for odd k. Thus the characteristic function of such a distribution is written in the form:

$$\theta(iv) = \exp\left\{\sum_{k=1}^{\infty}\frac{1+\mu}{(2k)!}(iv)^{2k} + \sum_{k=1}^{\infty}\frac{1-\mu}{(2k+1)!}(iv)^{2k+1}\right\}. \quad (20)$$

Taking into account the expansion of the functions cos ν and sin ν in a power series, we write down

$$\theta(iv) = \exp\left[(\lambda+\mu)(\cos v - 1) + i(\lambda-\mu)\sin v\right]. \quad (21)$$

Using the known relations for the Bessel function,[7] we reduce Eq. (21) to the form

$$\theta(iv) = \exp\{-(\lambda+\mu)\}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} I_m(\lambda+\mu)\, J_n(\lambda-\mu)\times$$
$$\times \exp\{i(n-m)v\}. \quad (22)$$

To derive the analytical expression for the distribution density of random variable n_x, we take the Fourier transform of Eq. (22)

$$P(x) = \exp\{-(\lambda+\mu)\}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} I_m(\lambda+\mu)\, J_n(\lambda-\mu)\times$$
$$\times\int_{-\infty}^{\infty}\exp\{i(n-m)v - ivx\}\,dv. \quad (23)$$

As a result, we obtain

$$P(x) = \exp\{-(\lambda+\mu)\}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} I_m(\lambda+\mu)\, J_n(\lambda-\mu)\times$$
$$\times\delta\{x-(n-m)\}. \quad (24)$$

For fixed values of the random variable n_x (precisely this case is of interest for us) based on physical formulation of the problem, we have

$$P(k) = \exp\{-(\lambda+\mu)\}\sum_{n=-\infty}^{\infty} I_n(\lambda+\mu)\, J_{n-k}(\lambda-\mu) \quad (25)$$

at $n - m = k$.

Let us consider some properties of Eq. (25). To do this, we use the series expansion of the modified Bessel function

$$I_n(\lambda+\mu) = \sum_{l=0}^{\infty}\frac{(\lambda+\mu)^l}{l!} J_{n+l}(\lambda+\mu). \quad (26)$$

Then

$$P(k) = \exp\{-(\lambda+\mu)\}\sum_{l=0}^{\infty}\frac{(\lambda+\mu)^l}{l!} J_{n-l}(\lambda+\mu)\, J_{n-k}(\lambda-\mu). \quad (27)$$

Applying the Neumann addition theorem

$$J_m(U-V) = \sum_{p=0}^{\infty} J_{m+p}(U)\, J_p(V), \quad (28)$$

at $m + p = n + l$ ($p = n - k$, $m = l - k$), $U = \lambda + \mu$, and $V = \lambda - \mu$, we derive the final expression

$$P(k) = \exp\{-(\lambda+\mu)\}\sum_{l=0}^{\infty}\frac{(\lambda+\mu)^l}{l!} J_{l+k}(2\mu). \quad (29)$$

It is obvious that

$$\lim_{\mu\to 0} P(k) = \lim_{\mu\to 0}\left[\exp\{-(\lambda-\mu)\}\sum_{l=0}^{\infty}\frac{(\lambda+\mu)^l}{l!} J_{l+k}(2\mu)\right] =$$
$$= \frac{\lambda^l}{l!}\exp(-\lambda) \quad (30)$$

because $J_{l+k}(0) = 0$ only at $l = -k$.

Thus without a signal in a channel of the quadrantal photodetector of the Hartmann sensor, only the Poisson signal is at its output (mixture of the Poisson signal and Poisson noise).

Similarly we can obtain

$$\lim_{\mu\to\lambda} P(k) = \lim_{\mu\to\lambda}\left[\exp\{-(\lambda+\mu)\}\sum_{l=0}^{\infty}\frac{(\lambda+\mu)^l}{l!} J_{l+k}(2\mu)\right] =$$
$$= \exp\{-2\lambda\}\, I_n(2\lambda). \quad (31)$$

Expressions (30) and (31) allow us to prove the reliability of obtained results. Thus, the Poisson distribution follows from Eq. (29). Considering Eq. (29), we can make a conclusion that because of nonzero odd cumulants, the distribution density is asymmetric. In this case, the density maximum will occur in the region of expectation $m_x = \lambda - \mu$, but will differ from it in general. As $\mu \to 0$, density degenerates into the Poisson one, and as $\mu \to \lambda$ we obtain the distribution density of the difference between two Poisson values with identical distribution parameters, corresponding to the case of the plane phase front incident on a subaperture of the Hartmann sensor.

4. CALCULATION OF THE LIKELIHOOD RATIO LOGARITHM

To obtain optimum estimate of $\lambda - \mu$, it is necessary to calculate the likelihood ratio logarithm. Let us convert the density obtained (Eq. (29)) with the use of the theorem on the Bessel function multiplication[7]

$$\sum_{k=0}^{\infty}\frac{(-1)^k(y^2-1)^k(z/2)^k}{k!} J_{n+k} = \frac{J_n(yz)}{y^n}. \quad (32)$$

Let us set $y = i\sqrt{\lambda/\mu}$. Then we derive:

$$P(n) = \exp\{-(\lambda+\mu)\}\sum_{l=0}^{\infty}\frac{(\lambda+\mu)^l}{l!} J_{l+n}(2\mu) =$$
$$= \exp\{-(\lambda+\mu)\}\frac{J_n\left(i\,2\sqrt{\lambda\mu}\right)}{\left(i\sqrt{\lambda/\mu}\right)^n} = \frac{\exp\{-(\lambda+\mu)\} I_n\left(2\sqrt{\lambda\mu}\right)}{(\lambda/\mu)^{n/2}}. \quad (33)$$

The multidimensional distribution density, because of independence of different readings, is written in the form

$$P(n_1, \ldots, n_N) = \prod_{i=1}^{N}\frac{\exp\{-(\lambda+\mu)\} I_{n_i}\left(2\sqrt{\lambda\mu}\right)}{(\lambda/\mu)^{n_i/2}}. \quad (34)$$

We obtain the optimum estimate of $\lambda - \mu$ by way of solution of the equation of the following form:

$$\frac{\partial \ln \Lambda(n_1, \ldots, n_N)}{\partial \mu} = 0, \quad (35)$$

where

$$\Lambda(n_1, \ldots, n_N) = \frac{\prod_{i=1}^{N}\exp\{-(\lambda+\mu)\} I_{n_i}\left(2\sqrt{\lambda\mu}\right)}{\prod_{i=1}^{N}\exp\{-2\lambda\} I_{n_i}(2\lambda)(\lambda/\mu)^{n_i/2}}. \quad (36)$$

Taking into account that $\lambda \ll 0$ and $\mu \ll 0$, we can write the Bessel functions in the form[7]

$$I_n(z) = (z/2)^n / n! \, . \quad (37)$$

Then substituting Eqs. (36) and (37) into Eq. (35), after simple transformation we can obtain

$$(\lambda \hat{-} \mu) = \sum_{i=1}^{N} n_i \frac{\ln \mu - \ln \lambda}{N} \, . \quad (38)$$

Thus Eq. (38) specifies the optimum algorithm for signal processing in a channel of the Hartmann sensor.

Equation (38) can be represented as the recursion relation

$$(\lambda \hat{-} \mu)_j = \frac{\ln \mu_j - \ln \lambda_j}{j} \sum_{i=1}^{j} n_i \, ,$$
$$(\lambda \hat{-} \mu)_{j+1} = \frac{(\lambda - \mu)_j \, j}{j+1} + \frac{\ln \hat{\mu}_j - \ln \hat{\lambda}_{j+1}}{j+1} n_{j+1} \, . \quad (39)$$

Estimates of the parameters μ_j and λ_j can be obtained by known methods because they are properly the Poisson variables.

CONCLUSION

Thus as a result of the performed analytical analysis, expression (38) for optimum estimate of a signal at the output from a channel of the Hartmann sensor has been derived. Analysis of Eq. (38) shows that data processing algorithm being implemented at present and involving the simple summation of the photoelectron counts is essentially quasi–optimum. This being the case, it gives slightly overestimated values of tilts. The distribution density of a signal at the output from a channel of the Hartmann sensor is symmetric and unimodal in the case of the plane phase front; however, it differs essentially from Gaussian distribution for nonzero higher cumulants. In the presence of a phase–front tilt, the distribution density remains unimodal; however, it shifts along the x axis, and optimum estimate of the phase front should be found in this case from Eq. (38). The derived expressions for the characteristic functions given by Eqs. (13) and (22) also can be used for an analysis of performance of adaptive optical systems of phase conjugation. In conclusion, it should be specially emphasized that suggested approach is optimum only for weak signals when both signal and noise are well approximated by the Poisson distributions. However, in the case of deviation of the noise and signal distribution densities from the Poisson ones, derivation of analogous expressions for optimum estimates is possible on the basis of suggested approach to an analysis of cumulants of corresponding random variables and processes.

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