EXTINCTION MATRIX OF AN ENSEMBLE OF PARTICLES WITH ARBITRARY SHAPES AND ORIENTATION

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*The **T**–matrix approach is used to develop a rigorous analytical method to compute the extinction matrix of an ensemble of arbitrary shaped particles with an arbitrary square integrable function of distribution over orientation*

INTRODUCTION

Extinction and polarization of light passed through a layer of particles in the atmosphere or interstellar space is an important source of information on the particles properties. The choice of a model of a particle ensemble is important for correct interpretation of such data, development of the express methods for estimating optical properties of particles and their identification.

As a rule, in real situations particles are nonspherical, and, in addition, many physical factors such as magnetic field, gravitation, air flows, and so on effect on the orientation structure within the particle ensemble described by an arbitrary function of distribution over orientation.

In this paper we derive analytical expression for the extinction matrix of an ensemble of arbitrary shaped particles with an arbitrary square integrable function of distribution over orientation using the **T**–matrix method.

1. T–MATRIX METHOD

When solving problems on the electromagnetic waves diffraction on nonspherical particles the **T**–matrix method by Waterman[1,2] is widely used though it was developed for solving the problems on scattering of electromagnetic radiation (see, for example, Ref. 3).

Very often in literature it is referred to as EBCM (extended boundary conditions method). An alternative justification of the **T**–matrix method is given in Ref. 5 using Shchelkunoff's equivalence principle.[4] It should be noted that this method can be naturally and consistently used in the case of inhomogeneous particles[6–8] as well different systems of vector spherical harmonics (linearly independent solutions of the vector Helmholtz equation[9]) used by different authors produce different representations of the **T**–matrices.[5,8,10] It should be noted that the choice of spherical harmonics in Refs. 5 and 8 is poor from the standpoint of further development and application of the **T**–matrix method, for example, to ensembles of particles with different orientation structure. Taking into account the invariance of the vector Helmholtz equation with respect to rotations of the coordinate system,[11] the choice of spherical harmonics should be done based on the invariance property (in the sense of closeness), namely, the spherical harmonics of the types $\mathbf{M}_{\tau mn}$ and $\mathbf{N}_{\tau mn}$ (Ref. 9) should be transformed independently when rotating the coordinate system.

The following vector spherical harmonics[10,12] satisfy the sought invariance properties:

$$\mathbf{M}_{mn}(k\mathbf{r}) = (-1)^m\, d_n\, h_n^{(1)}(k\, r)\, \mathbf{C}_{mn}(\theta)\, \exp(i\, m\, \varphi); \quad (1)$$

$$\mathbf{N}_{mn}(k\, \mathbf{r}) = (-1)^m\, d_n \left(\frac{n(n+1)}{k\, r}\, h_n^{(1)}(k\, r)\, \mathbf{P}_{mn}(q) + \right.$$

$$\left. + \frac{1}{k\, r} \left[kr\, h_n^{(1)}(kr) \right]' \mathbf{B}_{mn}(q) \right) \exp(i\, m\, \varphi); \quad (2)$$

$$\mathbf{B}_{mn}(\theta) = \mathbf{i}_\theta \frac{d}{dq}\, d^{\,n}_{0m}(\theta) + \mathbf{i}_\varphi \frac{i\, m}{\sin q}\, d^{\,n}_{0m}(\theta); \quad (3)$$

$$\mathbf{C}_{mn}(\theta) = \mathbf{i}_\theta \frac{i\, m}{\sin q}\, d^{\,n}_{0m}(\theta) - \mathbf{i}_\varphi \frac{d}{dq}\, d^{\,n}_{0m}(\theta); \quad (4)$$

$$\mathbf{P}_{mn}(\theta) = \mathbf{i}_r\, d^{\,n}_{0m}(\theta); \quad (5)$$

$$d_n = \left[\frac{(2\, n+1)}{4\, n(n+1)} \right]^{1/2}.$$

Definition and principal properties of the Wigner functions[13] $d^{\,n}_{0m}(q)$ are given in Appendix, $h_n^{(1)}(k\, r)$ is the Hankel spherical function of the first kind; $\mathbf{i}_r$, $\mathbf{i}_h$, and $\mathbf{i}_z$ are the orths of the spherical coordinate system; $k = 2\pi/\lambda$ is the wave number, and λ is the wavelength of radiation. The harmonics Rg $\mathbf{M}_{mn}$ and Rg $\mathbf{N}_{mn}$ are defined analogously with the Hankel spherical functions substituted by Bessel spherical functions $j_n(k\, r)$.

The series expansion of a plane electromagnetic wave incident on a particle has the form (hereinafter the factor $\exp(-\, i\, \omega\, t)$ is omitted):

$$\mathbf{E}^i(\mathbf{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[a_{mn}\, \mathrm{Rg}\, \mathbf{M}_{mn}(k\, \mathbf{r}) + b_{mn}\, \mathrm{Rg}\, \mathbf{M}_{mn}(k\, \mathbf{r}) \right]. \quad (6)$$

The coefficients of series expansion of the incident plane electromagnetic wave propagating along the (u, υ) direction have the form[10]

$$a_{mn} = 4(-1)^m\, i^n\, d_n\, \mathbf{C}^*_{mn}(u)\, \mathbf{E}_i \exp(-i\, m\, \mathrm{u});$$
$$b_{mn} = 4(-1)^m\, i^{n-1}\, d_n\, \mathbf{B}^*_{mn}(u)\, \mathbf{E}_i \exp(-i\, m\, \mathrm{u}), \quad (7)$$

where $\mathbf{E}_i$ is the vector of linear polarization.

For the scattered field we have the following expansion:

$$\mathbf{E}^s(\mathbf{r}) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[p_{mn}\, \mathbf{M}_{mn}(k\, \mathbf{r}) + q_{mn}\, \mathbf{N}_{mn}(k\, \mathbf{r}) \right], \quad r > r_0, \quad (8)$$

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where r_0 is the radius of the sphere circumscribed about the particle.

Linear transformation relating the expansion coefficients of incident and scattered fields is as follows[10]:

$$p_{mn}= \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[T^{11}_{mnm'n'}\, a_{m'n'} + T^{12}_{mnm'n'}\, b_{m'n'}\right];$$
$$q_{mn}= \sum_{n'=1}^{\infty} \sum_{m'=-n'}^{n'} \left[T^{21}_{mnm'n'}\, a_{m'n'} + T^{22}_{mnm'n'}\, b_{m'n'}\right]. \qquad (9)$$

Let us note that the representation of **T**–matrix method considered in Ref. 10 has some advantages over the representation in Ref. 5. They are in the use of the vector spherical harmonics that are invariant with respect to rotation of the coordinate system, and also in the symmetric form the main relationships.

2. ROTATION OF A COORDINATE SYSTEM

Arbitrary rotations of the coordinate system about its origin are completely determined by setting of three real parameters. The Euler angles α, β, and γ (see Ref. 13) are more often used as the parameters characterizing the rotation.

Let us designate the coordinate systems and the values given in these systems by the indices *1* and *2*. Then the Euler angles α, β, and γ determine the position of the second coordinate system with respect to the first one.

As was noted above, the vector spherical harmonics (1) and (2) are closed with respect to rotation of the coordinate system[10]

$$\mathbf{M}_{mn}(k\,r,\theta_1,\varphi_1) = \sum_{m'=-n}^{n} D^{n}_{m'm}(\alpha\,\beta\,\gamma)\,\mathbf{M}_{m'n}(k\,r,\theta_2,\varphi_2). \qquad (10)$$

The reverse transformation is

$$\mathbf{M}_{mn}(kr,\theta_2,\varphi_2) = \sum_{m'=-n}^{n} D^{n-1}_{m'm}(\alpha\,\beta\,\gamma)\,\mathbf{M}_{m'n}(k\,r,\theta_1,\varphi_1), \qquad (11)$$

where $D^{n}_{m'm}(\alpha\,\beta\,\gamma)$ are the Wigner *D*–functions[13] (see Appendix).

The relationships analogous to Eqs. (10) and (11) are valid for $\mathbf{N}_{mn}$, Rg $\mathbf{M}_{mn}$, and Rg $\mathbf{N}_{mn}$.

Let us represent Eq. (9) in the coordinate system *1* in the form

$$\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{T}^{11} & \mathbf{T}^{12}\\ \mathbf{T}^{21} & \mathbf{T}^{22}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}. \qquad (12)$$

Let us note that the **T**–matrix (12) in a fixed coordinate system is invariant with respect to the parameters of incident radiation. In the coordinate system *2*, taking into account Eqs. (10) and (11) and designating the linear transform with elements $D_{mnm'n'} = \delta_{nn'}\, D^{n}_{mm'}(\alpha\beta\gamma)$ by **D**, we obtain[10]

$$\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\mathbf{D} & 0\\ 0 & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{T}^{11} & \mathbf{T}^{12}\\ \mathbf{T}^{21} & \mathbf{T}^{22}\end{bmatrix}\begin{bmatrix}\mathbf{D}^{-1} & 0\\ 0 & \mathbf{D}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{a}\\ \mathbf{b}\end{bmatrix}, \qquad (13)$$

that is equivalent to

$$^{2}\mathbf{T}^{ij} = \mathbf{D}^{1}\,\mathbf{T}^{ij}\,\mathbf{D}^{-1}, \qquad i, j= 1, 2, \ldots, \qquad (14)$$

$$^{2}T^{ij}_{mnm'n'} = \sum_{m_1=-n}^{n} \sum_{m_2=-n'}^{n'} D^{n}_{mm_1}(\alpha\beta\gamma)\; {}^{1}T^{ij}_{m_1nm_2n'} D^{n'-1}_{m_2m'}(\alpha\beta\gamma). \qquad (15)$$

Using the unitarity property of the Wigner *D*–functions (A3), we can found the invariants of the **T**–matrix with respect to rotations of the coordinate system. Using Eqs. (A3) and (15) for fixed n and n' we obtain

$$\sum_{m=-n}^{n} {}^{2}T^{ij}_{mnmn} = \sum_{m_1=-n}^{n} \sum_{m_2=-n}^{n} {}^{1}T^{ij}_{m_1nm_2n} \sum_{m=-n}^{n} D^{n}_{mm_1}(\alpha\,\beta\,\gamma)\times$$
$$\times D^{n-1}_{m_2m}(\alpha\,\beta\,\gamma) = \sum_{m=-n}^{n} {}^{1}T^{ij}_{mnmn}; \qquad (16)$$

$$\sum_{m=-n}^{n}\sum_{m'=-n'}^{n'} \left|{}^{2}T^{ij}_{mnm'n'}\right|^2 = \sum_{m=-n}^{n}\sum_{m'=-n'}^{n'} \left|{}^{1}T^{ij}_{mnm'n'}\right|^2 \qquad (17)$$

and, hence,[12]

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n} {}^{2}T^{ij}_{mnmn} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} {}^{1}T^{ij}_{mnmn}, \qquad (18)$$

$$\sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=-n}^{n}\sum_{m'=-n'}^{n'} \left|{}^{2}T^{ij}_{mnm'n'}\right|^2 = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=-n}^{n}\sum_{m'=-n'}^{n'} \left|{}^{1}T^{ij}_{mnm'n'}\right|^2, \qquad (19)$$

$i, j = 1, 2$.

Thus, when rotating the coordinate system, the spur of submatrices (16) and their analogs (18) are the invariants of **T**–matrix as are the sums of squares of absolute values of the elements of submatrices (17) and submatrices $\mathbf{T}^{ij}$. The same property is characteristic of the **T**–matrix as a whole.

3. SCATTERING AMPLITUDE MATRIX

Let the direction of the radiation propagation be along the unit vector $\mathbf{n} = (\theta, \varphi)$, where the components of electromagnetic field θ and φ are designated by indices *1* and *2*, respectively.

Let us consider a plane electromagnetic wave

$$\mathbf{E}^{i}(\mathbf{r}) = \mathbf{E}_i \exp(i\,k\,\mathbf{n}_i\,\mathbf{r}) = \left(E^{i}_{1}\,\mathbf{i}_h + E^{i}_{2}\,\mathbf{i}_z\right)\exp(i\,k\,\mathbf{n}_i\,\mathbf{r}) \qquad (20)$$

being incident on a particle.

In the wave zone ($k\,r > 1$) the scattered wave has the following components[10] (LP–representation):

$$\begin{bmatrix}E^{s}_{1}\\ E^{s}_{2}\end{bmatrix} = \frac{\exp(i\,k\,r)}{k\,r}\,\mathbf{S}(\mathbf{n}_s;\,\mathbf{n}_i)\begin{bmatrix}E^{i}_{1}\\ E^{i}_{2}\end{bmatrix}, \qquad (21)$$

where **S** is the scattering amplitude matrix.

Circular components of the electric field are determined as follows[14] (CP–representation):

$$\begin{bmatrix}E_{+1}\\ E_{-1}\end{bmatrix} = 2^{-1/2}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}\begin{bmatrix}E_1\\ E_2\end{bmatrix}. \qquad (22)$$

The corresponding amplitude matrix has the following form in CP–representation:

$$\mathbf{C} = \begin{bmatrix}C_{+1+1} & C_{+1-1}\\ C_{-1+1} & C_{-1-1}\end{bmatrix} =$$
$$=\frac{1}{2}\begin{bmatrix}S_{11} + iS_{12} - iS_{21} + S_{22} & S_{11} - iS_{12} - iS_{21} - S_{22}\\ S_{11} + iS_{12} + iS_{21} - S_{22} & S_{11} - iS_{12} + iS_{21} + S_{22}\end{bmatrix}. \qquad (23)$$

Using Eqs. (6)–(9) and (21) as well as the asymptotics of the function $h^{(1)}_{n}(k\,r)$ at infinity, we obtain the dyadic representation of the scattering amplitude matrix $\mathbf{S}(\mathbf{n}_s; \mathbf{n}_i)$ (Refs. 3, 10, and 13)

$$\mathbf{S}(\mathbf{n}_s; \mathbf{n}_i) = 4 \sum_{mnm'n'} i^{n'-n-1} (-1)^{m+m'} d_n d_{n'} \exp[i(m\varphi_s - m'\varphi_i)] \times$$
$$\times \{[T^{11}_{mnm'n'} C_{mn}(\theta_s) + i T^{21}_{mnm'n'} B_{mn}(\theta_s)] C^*_{m'n'}(\theta_i) +$$
$$+ [T^{12}_{mnm'n'} C_{mn}(\theta_s) + i T^{22}_{mnm'n'} B_{mn}(\theta_s)] B^*_{m'n'}(\theta_i)/i\}. \quad (24)$$

Omitting simple but cumbersome derivation, we write the expressions for elements of the amplitude matrix in CP–representation using Eqs. (23), (24), and (A6)

$$C_{+1+1} = -\frac{1}{2} \sum_{mnm'n'} t_{nn'} A_{mm'}(\varphi_s, \varphi_i) d^n_{1m}(\theta_s) d^{n'}_{1m'}(\theta_i) \times$$
$$\times (T^{11} + T^{12} + T^{21} + T^{22});$$
$$C_{+1-1} = -\frac{1}{2} \sum_{mnm'n'} t_{nn'} A_{mm'}(\varphi_s, \varphi_i) d^n_{1m}(\theta_s) d^{n'}_{-1m'}(\theta_i) \times$$
$$\times (T^{11} - T^{12} + T^{21} - T^{22}), \quad (25)$$
$$C_{-1+1} = -\frac{1}{2} \sum_{mnm'n'} t_{nn'} A_{mm'}(\varphi_s, \varphi_i) d^n_{-1m}(\theta_s) d^{n'}_{1m'}(\theta_i) \times$$
$$\times (T^{11} + T^{12} - T^{21} - T^{22});$$
$$C_{-1-1} = -\frac{1}{2} \sum_{mnm'n'} t_{nn'} A_{mm'}(\varphi_s, \varphi_i) d^n_{-1m}(\theta_s) d^{n'}_{-1m'}(\theta_i) \times$$
$$\times (T^{11} - T^{12} - T^{21} + T^{22}),$$

where

$$t_{nn'} = i^{n'-n-1}[(2n+1)(2n'+1)]^{1/2};$$
$$A_{mm'}(\varphi_s, \varphi_i) = (-1)^{m+m'} \exp[i(m\varphi_s - m'\varphi_i)]; \quad (26)$$

Subscripts of $\mathbf{T}^{ij}$–matrix elements are omitted for brevity.

4. RADIATIVE TRANSFER EQUATION

Let us use in this paper the Stokes parameters of incident and scattered fields in CP–representation[14]

$$I_2 = E_{-1} E^*_{+1} = 1/2\,(Q - iU),\ I_0 = E_{+1} E^*_{+1} = 1/2\,(I - V),$$
$$I_{-0} = E_{-1} E^*_{-1} = 1/2\,(I + V),\ I_{-2} = E_{+1} E^*_{-1} = 1/2\,(Q + iU), \quad (27)$$

where I, Q, U, and V are the Stokes parameters in LP–representation.[15]

Taking into account the polarization, propagation of electromagnetic radiation in a rarefied medium of scatters arbitrarily located in space is described by the transfer equation[15,16]

$$\frac{d\mathbf{I}(\mathbf{r}, \mathbf{n}_i)}{ds} = n_d(\mathbf{r}) \left[\mathbf{K}(\mathbf{r}, \mathbf{n}_i)\, \mathbf{I}(\mathbf{r}, \mathbf{n}_i) + \int_{4\pi} d\mathbf{n}_s\, \mathbf{Z}(\mathbf{r}, \mathbf{n}_i, \mathbf{n}_s)\, \mathbf{I}(\mathbf{r}, \mathbf{n}_s) \right], \quad (28)$$

where $\mathbf{I}(\mathbf{r}, \mathbf{n})$ is the Stokes vector, $\mathbf{K}(\mathbf{r}, \mathbf{n})$ is the extinction matrix, $\mathbf{Z}(\mathbf{r}, \mathbf{n}_i, \mathbf{n}_s)$ is the scattering matrix at the point $\mathbf{r}$, and $n_d(\mathbf{r})$ is the number density of particles. Derivative in the left–hand side is taken with respect to the direction $\mathbf{n}_i$.

Let us consider the equation for the coherent component in the direction $\mathbf{n}_i$, ignoring the second term that describes the multiple scattering effects. In this case the transfer equation has the form

$$\frac{d\mathbf{I}(\mathbf{r}, \mathbf{n}_i)}{ds} = n_d(\mathbf{r})\, \mathbf{K}(\mathbf{r}, \mathbf{n}_i)\, \mathbf{I}(\mathbf{r}, \mathbf{n}_i). \quad (29)$$

Numerical solution of Eq. (29) is not a problem provided that $n_d(\mathbf{r})$ and $\mathbf{K}(\mathbf{r}, \mathbf{n})$ are known. For the parametrization $\mathbf{r} = \mathbf{r}(s)$ and under the initial condition

$$\mathbf{I}(\mathbf{r}(0), \mathbf{n}_i) = \mathbf{I}_0 \quad (30)$$

for homogeneous medium the solution of Eq. (29) has the form

$$\mathbf{I}(\mathbf{r}(s), \mathbf{n}_i) = \exp\left[\int_0^s ds'\, n_d(\mathbf{r}(s'))\mathbf{K}(\mathbf{r}(s'), \mathbf{n}_i) \right] \mathbf{I}_0. \quad (31)$$

Exponential function having a matrix as an argument is defined as follows:

$$\exp(\mathbf{A}) = \sum_{n=0}^{\infty} \mathbf{A}^n/n!\,. \quad (32)$$

Equation (31) can be considered as the generalized Bouguer law for a polarized radiation.

5. EXTINCTION MATRIX OF AN ENSEMBLE OF PARTICLES WITH ARBITRARY SHAPES AND ORIENTATION

Coming back to the initial problem, let us note that in order to determine the extinction matrix of the ensemble of particles with different orientations for the coherent component it is necessary to average relevant single–particle matrices taking into account the density function of distribution of particles over orientation.

Using LP–representation of the extinction matrix[16] and formulas of its transformation to CP–representation we obtain the following CP–representation for the extinction matrix

$$\mathbf{K}(\mathbf{n}_i) = -i\frac{2\pi}{\kappa^2} \begin{bmatrix} \langle C_{+1+1} + C^*_{-1-1} \rangle & \langle C_{+1-1} \rangle & \langle -C^*_{-1+1} \rangle & 0 \\ \langle C_{-1+1} \rangle & \langle C_{-1-1} - C^*_{-1-1} \rangle & 0 & \langle -C_{-1+1} \rangle \\ \langle -C^*_{+1-1} \rangle & 0 & \langle C_{+1+1} - C^*_{+1+1} \rangle & \langle C_{+1-1} \rangle \\ 0 & \langle -C^*_{+1-1} \rangle & \langle C_{-1+1} \rangle & \langle C_{-1-1} - C^*_{+1+1} \rangle \end{bmatrix}, \quad (33)$$

where angular brackets in the expression mean averaging over corresponding expressions (25) with the argument $(\mathbf{n}_i, \mathbf{n}_i)$ and taking into account the distribution over orientations. To estimate the extinction matrix (33) elements it is necessary and sufficient to know $\langle T^{ij}_{mnm'n'} \rangle$ for their subsequent substitution into Eq. (33).

Let some coordinate system to exist where the directions of scattered and incident radiation are described by the aforementioned parameters, and let it be called the laboratory coordinate system, the choice of which depends on the observational conditions and certain theoretical considerations.

Let $p(\alpha\ \beta\ \gamma)$ be an arbitrary function of particles orientation distribution density square integrable within the range $[0, 2\pi]\times[0, \pi]\times[0, 2\pi]$. In this case the following expansion is valid[13]:

$$p(\alpha\beta\gamma) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{m'=-n}^{n} \frac{2\,n+1}{8\pi^2}\, p^n_{mm'}\, D^n_{mm'}(\alpha\beta\gamma), \quad (34)$$

where the expansion coefficients [see Eq. (A4)] are

$$p^n_{mm'} = \int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\, d\beta \int_0^{2\pi} d\gamma\, p(\alpha\beta\gamma)\, D^{n*}_{mm'}(\alpha\beta\gamma). \quad (35)$$

Let **A** be the coordinate system related to the particle. The **T**–matrix of the particle is calculated in this system. The particle orientation is determined by the angles α, β, and γ that describe the rotation of the laboratory coordinate system with respect to the coordinate system **A**.

Taking into account Eqs. (15), (A2), (A3), (A5), (A7), and (A8) as well as the orthogonality property of the functions $\exp(im\alpha)$ and $\exp(im\gamma)$, the orientation–averaged $\mathbf{T}^{ij}$–matrices, in the laboratory coordinate system, have the form

$$<T^{\,ij}_{mnm'n'}> = \sum_{m_1 m_2} (-1)^{m'-m_2} \sum_{l=|n-n'|}^{n+n'} p^l_{m-m'\ m_2-m_1} \times$$

$$\times\, C^{\,lm-m'}_{mn\ n'-m'}\, C^{\,lm_1-m_2}_{nm_1\ n'-m_2}\, T^{\,ij}_{m_1nm_2n'}(A), \quad (36)$$

where $C^{\,mn}_{n_1m_1\ n_2m_2}$ are the Clebsch–Gordan coefficients.[13]

The extinction matrix of the ensemble of particles can be obtained by substitution of Eq. (36) into Eqs. (25) and (33).

Let us consider some consequences of Eq. (36).

a) The particular case of $p(\alpha\beta\gamma) = p(\beta)$ was considered in Ref. (19). Taking into account Eq. (35), the formula (36) can be reduced to[19] ($m = m'$, $m_1 = m_2$)

$$<T^{\,ij}_{mnm'n'}> = \delta_{mm'} \sum_{m_1=-M}^{M} (-1)^{m-m_1} \sum_{l=|n-n'|}^{n+n'} p^l_{00}\, C^{\,l0}_{nm\ n'\ -m}\times$$

$$\times\, C^{\,l0}_{nm_1\ n'-m_1}\, T^{\,ij}_{m_1nm_1n'}(A), \quad (37)$$

where $M = \min(n, n')$. This case takes place when only one factor of orientation (magnetic field in Ref. 19) works.

b) For arbitrarily oriented particles $p(\alpha\beta\gamma) = 1/8\pi^2$, and only one coefficient in Eq. (29) differs from zero: $p^0_{00} = 1$. Let us make use of the relationship[13]

$$C^{\,00}_{nm\ n-m} = (-1)^{n-m}(2\,n+1)^{-1/2}, \quad (38)$$

and also choose the direction of incident radiation along the z–axis of the laboratory coordinate system with $\theta_i = \theta_s = 0$ and $\varphi_i = \varphi_s = 0$. In this case the extinction matrix is invariant with respect to the direction of incident radiation if Stokes vectors of incident and scattered radiation are set in the same plane. This can be reached by rotating of the reference plane, for example, that of the Stokes vector of incident radiation.

Transformation of the Stokes vector parameters at rotating the reference plane by an angle ψ clockwise with respect to the propagation direction is described as[17]

$$I_n(\psi) = \exp(in\psi)\, I_n(0), \qquad n = 2,\ 0,\ -0,\ -2. \quad (39)$$

Taking into account the relationships[13]

$$d^{\,n}_{-11}(0) = d^{\,n}_{1-1}(0) = 0, \qquad d^{\,n}_{11}(0) = d^{\,n}_{-1-1}(0) = 1 \quad (40)$$

the extinction matrix has the diagonal form, where

$$<C_{+1+1}> = \frac{1}{2}\, i \sum_{mn} \Big(T^{\,11}_{mnmn}(A) + T^{\,12}_{mnmn}(A) + T^{21}_{mnmn}(A) + T^{\,22}_{mnmn}(A)\Big);$$

$$<C_{-1-1}> = \frac{1}{2}\, i \sum_{mn} \Big(T^{\,11}_{mnmn}(A) - T^{\,12}_{mnmn}(A) - T^{\,21}_{mnmn}(A) + T^{\,22}_{mnmn}(A)\Big). \quad (41)$$

The expressions (41) are invariant with respect to a coordinate system **A** (see Eq. (18)).

The extinction matrix in LP–representation also has the diagonal form with the diagonal elements C_{ext} (extinction cross section); C_{ext} being the half–sum of K_{00} and K_{-0-0} elements of the extinction matrix in CP–representation and[20]

$$C_{ext} = -\frac{2p}{k^2}\, \mathrm{Re} \sum_{mn} \Big(T^{\,11}_{mnmn}(A) + T^{\,22}_{mnmn}(A)\Big). \quad (42)$$

c) For an ensemble of uniformly oriented particles with $p(\alpha\beta\gamma) = \delta(\alpha-\alpha_0)\,\delta(\cos\beta-\cos\beta_0)\,\delta(\gamma-\gamma_0)$, according to Eq. (35) and the definition of the Dirac delta function $\delta(x)$, $p^{\,n}_{mm'} = D^{\,n*}_{mm'}(\alpha_0\beta_0\gamma_0)$. Using the properties of the Wigner D–functions (see Appendix) and Eq. (15) we obtain, in the right–hand side of Eq. (37), the expression for $\mathbf{T}^{ij}$–matrix of a single particle with the orientation $\alpha_0\beta_0\gamma_0$ in the laboratory coordinate system.

Let us present, using the results obtained, the derivation of the formula for the extinction cross section of an ensemble of arbitrarily shaped particles with the distribution over orientations described by Eq. (34).

6. EXTINCTION CROSS SECTION OF AN ENSEMBLE OF PARTICLES

Let us write the formula for the extinction cross section of a single particle in the laboratory coordinate system in the form[21]

$$C_{ext} = -\frac{\pi}{\kappa^2}\, \mathrm{Re}\Big\{\Big(\mathbf{a}^*[\mathbf{T}^{11}\,\mathbf{a} + \mathbf{T}^{12}\,\mathbf{b}]\Big) + \Big(\mathbf{b}^*[\mathbf{T}^{21}\,\mathbf{a} + \mathbf{T}^{22}\,\mathbf{b}]\Big)\Big\}, \quad (43)$$

where the scalar product $(\mathbf{a}^*\mathbf{p})$ is defined by the formula

$$(\mathbf{a}^*\mathbf{p}) = \sum_{mn} a^*_{mn}\, p_{mn}. \quad (44)$$

Taking into account the linearity of Eq. (43) with respect to the **T**–matrix elements for **a** and **b** fixed in a selected coordinate system, the expression for the extinction cross section of an ensemble of particles has the same form, (43), where $\mathbf{T}^{\,ij}$–matrices are replaced by $<\mathbf{T}^{\,ij}>$–matrices with all the consequences ensuing therefrom.

Let us give in an explicit form the coefficients of expansion of the incident plane electromagnetic wave propagating along an arbitrary direction and of arbitrary polarization in the laboratory coordinate system. Then we believe that the rotation of the laboratory coordinate system making it coincident with the coordinate system, in which the z–axis corresponds to the direction of wave propagation and x–axis corresponds to the polarization vector ($\mathbf{E}_i = \mathbf{i}_u$), is described by the Euler angles α_i, β_i, and γ_i. The coefficients of expansion (7) have a simple form in the coordinate system connected with the propagation of radiation polarized along x–axis (see Eqs. (3), (4), (7), and (A6))

$$a_{mn} = \delta_{m\pm1}\, i^{\,n+1}(2\,n + 1)^{1/2};$$
$$b_{mn} = m\, \delta_{m\pm1}\, i^{\,n+1}(2\,n + 1)^{1/2}. \qquad (45)$$

Taking into account Eq. (10), the expansion coefficients in the laboratory coordinate system have the form

$$a_{mn} = i^{\,n+1}(2\,n + 1)^{1/2}\left[D^{\,n}_{m1}(\alpha_i\,\beta_i\,\gamma_i) + D^{\,n}_{m-1}(\alpha_i\,\beta_i\,\gamma_i)\right];$$
$$b_{mn} = i^{\,n+1}(2\,n + 1)^{1/2}\left[D^{\,n}_{m1}(\alpha_i\,\beta_i\,\gamma_i) - D^{\,n}_{m-1}(\alpha_i\,\beta_i\,\gamma_i)\right] \qquad (46)$$

and for polarization along the y–axis ($\mathbf{E}_i = \mathbf{i}_v$)

$$a_{mn} = i^{\,n}(2\,n + 1)^{1/2}\left[D^{\,n}_{m1}(\alpha_i\,\beta_i\,\gamma_i) - D^{\,n}_{m-1}(\alpha_i\,\beta_i\,\gamma_i)\right];$$
$$b_{mn} = i^{\,n}(2\,n + 1)^{1/2}\left[D^{\,n}_{m1}(\alpha_i\,\beta_i\,\gamma_i) + D^{\,n}_{m-1}(\alpha_i\,\beta_i\,\gamma_i)\right]. \qquad (47)$$

For an arbitrary elliptic polarization the incident radiation can be presented in the form of linear combination of two coherent waves. The same is valid for corresponding extinction cross sections. For an unpolarized incident radiation the extinction cross section is equal to the half–sum of the extinction cross sections for orthogonally polarized incident waves.

7. DISCUSSION AND CONCLUSIONS

As was shown, the **T**–matrix method combined with the quantum theory of angular momentum[13] is an adequate method for estimating the matrices and cross sections of extinction of an ensemble of particles with arbitrary shapes and orientation, and it provides a possibility of replacing the cumbersome integration by an analytical method. Let us note that the same is valid for the estimation of the extinction cross sections.[21]

The **T**–matrix method is effective for the particles with the shape of body of revolution with a smooth surface (the numerical version is discussed in Ref. 22). For ellipsoidal particles[23] the time of computation of the **T**–matrix is two orders of magnitude greater than that for the spheroidal particles.[24]

For axially symmetric particles, in the case when z–axis of the coordinate system **A** is the rotation axis of the particle, some simplifications are possible connected with the relationships[10,12]

$$T^{\,ij}_{mnm'n'} = \delta_{mm'}\, T^{\,ij}_{mnn'}, \qquad T^{\,ij}_{-mnn'} = (-1)^{i+j}\, T^{\,ij}_{mnn'}, \qquad (48)$$

that makes it possible to reduce the number of summation indices in Eq. (36) by one and to halve the bulk of calculations.

For axially symmetric particles (for example, spheroids, cylinders, etc.) with the size less than the wavelength of incident radiation, in the Rayleigh approximation ($n = n' = 1$); m, $m' = -1,\ 0\ ,1$) the **T**–matrix elements are expressed in an explicit form[10,24]

$$T^{\,ij}_{mnm'n'} = \delta_{i2}\,\delta_{j2}\,\delta_{mm'}\;\; T_{m}. \qquad (49)$$

In this case all the parameters considered in the paper have the explicit form. Their use makes it possible to study the regularities of propagation of the polarized radiation, for example, in media with anysotropy caused by the molecules orientation, on the basis of the generalized Bouguer law (31).

The results obtained can be useful for theoretical studies in atmospheric optics for estimating the extinction and polarization of radiation passing through a layer of particles with different orientation structure. For a more effective calculations the radiation propagation direction can be chosen along z–axis, that allows significant simplifications (see Eqs. (25), (26), and (40)) of the calculations to be achieved.

APPENDIX: WIGNER *D*–FUNCTIONS

Wigner D–functions $D^{\,n}_{mm'}(\alpha\beta\gamma)$ are defined as matrix elements of irreducible representation of the weight n on the rotation group[11,13] or as matrix elements of the rotation operator $\mathbf{D}(\alpha\beta\gamma)$ in JM–representation[13]

$$\langle J\ M\ |\mathbf{D}(\alpha\ \beta\ \gamma)|\ J'\ M'\rangle = \delta_{JJ'}\, D^{\,J}_{mm'}(\alpha\ \beta\ \gamma). \qquad (A1)$$

Functions $D^{\,n}_{mm'}(\alpha\beta\gamma)$ are written in the form of a product of three factors, each of which depends only on one Euler angle,[13]

$$D^{\,n}_{mm'}(\alpha\ \beta\ \gamma) = \exp(-i\,m\,\alpha)\; d^{\,n}_{mm'}(\beta)\; \exp(-i\,m'\gamma), \qquad (A2)$$

where $d^{\,n}_{mm'}(\beta)$ are the Wigner functions[13] that satisfy the conditions of unitarity[13]

$$\left[\mathbf{D}^{-1}(\alpha\ \beta\ \gamma)\right]^{n}_{mm'} = \left[\mathbf{D}^{*}(\alpha\ \beta\ \gamma)\right]^{n}_{m'm};$$
$$\sum_{m=-n}^{n} D^{n}_{mm'}(\alpha\ \beta\ \gamma)\, D^{\,n*}_{mm_1}(\alpha\ \beta\ \gamma) =$$
$$= \sum_{m=-n}^{n} D^{n}_{mm'}(\alpha\ \beta\ \gamma)\, D^{\,n-1}_{m_1m}(\alpha\ \beta\ \gamma) = \delta_{m'm_1} \qquad (A3)$$

and orthogonality

$$\frac{2\,n + 1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi} \sin\beta\ d\beta \int_0^{2\pi} d\gamma\ D^{\,n}_{mm'}(\alpha\ \beta\ \gamma)\ \times$$
$$\times\ D^{\,n_1^{*}}_{m_1m_1'}(\alpha\ \beta\ \gamma) = \delta_{nn_1}\,\delta_{mm_1}\,\delta_{m'm_1'} \qquad (A4)$$

for the functions $d^{\,n}_{mm'}(\beta)$

$$\int_0^{\pi} \sin\beta\ d\beta\ d^{\,n}_{mm'}(\beta)\ d^{\,n'}_{mm'}(\beta) = \frac{2}{2\,n + 1}\,\delta_{nn'}. \qquad (A5)$$

Functions $d^{\,n}_{mm'}(\beta)$ satisfy the following relations[13]:

$$\frac{m}{\sin b}\, d^{\,n}_{0m}(\beta)\Big|_{\beta=0} = 1/2\ \delta_{m\pm1}[n(n + 1)]^{1/2},$$
$$\frac{d}{d\ b}\, d^{\,n}_{0m}(\beta)\Big|_{\beta=0} = 1/2\ m\ \delta_{m\pm1}[n(n + 1)]^{1/2}, \qquad (A6)$$
$$\frac{m}{\sin b}\, d^{n}_{0m}(\beta) = 1/2\ [n(n + 1)]^{1/2}\left[d^{\,n}_{1m}(\beta) + d^{\,n}_{-1m}(\beta)\right],$$

$$\frac{d}{d\,b}\, d^{\,n}_{0m}(\beta) = 1/2\,[n(n+1)]^{1/2}\Big[d^{\,n}_{1m}(\beta) - d^{\,n}_{-1m}(\beta)\Big]$$

as well as the multiplication theorem in the form

$$d^{\,n}_{mm'}(\beta)\, d^{\,n'}_{m_1m'_1}(\beta) = \sum_{n_1=|n-n'|}^{n+n'} C^{\,n_1m+m_1}_{\,nmn'm_1}\, C^{\,n_1m'+m'_1}_{\,nm'n'm'_1}\, d^{\,n_1}_{\,m+m_1\; m'+m'_1}(\beta) \tag{A7}$$

and the symmetry relations

$$d^{\,n}_{mm'}(\beta) = (-1)^{m'-m}\, d^{\,n}_{-m-m'}(\beta) = (-1)^{m'-m}\, d^{\,n}_{m'm}(\beta). \tag{A8}$$

The product of two D–functions $D^{\,n_1}_{\,m_1m'_1}(\alpha\beta\gamma)$ and $D^{\,n_2}_{\,m_2m'_2}(\alpha\beta\gamma)$ can be written in the form of the following sum referred to as the Clebsch–Gordan series[13]:

$$D^{\,n_1}_{\,m_1\,m'_1}(\alpha\;\beta\;\gamma)\, D^{\,n_2}_{\,m_2\,m'_2}(\alpha\;\beta\;\gamma) =$$

$$= \sum_{n_3=|n_1-n_2|}^{n_1+n_2} C^{\,n_3m_1+m_2}_{\,n_1m_1n_2m_2}\, D^{n_3}_{\,m_1+m_2\; m'_1+m'_2}(\alpha\;\beta\;\gamma)\, C^{\,n_3m'_1+m'_2}_{\,n_1m'_1\;n_2m'_2}. \tag{A9}$$

Recurrent relations for calculating the Wigner D–functions and the Clebsch–Gordan coefficients are given in Ref. 13.

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