Light scattering by cylindrical particles in the Rayleigh–Gans–Debye approximation. Part 1. Rigorously oriented particles

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The equations for calculation of the amplitude and cross section of light scattering by optically "soft" hexagonal cylindrical particles of finite length in the Rayleigh–Gans–Debye approximation are obtained. Numerical results on the light scattering phase function (or the element f_{11} of the scattering phase matrix) for infinitely long cylinder (rigorous solution) and for finite circular and hexagonal cylinders in the Rayleigh–Gans–Debye approximation are compared. The approximate equations for the light scattering cross section of circular and hexagonal cylinders in Rayleigh–Gans–Debye approximation for small diffraction parameters $ka < 1$, $kH < 1$ (where k, H, a are the wavenumber, height, and radius of the cylinder, respectively) are obtained.

Different approximations[1,2] are used in many research areas, such as optics of dispersed media, remote sensing, etc. for accurate and fast analysis of the light scattering characteristics of particles of arbitrary shape and structure. The Rayleigh–Gans–Debye (RGD), anomaly diffraction (AD), and Fraunhofer diffraction approximations are often applied to the optically "soft" particles, ($|n-1| \ll 1$, where n is the relative refractive index of the particulate matter). The formulas were obtained in Refs. 3 and 4 for cylindrical particles of round and hexagonal cross section in the AD approximation. Similar calculations in the RGD approximation have been made only for round cylinders.[5]

The purpose of this paper is analysis and comparison of some characteristics of light scattering by round and hexagonal cylinders calculated in the RGD approximation.

1. Scattering amplitude and scattering phase function

We use the integral representation of the scattering amplitude in the RGD approximation[6]:

$$\mathbf{f}(\mathbf{s},\mathbf{i}) = \frac{k^2}{4\pi}\left[-\mathbf{s}\times(\mathbf{s}\times\mathbf{e}_\mathrm{i})\right]\int(n^2-1)\exp\left[ik\mathbf{r}'(\mathbf{i}-\mathbf{s})\right]\mathrm{d}V', \tag{1}$$

where $\mathbf{s}$ and $\mathbf{i}$ are the unit vectors along the direction of scattering and propagation of light, respectively, $k = 2\pi/\lambda$ is the wavenumber, λ is the wavelength in the disperse medium, $\mathbf{e}_\mathrm{i}$ is the unit vector along the direction of polarization of the incident wave, $\mathbf{r}'$ is the radius vector of a point inside the particle. After integration, we have for a round cylinder of the height H and the radius a[6]:

$$\mathbf{f}(\mathbf{s},\mathbf{i}) = \frac{k^2}{4\pi}\left[-\mathbf{s}\times(\mathbf{s}\times\mathbf{e}_\mathrm{i})\right](n^2-1)\times$$

$$\times V_\mathrm{r}\frac{2J_1\left(\sqrt{k_1^2+k_2^2}\,a\right)}{\sqrt{k_1^2+k_2^2}\,a}j_0\left(\frac{k_3H}{2}\right), \tag{2}$$

where $V_\mathrm{r} = \pi a^2H$ is the volume of the round cylinder, $J_1(x)$ is the Bessel function of the first order, $j_0(x) = \sin(x)/x$ is the spherical Bessel function of the zeroth order;

$$k_1 = k[\sin(\theta_\mathrm{i})\cos(\phi_\mathrm{i}) - \sin(\theta_\mathrm{s})\cos(\phi_\mathrm{s})],$$

$$k_2 = k[\sin(\theta_\mathrm{i})\sin(\phi_\mathrm{i}) - \sin(\theta_\mathrm{s})\sin(\phi_\mathrm{s})],$$

$$k_3 = k[\cos(\theta_\mathrm{i}) - \cos(\theta_\mathrm{s})],$$

θ_i, ϕ_i, θ_s, ϕ_s are the angles in the spherical coordinate system indicating the direction of the incident $\mathbf{i}$ and scattered $\mathbf{s}$ waves, respectively.

Then, after integration, we obtain the formula for a hexagonal cylinder (Fig. 1)

$$\mathbf{f}(\mathbf{s},\mathbf{i}) = \frac{k^2}{4\pi}\left[-\mathbf{s}\times(\mathbf{s}\times\mathbf{e}_\mathrm{i})\right](n^2-1)V_\mathrm{h}F(H,a,\theta_\mathrm{i},\phi_\mathrm{i},\theta_\mathrm{s},\phi_\mathrm{s}), \tag{3}$$

where $V_\mathrm{h} = \frac{3\sqrt{3}}{2}a^2H$ is the volume of the hexagonal cylinder;

$$F(H,\, a,\, \theta_\mathrm{i},\, \phi_\mathrm{i},\, \theta_\mathrm{s},\, \phi_\mathrm{s}) = \frac{2}{3}j_0\left(\frac{k_3H}{2}\right)[F_1+F_2+F_3],$$

$$F_1 = j_0\left(\frac{k_1a}{2}\right)j_0\left(\frac{k_2\sqrt{3}a}{2}\right),$$

$$F_2 = \frac{1}{4}\left(1-\sqrt{3}\frac{k_1}{k_2}\right)j_0\left(\frac{\sqrt{3}a(k_2-\sqrt{3}k_1)}{4}\right)j_0\left(\frac{a(k_1+\sqrt{3}k_2)}{4}\right),$$

$$F_3 = \frac{1}{4}\left(1+\sqrt{3}\frac{k_1}{k_2}\right)j_0\left(\frac{\sqrt{3}a(k_2+\sqrt{3}k_1)}{4}\right)j_0\left(\frac{a(k_1-\sqrt{3}k_2)}{4}\right).$$

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a *b* *c*

Fig. 1. Round (*a*) and hexagonal (*b*) cylindrical particles of the radius *a* and height *H*, and the coordinate system used in the calculations (*c*).

The light scattering phase function (or the element f_{11} of the scattering phase matrix) of natural light (the polarization is chaotic) was calculated by the formula:

$$f_{11}(\beta) = \frac{1+\cos^2(\beta)}{2} k^2 |f(\beta)|^2, \qquad (4)$$

where $|f(\beta)|^2$ is the square of the absolute value of the light scattering amplitude determined by Eqs. (2) or (3); β is the scattering angle counted from the forward scattering direction.

Calculations for an infinitely long cylinder (exact solution) were carried out following the algorithm from Ref. 8 in the RGD approximation by formulas (2) and (3). The scattering phase function was normalized to that for the forward scattering direction. The relative error here and below was calculated as $(F_{approx}/F_{exact} - 1) \cdot 100\%$.

Numerical comparison has revealed that the value of the relative error in the scattering phase function of both round and hexagonal RGD cylinders does not exceed 9% as compared with the results of the exact solution for the infinitely long cylinder in the angular range up to 30°. The scattering phase function of the cylinder of the size $ka = 1$ and $kH = 2$, for light incident perpendicularly to the cylinder symmetry axis ($\theta_i = 90°$, $\phi_i = 0°$, $\beta = \phi_s$) is shown in Fig. 2*a*, and along the cylinder symmetry axis ($\theta_i = 0°$, $\phi_i = 0°$, $\beta = \theta_s$) is in Fig. 2*b*, respectively.

Let us also note that one can simplify formula (3) at small diffraction parameter ($ka < 1$), using the equality[7]:

$$j_0(x-y) = \sum_{m=0}^{\infty} (2m+1) j_m(x) j_m(y), \qquad (5)$$

or, at $x < 1$ and $y < 1$ one can consider only the first term, then

$$j_0(x-y) \approx j_0(x) j_0(y).$$

Then, at $ka < 1$ it follows from Eq. (3) after the transformations that

$$F(H, a, \theta_i, \phi_i, \theta_s, \phi_s) \approx j_0\left(\frac{k_3 H}{2}\right) j_0\left(\frac{k_1 a}{2}\right) j_0\left(\frac{k_2\sqrt{3}a}{2}\right). \qquad (6)$$

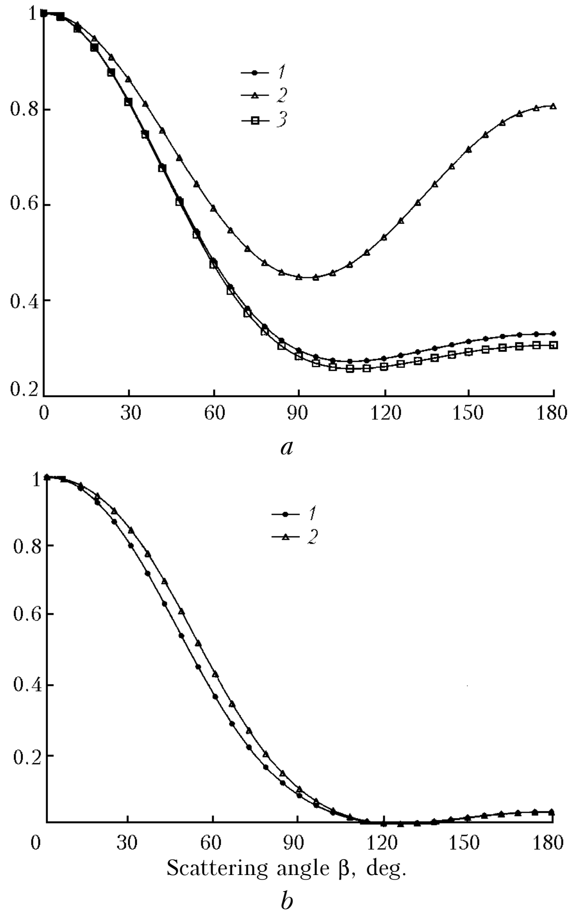

Fig. 2. The scattering phase functions $f_{11}(\beta)/f_{11}(0)$ of round RGD (*1*), hexagonal RGD (*2*) and infinitely long (*3*) cylinder with the relative refractive index $n = 1.027 - i0.068$ as a function of the scattering angle β at $ka = 1$, $kH = 2$ and for the light incident perpendicularly (*a*) and along (*b*) the cylinder axis of symmetry.

2. Light scattering cross section

The light scattering cross section σ_s is equal to[6,8]:

$$\sigma_s = \int\limits_{4\pi} |\mathbf{f}(\mathbf{s},\mathbf{i})|^2 \, d\omega, \qquad (7)$$

where $d\omega$ is the element of the solid angle (in the spherical coordinate system $\sin(\theta_s)d\theta_s d\phi_s$).

Assuming that light incident on the particle is natural (the polarization is chaotic), expanding the amplitude (2) into the Taylor series at small diffraction parameters $ka < 1$ and $kH < 1$ and integrating, we obtain the light scattering cross section σ_s of a round cylinder

$$\sigma_s = \frac{(ka)^4 H^2 (n^2-1)^2}{16} UC(H,a,\theta_i,\phi_i), \qquad (8)$$

where

$$UC(H, a, \theta_i, \phi_i) = UC_1 + UC_2,$$

$$UC_1 = \frac{2\pi}{3}\left\{4 - (kH)^2\left(\frac{2}{15} + \frac{\cos^2(\theta_i)}{3}\right) - (ka)^2\left(\frac{3}{5} + \sin^2(\theta_i)\right)\right\},$$

$$UC_2 = \frac{2\pi}{3}\left\{(kH)^2(ka)^2\left(\frac{1}{84} + \frac{\sin^2(\theta_i)\cos^2(\theta_i)}{12} + \frac{\sin^2(\theta_i)}{30} + \frac{\cos^2(\theta_i)}{20}\right)\right\}.$$

By analogy, we obtain from Eqs. (3) and (6), taking into account the expansion in the Taylor series at small diffraction parameters $ka < 1$ and $kH < 1$, for the hexagonal cylinder that:

$$\sigma_s = \frac{27(ka)^4 H^2 (n^2-1)^2}{64\pi^2} UH(H,a,\theta_i,\phi_i), \qquad (9)$$

where

$$UH(H, a, \theta_i, \phi_i) = UH_1 + UH_2,$$

$$UH_1 = \frac{2\pi}{3}\left\{4 - (kH)^2\left(\frac{2}{15} + \frac{\cos^2(\theta_i)}{3}\right) - (ka)^2\left(\frac{2}{5} + \frac{\sin^2(\theta_i)\left[3\sin^2(\phi_i) + \cos^2(\phi_i)\right]}{3}\right)\right\},$$

$$UH_2 = \frac{2\pi}{3}\left\{(kH)^2(ka)^2\left(\frac{1}{126} + \frac{\sin^2(\theta_i)}{18} \times \left(\left[3\sin^2(\phi_i) + \cos^2(\phi_i)\right]\left[\frac{1}{5} + \frac{\cos^2(\theta_i)}{2}\right]\right) + \frac{\cos^2(\theta_i)}{30}\right)\right\}.$$

In the Rayleigh case ($ka \ll 1$, $kH \ll 1$) it follows from expression (8) for the round cylinder that

$$\sigma_s = \frac{\pi(ka)^4 H^2 (n^2-1)^2}{6}, \qquad (10)$$

which agrees with the conclusion drawn in Ref. 5, and from (9) for the hexagonal cylinder we obtain

$$\sigma_s = \frac{9(ka)^4 H^2 (n^2-1)^2}{8\pi}. \qquad (11)$$

The values of the approximate scattering cross sections obtained by Eqs. (8) and (9) were also numerically compared with the light scattering cross section of the RGD cylinder obtained by direct integration following expression (7). Then the absolute value of the relative error for homogeneous RGD cylinders does not exceed 12% at $ka < 1$ and $kH < 1$ (Table) and does not depend on the refractive index of the particulate matter.

Table. Relative error of the values of the light scattering cross section calculated by the approximate formulas (8) and (9) for round and hexagonal RGD cylinders at $\phi_i = 0°$

ka	Round cylinder				Hexagonal cylinder			
	kH = 0.3		*kH* = 0.7		*kH* = 0.3		*kH* = 0.7	
	$\theta_i = 0°$	$\theta_i = 90°$	$\theta_i = 0°$	$\theta_i = 90°$	$\theta_i = 0°$	$\theta_i = 90°$	$\theta_i = 0°$	$\theta_i = 90°$
0.05	−0.008	−0.001	−0.251	−0.017	−0.002	0.037	−0.245	0.020
0.10	−0.008	−0.002	−0.252	−0.018	0.017	0.149	−0.226	0.133
0.15	−0.009	−0.006	−0.252	−0.022	0.048	0.336	−0.195	0.319
0.20	−0.010	−0.017	−0.254	−0.034	0.091	0.596	−0.152	0.580
0.25	−0.013	−0.040	−0.257	−0.057	0.146	0.929	−0.097	0.913
0.30	−0.018	−0.084	−0.262	−0.101	0.213	1.333	−0.030	1.317
0.40	−0.039	−0.268	−0.284	−0.286	0.379	2.346	0.135	2.330
0.50	−0.085	−0.666	−0.331	−0.686	0.584	3.613	0.341	3.598
0.60	−0.169	−1.416	−0.417	−1.439	0.823	5.105	0.579	5.090
0.70	−0.309	−2.699	−0.560	−2.729	1.088	6.779	0.843	6.763
0.80	−0.530	−4.756	−0.783	−4.795	1.367	8.576	1.123	8.558
0.85	−0.678	−6.168	−0.934	−6.214	1.510	9.497	1.264	9.477
0.90	−0.858	−7.895	−1.116	−7.949	1.651	10.418	1.405	10.396
0.95	−1.073	−9.988	−1.333	−10.052	1.789	11.325	1.543	11.300
1.00	−1.328	−12.506	−1.592	−12.581	1.923	12.201	1.675	12.173

Conclusions

The formulas have been obtained for calculation of the amplitude and cross section of scattering by optically "soft" cylindrical particles of finite length and hexagonal cross section in the RGD approximation. The numerical results on the scattering phase functions of round and hexagonal cylinders in RGD approximation are compared with the results for an infinitely long cylinder (exact solution). It has been revealed that the relative error of the RGD scattering phase function of round and hexagonal cylinders does not exceed 9% in the angular range up to 30°. Numerical comparison of the scattering cross sections of round and hexagonal RGD cylinders and the values of the scattering cross section calculated by the approximate formulas at small diffraction parameters ($ka < 1$ and $kH < 1$) of the RGD cylinders provides for the absolute value of the relative error no more than 12%.

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