Application of the sensitivity theory to studying tendencies for climate change

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Up-to-date approaches to modeling the climate and tendencies of its changes are considered, their merits and demerits are analyzed, and main difficulties arising in solution of the modeling problems are pointed out. An approach is proposed to estimate climate change tendencies; it is based on the use of ideas and methods of the sensitivity theory of distributed systems. New methods of the sensitivity theory are developed, and models of sensitivity of climatic models to variations of their parameters are constructed. The equations for the functional sensitivity depending on solution of the equations of fluid-dynamics models are derived.

Introduction

The problems of studying the tendencies in climate change both on the global and local scales attract now the growing attention of scientists. This is caused, on the one hand, by growing anthropogenic impact on the environment. Climatic systems response to this impact, and the consequences of the response for social and economic development of individual countries and humanity as a whole are not obvious and therefore call for thorough study. These consequences include global warming of the lower atmosphere due to growing atmospheric emissions of carbon dioxide, increase in the flux of UV radiation coming to the Earth's due the effect called the ozone hole, etc.

On the other hand, a great attention to evolution of climatic systems is caused by the need in planning of the social and economic development of regions and solution of geopolitical problems.

Approaches to climate modeling

In modeling of the climate and its tendencies, as well as in modeling of other atmospheric processes, three approaches can be distinguished: physical-statistical, hydrodynamic, and dynamic-stochastic.

The first approach, developed largely in the papers by Borisenkov and Budyko, is based on the time extrapolation of the information on the state of some or other climatic system. However, the conditions of climate evolution in the past and future cannot be thought the same, because, for example, the anthropogenic impact on the global atmospheric processes was negligible by the end of the 19th century, and now it is an important factor affecting the climate evolution. In this connection, prognostic estimates obtained in the framework of this approach are characterized by low reliability.

An attempt to take into account the variation of factors affecting the climate evolution, leads to the need of using fundamental physical laws expressed by the system of equations of atmospheric hydrothermodynamics written with some or other assumptions. Depending on whether the model parameters are random objects or not, the results of modeling also are random or nonrandom objects (parameters, vectors, fields). In the first case, we deal with dynamic-stochastic modeling; otherwise, it is a hydrodynamic modeling.

Two latter approaches are obviously more constructive, because they allow different scenarios of variation of the affecting factors. However, implementation of these approaches faces a number of difficulties.

Let us introduce designations traditional for climatic models: $\Phi_{<m>}$ is the state vector, whose components are the fields of the state parameters

$$\Phi_{<m>} = \Phi_{<m>}(X_{<3>}, t; \Psi_{<n>}),$$

where $X_{<3>}$ is the vector of spatial coordinates; t is time; $\Psi_{<n>}$ is the vector of model parameters, whose components are generally the fields determined from the physical formulation of the problem and its particular mathematical formalization,

$$\Psi_{<n>} = \Psi_{<n>}(X_{<3>}, t).$$

Then the general formulation of the model looks as (subscripts are omitted for brevity)

$$B\frac{\partial \Phi}{\partial t} + G(\Phi, \Psi) = 0;$$

$$\Phi \in Q(D_t), \quad \Psi \in R(D_t), \qquad (1)$$

where $B = B_{[m,m]}$ is an $m \times m$ diagonal matrix; $G(\Phi,\Psi) = G_{<m>}(\Phi, \Psi)$ is a nonlinear matrix differential operator; D_t is the domain of variability of

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the space and time coordinates; $Q(D_t)$ is the space of functions meeting the boundary conditions entering into the formulation of the problem (the solution of the problem belongs to this space); $R(D_t)$ is the domain of accessible values of the parameters.

The operator G in Eq. (1) must be a nonlinear differential operator, because it is obvious now that linear (linearized) models fail to realistically describe climatic systems.

Since by climatic values are meant some established values of solutions themselves (1), or some their functionals, implementation of a climatic model in the hydrodynamic approach consists in solution of Eq. (1) with the use of mean values of the model parameters (their mathematic expectations). In addition to high resource-consumption of such a process, it should be noted that, because of nonlinearity of the operator G, the solution of the problem (1) is shifted with respect to the mathematic expectation $\bar{\Phi}_{<m>}$ (or its functional), which is just an estimate of the future state of the climate. This explains low adequacy of hydrodynamic climatic models.

In the dynamic-stochastic approach, it is assumed that the parameters of the model (1) are random objects (parameters, vectors, fields) with a known distribution, and the sought parameter is the distribution $\hat{\Phi}_{<m>}$ (^ denotes a random value) or some numerical characteristics of this distribution. The approach gives more realistic estimates of climatic characteristics, but its implementation also faces some difficulties of applied and theoretical character.

The approach based on imitating modeling of the climate evolution can be used as a way out of the situation. In this case, realizations of $\hat{\Psi}_{<m>}$ are modeled, based on the information on the distribution of the parameters of the model (1), with the help of a generator of random objects. Then the problem (1) is solved many times for these realizations, whereupon numerical characteristics of the distribution $\hat{\Phi}_{<m>}$ are evaluated. However, the approach requires so much computational resources (taking into account the dimension of the vector of parameters) that its applicability becomes unrealistic.

Our purpose was to develop the mathematical apparatus allowing us to overcome the difficulties listed above. This apparatus is based on the ideas and methods of the sensitivity theory of distributed systems and imitating modeling.

Methods for studying the sensitivity of climatic models

Hereinafter, we will consider the sensitivity of climatic models, and it will be clear from the context what is meant: the sensitivity of solutions of the system of model equations or sensitivity of functionals of these solutions. Besides, we will discuss the sensitivity of discrete models, i.e., continuum models digitized in some or other way.

The methods of studying the sensitivity of mathematic models have received wide development in the theory of optimal control and identification of systems.[1–4] In meteorology, these methods were widely used in solution of the problems associated, in their idea, to inverse problems of mathematical physics and theory of identification of distributed systems, as well as to problems connected with estimation of initial fields and parameters of models from the experimental information.[5–10,etc.]

The discrete analog of Eq. (1) obtained with the use of some discretization method (the detailed review of these methods can be found in Refs. 11 and 12) has the following form:

$$B\Lambda_t \Phi_h + G^h(\Phi_h,\Psi_h) = 0;$$
$$\Phi_h \in Q^h(D_t^h),\ \Psi_h \in R^h(D_t^h), \qquad (2)$$

where $\Lambda_t \Phi_h$ and $G^h(\Phi_h,\Psi_h)$ are discrete analogs of $\frac{\partial \Phi}{\partial t}$ and $G(\Phi, \Psi)$; D_t^h is the grid domain corresponding to D_t; $Q^h(D_t^h)$ is the space of grid functions satisfying the discrete analogs of boundary conditions; $R^h(D_t^h)$ is the domain of admissible values of the grid functions being the parameters of model (2).

The discrete model of the form (2) can be obtained, for example, with the use of the variational formulation of the problem as some integral identity following from the definition of the generalized solution of the problem (1) (Ref. 12):

$$I(\Phi, \Psi, \Phi^*) = 0,\ \ \Phi \in Q(D_t).$$
$$\Phi^* \in Q^*(D_t),\ \ \Psi \in R(D_t), \qquad (3)$$

where $\Phi^* = \Phi^*_{<m>}$ is some sufficiently smooth arbitrary vector-function, and the functional $I(\Phi, \Psi, \Phi^*)$ is chosen so that the descriptions of the model in the forms (1) and (3) are equivalent in the classes of sufficiently smooth functions. The process of construction of discrete analogs of the models in this case is reduced to discretization of Eq. (3) in the corresponding functional spaces, i.e., to derivation of the following equations:

$$I^h(\Phi_h, \Psi_h, \Phi_h^*) = 0,\ \Phi_h \in Q^h(D_t^h),$$
$$\Phi_h^* \in Q^{*h}(D_t^h),\ \Psi_h \in R^h(D_t^h), \qquad (4)$$

and to writing down the stationarity conditions for the discrete analog $I^h(\Phi_h, \Psi_h, \Phi_h^*)$ of the summatory functional $I(\Phi, \Psi, \Phi^*)$.

The dimension of the grid functions $\Phi_h \in Q^h(D_t^h)$ and $\Psi_h \in R^h(D_t^h)$ is expressed through the dimension of the grid domain $D_t^h \subset D_t$, and the number of different, in the physical sense, fields being the components of the vectors $\Phi_{<m>}$ and $\Psi_{<n>}$, respectively. Taking into account the structure of the discrete representation of

Eq. (2) or (4), we renumber one after another all the components of the vectors Ψ_h and Φ_h

$$\Psi_h = \{\psi_{hi}\},\ i = 1(1)N;$$
$$\Phi_h = \{\varphi_{hi}\},\ i = 1(1)M, \qquad (5)$$

where N and M are the total numbers of components of the corresponding vectors. Hereinafter, we omit the subscript h for simplicity.

Believing that the problem (1) is well posed, we can assume[11–14] that the class of vector-functions $\Phi = \Phi(X, t, \Psi)$ $((X, t) \in D_t^h,$ $\Psi \in R^h(D_t^h))$ continuously depends on Ψ, i.e., small perturbations $\delta\Psi$ of the vector Ψ correspond to small perturbations $\delta\Phi$ of the vector Φ. As the vector $\delta\Psi$ we take

$$\delta\Psi = \{\delta\psi_i\},\ i = 1(1)N,\ \Psi + \delta\Psi \in R^h(D_t^h)$$

under the condition $\| \delta\Psi \| \ll \| \Psi \|$, where $\| \bullet \|$ denotes norm, and the vector $\delta\Phi$ is

$$\delta\Phi = \Phi(X, t, \Psi + \delta\Psi) - \Phi(X, t, \Psi).$$

Then the problem of determination of the sensitivity functions[1] reduces mathematically to calculation of partial derivatives of the sought solution or the functionals determined at the set of solutions with respect to the parameters Ψ of the model in the vicinity of some unperturbed values Ψ_0. Taking into account the peculiarities of dynamic processes occurring in the atmosphere[15] and the specificity of the problem to be solved, we restrict our consideration to estimation of the first-order sensitivity functions.

The sensitivity functions of the solutions of the problem (1) can be estimated by one of the three ways:

– by the method of direct modeling;
– with the use of variational equations;
– with the use of direct differentiation of Eq. (1).

In the case considered here, the problem of estimation of the sensitivity functions is the problem of estimation (with regard for numeration (5)) of elements of the matrix

$$H_{[M,N]} = \{h_{ij}\} = \left.\frac{\partial\Phi_{<M>}}{\partial\Psi_{<N>}}\right|_{\Psi_{<N>0}} = \left\{\frac{\partial\varphi_i}{\partial\psi_j}\right\},$$
$$i = 1(1)M,\ j = 1(1)N. \qquad (6)$$

The main idea forming the basis of the method of direct modeling is the replacement of the partial derivatives in Eq. (6) by the ratio of finite differences. First, we find the value of $\Phi(\Psi_0)$ from Eq. (2). Then, adding some increment $\delta\psi_i$ to the jth component of the vector Ψ_0, we obtain, upon solution of Eq. (2), the value of $\Phi(\Psi)$, where all the components of the vector Ψ are equal to the components of the vector Ψ_0, except for the jth component, which is equal to $\psi_{j0} + \delta\psi_j$. Then the jth column of the matrix H is roughly equal to the column vector

$$\left\{\begin{matrix} \dfrac{\varphi_1(\Psi) - \varphi_1(\Psi_0)}{\delta\psi_j} \\ \dfrac{\varphi_2(\Psi) - \varphi_2(\Psi_0)}{\delta\varphi_j} \\ \\ \dfrac{\varphi_M(\Psi) - \varphi_M(\Psi_0)}{\delta\psi_j} \end{matrix}\right\}$$

All the columns of the matrix $\tilde{H}_{[M,N]}$ being an estimate of the sensitivity matrix $H_{[M,N]}$ are determined in the similar way.

In spite of universality of the method of direct modeling, it has some disadvantages significantly restricting its domain of applicability. Presetting small variations $\delta\psi_j$, $j = 1(1)N$, one should expect small variations $\delta\Phi = \Phi(\Psi) - \Phi(\Psi_0)$, which, generally speaking, may be comparable with the errors of numerical integration of model equations. This causes low accuracy of approximation of the matrix $H_{[M,N]}$ by the matrix $\tilde{H}_{[M,N]}$. When presetting large variations $\delta\psi_j$, $j = 1(1)N$, the estimate $\tilde{H}_{[M,N]}$ is coarsened in the case of the nonlinear dependence of Φ on Ψ. What's more, this method requires voluminous calculations, and thus it can hardly be implemented in practice.

With regard for the above-said, the method of direct modeling can be recommended for estimating the sensitivity of the solutions of Eq. (2), which depend linearly on a small number of parameters.

The method based on solution of variational equations proposed in Ref. 12 is free of the two first disadvantages mentioned above. In Ref. 12, the equation relating the variations $\delta\Phi$ with the variations $\delta\Psi$ was obtained from the condition of stationarity of the summatory functional (4) with respect to the variations $\delta\Phi$ and $\delta\Psi$, i.e., from the condition

$$\lim_{\xi\to 0}\frac{\partial}{\partial\xi} I^h(\Phi_0 + \xi\delta\Phi,\ \Psi_0 + \xi\delta\Psi,\ \Phi^*) = 0. \qquad (7)$$

Assuming the stationarity of the left-hand side of Eq. (7) with respect to the variations of Φ^*, we obtain

$$B\Lambda_t\,\delta\Phi + A^h(\Phi_0,\Psi_0)\,\delta\Phi + C^h(\Phi_0,\Psi_0)\,\delta\Psi = 0, \qquad (8)$$

where the $M \times M$ matrix A^h and the $M \times N$ matrix C^h are determined as follows:

$$A^h_{[M,M]}(\Phi_{<M>0},\ \Psi_{<N>0}) =$$
$$= \frac{\partial}{\partial\Phi'_{<M>}}\lim_{\xi\to 0}\frac{\partial}{\partial\xi} G_{<M>}(\Phi_{<M>0} + \xi\Phi'_{<M>},\ \Psi_{<N>0});$$
$$C^h_{[M,M]}(\Phi_{<M>0},\ \Psi_{<N>0}) =$$
$$= \frac{\partial}{\partial\Psi'_{<M>}}\lim_{\xi\to 0}\frac{\partial}{\partial\xi} G_{<M>}(\Phi_{<M>0},\ \Psi_{<N>0} + \xi\Psi'_{<N>}),$$

where $\Phi'_{<M>}$ and $\Psi'_{<N>}$ are arbitrary vectors with the components φ'_i, $i = 1(1)M$, and ψ'_j, $j = 1(1)N$, respectively, such that $\Phi_{<M>0} + \xi\Phi'_{<M>} \in Q^h(D_t^h)$, $\Psi_{<N>0} + \xi\Psi'_{<N>} \in R^h(D_t^h)$ at small ξ.

Equation (8) is linear with respect to $\delta\Phi_{<M>}$ and $\delta\Psi_{<N>}$. Taking into account the definition (6) of the sensitivity functions and the fact that $\delta\Phi_{<M>} = \Phi_{<M>}(\Psi_{<N>0} + \delta\Psi_{<N>}) - \Phi(\Psi_{<N>0})$, we have

$$H_{[M,N]} = \left.\frac{\partial\Phi_{<M>}}{\partial\Psi_{<N>}}\right|_{\Psi_{<N>0}} = \left.\frac{\partial\delta\Phi_{<M>}}{\partial\delta\Psi_{<N>}}\right|_{\delta\Psi_{<N>0}}.$$

Upon differentiation of Eq. (8) with respect to $\delta\Psi_{<N>}$, we have

$$B\Lambda_t H + A^h(\Phi_0,\Psi_0)\, H = -\, C^h(\Phi_0,\Psi_0)\, E, \tag{9}$$

where E is the unit $N \times N$ matrix (subscripts in Eq. (9) are omitted for simplicity).

The set of equations (2) and (9) forms the set of equations of sensitivity of the discrete model corresponding to the continuous model (1).

As was already noted, the method based on the use of the variational equations is free of disadvantages inherent in the method of direct modeling, except only for cumbersome scheme of realization. Actually, the number of calculations of the function $\Phi(\Psi + \delta\Psi)$ by the method of direct modeling is equal to the number of equations with respect to the functions $\Phi_0(\Psi_0)$ and $\partial\Phi(\Psi)/\partial\psi_j$, $j = 1(1)N$ in the set (2) and (9).

This disadvantage can be eliminated taking into account the following circumstances. Usually, the study of the sensitivity of models is a preparatory stage for identification of the parameters of the models. As a result of solution of the identification problem, we have the vector $\Psi^{\rm opt}_{<N>}$ of the best (in some sense) values of the parameters. The dimension of this vector is usually rather large. Therefore, for its convenient storage and further use, $\Psi^{\rm opt}_{<N>}$ (being a grid function) is usually approximated by some few-parameter spatiotemporal dependence. That is, the function

$$\tilde{\Psi}^{\rm opt}_{<N>} = \tilde{\Psi}_{<n>}\,(X_{<3>}, t;\, \Pi^{\rm opt}_{<N>}), \tag{10}$$

is determined, whose projection onto the space of grid functions, to which $\Psi_{<N>}$ belongs, is close to the grid function $\Psi^{\rm opt}_{<N>}$ in a pre-fixed sense and satisfies the condition

$$k \ll N.$$

In addition, some parameters entering into Eq. (1) (for example, dissipation parameters, heat influxes, etc.) are often available in the analytical form

$$\Psi'_{<n'>} = \Psi'_{<n'>}\,(X_{<3>},\, t;\, \Pi'_{<k'>}), \tag{11}$$

and

$$\Psi_{<n>} = \langle\Psi'_{<n'>},\, \Psi''_{<n''>}\rangle, \tag{12}$$

where $n' + n'' = n$, and $\Psi''_{<n''>}$ is the vector of parameters not given in the analytical form (11). Note that in Eqs. (10) and (11) the components of the vectors $\Pi^{\rm opt}_{<k>}$ and $\Pi'_{<k'>}$ are scalars.

When solving the identification problems with following determination of the dependence of the form (10), this problem can be reformulated into the problem of identification of just the parameters $\Pi_{<k>}$ entering into the continuous model of the atmosphere

$$B\,\frac{\partial\Phi(X,\, t,\, \Pi)}{\partial t} + G(X,\, t,\, \Pi,\, \Phi) = 0; \tag{13}$$

$$\Phi \in Q(D_t),\;\; \Pi \in R(D_t).$$

Statements of such identification problems and their solutions are considered in Refs. 10 and 16. The problem of studying the sensitivity in this formulation is considered in Ref. 9.

Equation (1) takes the similar form in the case (12)

$$B\,\frac{\partial\Phi(X,\, t,\, \Pi',\, \Psi'')}{\partial t} + G(X,\, t,\, \Pi',\, \Phi,\, \Psi'') = 0; \tag{14}$$

$$\Phi \in Q(D_t),\;\; \Pi' \in R'(D_t),\;\; \Psi'' \in R''(D_t).$$

Below we consider the problem (14), because it is more general. The problem of estimating the sensitivity is reduced in this case to estimation of the sensitivity of Φ to the parameters Ψ'' and Π'. The sensitivity of the solution of Eq. (2) to Ψ'' is studied by the methods considered above.

If the problem (14) is well posed and the conditions of continuous differentiability of the operator $G(X,\, t,\, \Pi',\, \Phi,\, \Psi'')$ with respect to Π' in the domain $R'(D_t)$ are fulfilled, then according to Ref. 5 the solution of the equation

$$B\,\frac{\partial}{\partial t}\,H'_i + \left.\frac{\partial}{\partial\pi'_i}\,G(X,\, t,\, \Pi',\, \Phi,\, \Psi'')\right|_{\substack{\Pi'=\Pi'_0\\ \Psi''=\Psi''_0}} = 0 \tag{15}$$

exists and is the function of the sensitivity of the solution of Eq. (14) to π'_i at $\Pi' = \Pi'_0$, $\Psi'' = \Psi''_0$, i.e.,

$$H'_i = H'_{<M>i} = \left.\frac{\partial\Phi_{<M>}}{\partial\pi'_i}\right|_{\substack{\Pi'=\Pi'_0\\ \Psi''=\Psi''_0}}.$$

Upon differentiation of the second term in the right-hand side with regard for the conditions $\Pi' = \Pi'_0$ and $\Psi'' = \Psi''_0$, we obtain

$$B\,\frac{\partial}{\partial t}\,H'_i + G'(X,\, t,\, \Pi'_0,\, \Phi_0,\, \Psi''_0,\, H'_i) = 0,\; i = 1(1)k', \tag{16}$$

where $\Phi_0 = \Phi(X,\, t,\, \Pi'_0,\, \Psi''_0)$, and $G' = G'_{<m>}$ is a vector differential operator obtained from differentiation of the operator G with respect to π'_i.

The set of equations (14) and (16) is the set of equations of the model of sensitivity of the model (1) to the components of the vector Π'. The function of sensitivity of the discrete model (2) to the components of the vector Π' at $\Pi' = \Pi'_0$, $\Psi'' = \Psi''_0$, is the $M \times k'$ matrix H':

$$H'_{[M,k']} = \{h'_{ij}\}, \quad i = 1(1)M, \quad j = 1(1)k', \qquad (17)$$

whose components are the values of the ith element of the solution of the jth equation of the set (16).

Denote the matrix of sensitivity of the discrete model (2) to variations of the parameters $\Psi''_{<n''>}$ ($N'' = n''\,L$, where L is the number of nodes of the grid domain) as $H''_{[M,N'']}$. The total number of equations of the form (9) and (16) to be solved for estimating the sensitivity of the model (2) in the case that the parameters are given in the form (11) and (12) is $k' + N''$. Taking into account the fact that $k' \ll N' = n'\,L$ and $N = N' + N''$, we can conclude that estimation of the sensitivity of the model (2) is far more efficient, if its parameters Ψ are given analytically as functions of spatial coordinates, time, and some numerical parameters.

If the matrices $H_{[M,N]}$ ($H'_{[M,k']}$ and $H''_{[M,N'']}$) are estimated, then the variation $\delta\Phi_{<M>}$ of the solution of Eq. (2) corresponding to the variation $\delta\Psi_{<N>}$ ($\delta\Pi'_{<k'>}$ and $\delta\Psi''_{<N''>}$) is calculated as

$$\delta\Phi_{<M>} = H_{[M,N]}\, \delta\Psi_{<N>} \qquad (18)$$

or

$$\delta\Phi_{<M>} = H'_{[M,k']}\, \delta\Pi'_{<k'>} + H''_{[M,N'']}\, \delta\Psi''_{<N''>}. \qquad (19)$$

Now we pass on to the solution of the problem of estimating the sensitivity of the functionals of the solutions of Eq. (2). Let $I(\Phi)$ be the functional of Φ, which is thought, for definiteness, to be directly dependent on the components of Φ. Because Φ depends on Ψ, this functional always depends indirectly on the components of the vector Ψ as well. The variation δI of the functional I is sought as a function of the variations $\delta\Phi$ and $\delta\Psi$.

On the assumption of the limited and continuous dependence of the functional I on Φ (just this case is of interest in solution of a wide variety of applied problems), in the general case we can determine the vectors

$$\mathrm{grad}_\Phi I(\Phi) = \left\{\frac{\partial I(\Phi)}{\partial \varphi_i}\right\}, \; i = 1(1)M, \qquad (20)$$

$$\mathrm{grad}_\Psi I(\Phi(\Psi)) = \left\{\frac{\partial I(\Phi(\Psi))}{\partial \psi_j}\right\}, \; j = 1(1)N \qquad (21)$$

and in the particular case, when Ψ is described by Eqs. (11) and (12), the vectors

$$\mathrm{grad}_\Phi I(\Phi) = \left\{\frac{\partial I(\Phi)}{\partial \varphi_i}\right\}, \; i = 1(1)M, \qquad (22)$$

$$\mathrm{grad}_{\Pi'} I(\Phi(\Pi')) = \left\{\frac{\partial I(\Phi(\Pi'))}{\partial \pi'_j}\right\}, \; j = 1(1)k', \qquad (23)$$

and

$$\mathrm{grad}_{\Psi''} I(\Phi(\Psi'')) = \left\{\frac{\partial I(\Phi(\Psi''))}{\partial \psi''_j}\right\}, \; j = 1(1)N''. \qquad (24)$$

Because of the direct dependence of I on Φ, the vectors (20) and (22) can be written down in the explicit form. Then

$$\delta I(\delta\Phi)\big|_{\Phi=\Phi_0} = (\mathrm{grad}_\Phi I(\Phi)\big|_{\Phi=\Phi_0}, \delta\Phi), \qquad (25)$$

where $(\bullet,\bullet)$ denotes a scalar product.

Similarly (in the general case)

$$\delta I(\delta\Psi)\big|_{\Psi=\Psi_0} = (\mathrm{grad}_\Psi I(\Phi(\Psi))\big|_{\Psi=\Psi_0}, \delta\Psi) \qquad (26)$$

or, taking into account that

$$\mathrm{grad}_\Psi I(\Phi(\Psi))\big|_{\Psi=\Psi_0} = \mathrm{grad}_\Psi I(\Phi)\big|_{\Phi=\Phi_0},$$

$$\delta I(\delta\Psi)\big|_{\Psi=\Psi_0} = (\mathrm{grad}_\Psi I(\Phi)\big|_{\Phi=\Phi_0}, \delta\Psi). \qquad (27)$$

In the case (11), (12)

$$\delta I(\delta\Pi')\big|_{\Pi'=\Pi'_0} = (\mathrm{grad}_{\Pi'} I(\Phi)\big|_{\Phi=\Phi_0}, \delta\Pi'), \qquad (27')$$

$$\delta I(\delta\Psi'')\big|_{\Psi''=\Psi''_0} = (\mathrm{grad}_{\Psi''} I(\Phi)\big|_{\Phi=\Phi_0}, \delta\Psi''). \qquad (27'')$$

We obtain the explicit equations for $\mathrm{grad}_\Psi I(\Phi(\Psi))$. Upon substitution of the expression for $\delta\Phi$ from Eq. (18) into Eq. (25), we have

$$\delta I(\delta\Phi)\big|_{\Phi=\Phi_0} = \delta I(\delta\Phi(\delta\Psi))\big|_{\Phi=\Phi_0} = \delta I(\delta\Psi)\big|_{\Phi=\Phi_0} =$$

$$= (\mathrm{grad}_\Phi I(\Phi)\big|_{\Phi=\Phi_0}, H\delta\Psi). \qquad (28)$$

Taking into account the Lagrange identity

$$(X, AY) = (A^T X, Y),$$

where T denotes transposition; X and Y are arbitrary vectors; A is the matrix of the dimension corresponding to the dimension of the vectors X and Y, Eq. (28) can be written as

$$\delta I(\delta\Phi)\big|_{\Phi=\Phi_0} = (H^T\, \mathrm{grad}_\Phi I(\Phi)\big|_{\Phi=\Phi_0}, \delta\Psi). \qquad (29)$$

Comparing Eqs. (29) and (25), we come to the conclusion:

$$\mathrm{grad}_\Psi I(\Phi(\Psi))\big|_{\Psi=\Psi_0} = H^T\, \mathrm{grad}_\Phi I(\Phi)\big|_{\Phi=\Phi_0}. \qquad (30)$$

Similarly,

$$\mathrm{grad}_{\Pi'} I(\Phi(\Pi'))\big|_{\Pi'=\Pi'_0} = H'^T\, \mathrm{grad}_\Phi I(\Phi)\big|_{\Phi=\Phi_0}, \qquad (31)$$

$$\mathrm{grad}_{\Psi''} I(\Phi(\Psi''))\big|_{\Psi''=\Psi''_0} = H''^T\, \mathrm{grad}_\Phi I(\Phi)\big|_{\Phi=\Phi_0}, \qquad (32)$$

where H' is the matrix determined by Eq. (17), and the matrix

$$H'' = H''_{[M,N'']} = \left\{ \left. \frac{\partial \varphi_i}{\partial \psi''_l} \right|_{\Phi=\Phi_0} \right\}, \; i = 1(1)M, \; l = 1(1)N''.$$

Equation (30) (or (31), (32)) allows one to calculate the sensitivity of the functionals of the solutions of Eq. (2) to variations of the parameters of atmospheric models. Note that the sensitivity of functionals to variations of the model parameters can be also estimated using the apparatus of the theory of conjugate equations in partial derivatives,[12,17,18] but this approach gives no significant advantages as compared to the use of Eqs. (20), (30), (31), and (32).

Note that in the above equations the gradient of the functional in its arguments was thought to be a column vector. This is in some disagreement with the assumptions used in differentiation of vector fields. Taking into account this remark, it is easy to write equations, in which gradient is a row vector. However, calculations in this case are very cumbersome due to a great number of transposition signs.

Conclusion

The proposed mathematical apparatus for estimating the sensitivity of climatic models presents a possibility of more adequate estimation of climate change tendencies under conditions of *a priori* uncertain parameters affecting the evolution of climatic system. In this case, the current state of the climatic system should be used as the initial state. This state can be evaluated by processing climatic information. In this connection, there is no need in its realistic reconstruction with some climatic model. The last circumstance is the obvious advantage of the proposed approach to the study of climate tendencies on both the global and local scales.

The scheme of estimating the climate tendencies with the use of the proposed methods is the following. First, using the sensitivity model, the sensitivity of the studied climatic characteristics to variations of the interesting parameters is estimated. The sensitivity model uses currently observed climatic characteristics and the values of parameters. Then, realizations of parameters of the factors affecting the evolution of the climatic system are modeled based on *a priori* information on their distributions. Then, using Eqs. (18) and (27), variations of the studied characteristics are estimated, and, finally, the obtained results are statistically processed in order to determine numerically the distributions of the characteristics of our interest. Such a scheme, unlike the similar one described above, keeps the advantages of the method of imitating modeling, but does not require huge computational resources.

To check the efficiency of the proposed method, we have conducted a series of numerical experiments based on the following assumptions.

As known, the middle atmosphere affects significantly the processes of formation of the global climate and determines the tendencies of its change. This is caused by the fact that most active gaseous constituents of the atmosphere, such as carbon dioxide and ozone, affecting the radiative balance of the atmosphere, are concentrated just in the middle atmosphere. On the other hand, atmospheric tides refer to the basic processes occurring in the middle atmosphere. In this connection, in our experiments we studied the sensitivity of the model of established tide motions to variations of the parameters.

The considered model takes into account all known physical phenomena affecting the thermodynamic conditions of the middle atmosphere, such as viscosity, thermal conductivity, radiative cooling in the longwave spectral regions, and hydrodynamic effects. The diffusion processes were parameterized within the framework of the Rayleigh friction.

The parameters entering into the description of dissipative effects, namely, the parameters determining the degree of atmospheric turbulence, affect most significantly the variations of the state of the middle atmosphere due to wave processes.

The tide models especially often use, as parameterization of turbulent diffusion, the parameterization, in which the contribution of the turbulent exchange to the Rayleigh friction coefficient is described by some function of height and three parameters: the value (at the turbopause) of the addition α_0 due to turbulence to the Rayleigh friction coefficient, the height z_1 of the turbopause bottom, and the height z_2 of the turbopause top. The values of these parameters proposed by different authors differ significantly. This is connected not only with imperfection of the techniques for determination of these parameters from the experimental data. It seems that such a spread in the values of these parameters has some physical basis. In this connection it is interesting how sensitive are tide characteristics to variations of these parameters.

Figure 1 shows the functions of sensitivity of the amplitude of diurnal variations of the zonal wind. These functions were obtained with the use of the sensitivity model developed using the proposed approach.

The analysis of the results shown in the figure suggests that they are in complete agreement with the theory stated, for example, in Refs. 19 and 20. Thus, the fields of the amplitudes are maximally sensitive to the parameter α_0 characterizing the degree of atmospheric turbulence at the height of the turbopause. The maximum sensitivity to the turbopause bottom height z_1 is below the turbopause. This is connected with the presence of tide modes reflected down from the turbopause in the wave fields. The sensitivity to the turbopause top height is almost absent below the turbopause. This indicates that the turbopause even with its insignificant thickness is the main factor of formation of wave motions in the height range from 50 to 90 km.

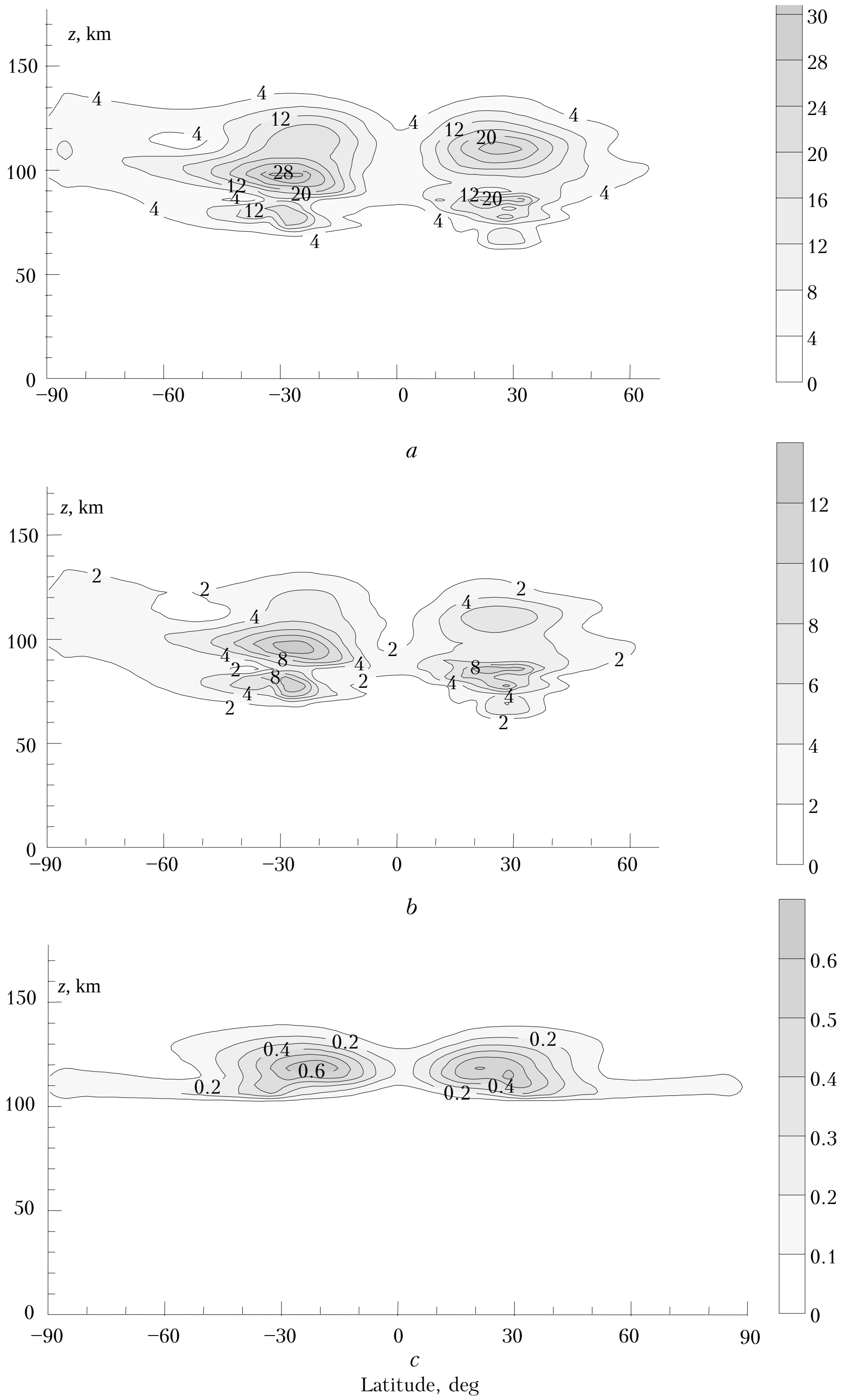

Fig. 1. Functions of sensitivity of the amplitude of diurnal tide variations of the zonal wind velocity to variations of the parameters: $\alpha_0 \cdot 10^5$, (m/s)/s^{-1} (*a*), z_1, (m/s)/km (*b*), z_2, (m/s)/km (*c*).

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