Account for the air-water interface in the problem on determining the source function of the radiative transfer equation within a spherical model of the atmosphere–ocean system

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A spherical model of the atmosphere–ocean system is represented by a mathematically equivalent model, in which light reflection, transmission, and refraction at the air–water interface is expressed through optical parameters of an elementary volume of the medium in the radiative transfer equation (RTE). These parameters are introduced using the theory of generalized functions and the Fermat principle. The source function for the RTE in this model is expressed through continuous linear functionals in the form corresponding to the absence of ray bending and refraction. Thus, the new method to account for refraction based on deformation of the system of spherical coordinates is generalized to the interface.

Introduction

Theoretical simulation of the field of solar radiation scattered by air and water shells of the atmosphere–ocean system (AOS) is based on solution of the boundary-value problem for the radiative transfer equation (RTE). Formulation of the problem on solar radiation transfer in the AOS assumes consideration of a large number of parameters determining the character of light interaction with a medium both inside it and at its boundaries, including the air–water interface. In this connection, approximate methods based on some simplifications, such as, for example, the plane medium geometry, absence of light refraction, single interaction of the radiation with the Fresnel interface, etc. (see Refs. 1–3 and references therein), have received wide acceptance. Radiation exchange between media with the plane Fresnel interface was considered in Ref. 4. The multiparameter character of the problem and difficulty of obtaining simple solutions motivate the search for new approaches accounting for the spherical geometry of the system, light refraction, and singular properties of the boundary conditions and the source function.

In this paper, we present the nontraditional approach to consideration of refraction and an interface in the problem of finding the source function in the RTE. If the radiative transfer problem is formulated in the spherical atmosphere–ocean system with the allowance made for ray bending, refraction, and reflection at the Fresnel interface, then this approach allows the source function to be represented in the form not including explicitly the refraction effect and the interface.

1. Source function of the radiative transfer equation accounting for refraction

In a medium with variable refractive index n, the RTE

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{I}{n^2}\right) = \varepsilon\left(\frac{B}{n^2} - \frac{I}{n^2}\right) \quad (1)$$

expresses the law of intensity I variation at the element ds of the ray trajectory.[5] In Eq. (1) the intensity $I = I(\mathbf{r}, \mathbf{s})$ is a function of the radius vector $\mathbf{r}$ of a point in space and the vector $\mathbf{s}$ of the direction of radiation propagation. The source function is[6]

$$B(\mathbf{r}, \mathbf{s}) = \frac{\Lambda}{4\pi}\int_{4\pi} x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')\, I(\mathbf{r}, \mathbf{s}')\, \mathrm{d}\Omega', \quad (2)$$

where $\mathbf{s}'$ is the direction vector of the ray incident on an elementary volume at the point $\mathbf{r}$, $\mathbf{s}\cdot\mathbf{s}' = \cos\gamma$, γ is the scattering angle; dΩ' is the element of the solid angle in the vicinity of the $\mathbf{s}'$ vector.

At every $\mathbf{r}$ point, the medium is characterized by the extinction coefficient ε, the quantum survival probability (single scattering albedo) $\Lambda = \sigma/\varepsilon$, where σ is the scattering coefficient, and the scattering phase function $x(\mathbf{s}\cdot\mathbf{s}')$ describing the angular distribution of the radiation scattered by the elementary volume.

The boundary conditions represent the character of intensity variation at the lower boundary (LB) of the AOS and at the air–water interface, as well as specify the solar radiation flux incident on the upper boundary (UB) of the atmosphere

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$$I_{0\lambda} = \pi F_\lambda\, \delta(\mathbf{s}' - \mathbf{s}_0)\ , \qquad (3)$$

where $\delta(\mathbf{s}' - \mathbf{s}_0)$ is the Dirac's delta; $\mathbf{s}_0$ is the vector of direction of the solar ray onto the atmospheric UB, πF_λ is the solar constant, and λ is the radiation wavelength. Hereinafter the subscript λ is omitted.

Taking into account that the source function $B = B(\mathbf{r}, \mathbf{s})$ accounting for refraction has the form (2) (Ref. 6) the same as with its neglect, let us separate its component $B_1 = B_1(\mathbf{r}, \mathbf{s})$ caused by nonscattered radiation. For this purpose, let us represent the RTE solution in the form of the sum $I = I_i + I_d$ of the direct $I_i = I_i(\mathbf{r}, \mathbf{s})$ and diffuse $I_d = I_d(\mathbf{r}, \mathbf{s})$ components and separate the equation for I_i from Eq. (1):

$$\frac{d}{ds}\left(\frac{I_i}{n^2}\right) = \varepsilon\left(-\frac{I_i}{n^2}\right). \qquad (4)$$

For the diffuse component intensity I_d, the radiative transfer equation

$$\frac{d}{ds}\left(\frac{I_d}{n^2}\right) = \varepsilon\left(\frac{B}{n^2} - \frac{I_d}{n^2}\right) \qquad (5)$$

includes the source function $B = B_d + B_1$ determined by the integral equation in the form of the sum of two terms:

$$B = \frac{\Lambda}{4\pi}\int_{4\pi} x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')\, I_d(\mathbf{r}, \mathbf{s}')\, d\Omega' + $$

$$+ \frac{\Lambda}{4\pi}\int_{4\pi} x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')\, I_i(\mathbf{r}, \mathbf{s}_0')\, d\Omega', \qquad (6)$$

where the diffuse component B_d (first term) keeps the form (2).

The component B_1 determined by the second term in Eq. (6) corresponds to the single scattering sources and includes the unknown singular function $I_i(\mathbf{r}, \mathbf{s}_0')$, where $\mathbf{s}_0'$ is the vector of direction of the nonscattered solar ray at the point $\mathbf{r}$. Obviously, in the AOS with the variable refractive index, the direction $\mathbf{s}_0'$ of the radiation incident on the elementary volume at the point $\mathbf{r}$ does not coincide with the initial direction $\mathbf{s}_0$ of the solar ray at the atmospheric UB because of refraction. The function $I_i(\mathbf{r}, \mathbf{s}_0')$ can be found by integrating Eq. (4) with the boundary condition (3). It should be noted that integration in Eq. (4) should be done over the actual curvilinear trajectory $\breve{L}$ of the solar ray as it passes from the atmospheric UB to a fixed point $P(\mathbf{r})$. The solution of Eq. (4)

$$I_i^o = \frac{I_i}{n^2} = \pi F\, \exp\left(-\int_{\breve{L}} \varepsilon\, ds\right)\, \delta(\mathbf{s}' - \mathbf{s}_0') =$$

$$= \pi F\, \exp(-\tau)\, \delta(\mathbf{s}' - \mathbf{s}_0') \qquad (7)$$

includes, as parameters, the optical distance $\tau = \tau(\mathbf{r})$ corresponding to the ray path up to the point P:

$$\tau = \int_{\breve{L}} \varepsilon\, ds, \qquad (8)$$

and the direction $\mathbf{s}_0' \neq \mathbf{s}_0$ depending on the actual distribution of the refractive index n along its trajectory. Let us use the concepts accepted in the theory of generalized functions[7] to define $\pi F \exp(-\tau)\, \delta(\mathbf{s}' - \mathbf{s}_0')$ as the density of the angular distribution of solar radiation incident onto an elementary volume and select $\frac{\Lambda}{4\pi} x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')$ to be the main function. Then determination of the single scattering source function $B_1(\mathbf{r}, \mathbf{s})$ in Eq. (6) can be reduced to calculation of the functional

$$B_1(\mathbf{r}, \mathbf{s}) = \frac{\Lambda}{4\pi}\int_{4\pi} x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}')\, \pi F\, \exp(-\tau)\, \delta(\mathbf{s}' - \mathbf{s}_0') d\Omega' =$$

$$= \frac{\Lambda}{4\pi} x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}_0')\, \pi F \exp(-\tau) \qquad (9)$$

at a fixed solar ray direction $\mathbf{s}_0'$ as a parameter of the scattering phase function $x(\mathbf{r}, \mathbf{s}\cdot\mathbf{s}_0')$.

Equation (9) specifies the scheme of determining the source function $B(\mathbf{r}, \mathbf{s})$ in the medium, when the refractive index $n(\mathbf{r})$ continuously depends on the spatial coordinates. The distribution $n(\mathbf{r})$ in the AOS has a discontinuity at the air–water interface, and the source function $B(\mathbf{r}, \mathbf{s})$ should account for this fact. Below we present a generalized scheme for determining the source function $B(\mathbf{r}, \mathbf{s})$ for multilayer and two-medium spherical models of the AOS with the stepwise distribution of the refractive index, $n(\mathbf{r})$, over radius.

2. Consideration of the interface in a multilayer spherical model of the AOS

Consider the AOS model consisting of spherically symmetric layers of finite geometric thickness with discretely varying value of the refractive index. The air–water interface is one of the numerous interfaces between layers with different n. The trajectory of quanta in such a medium is a broken line meeting the Fermat principle. The Fermat principle expresses the condition, according to which a ray in a medium with the variable refractive index chooses the extreme trajectory. Mathematically, this is reduced to the requirement that the first variation of the functional expressing the optical length of the ray l^o (the superscript "o" means "optical") along its path between two arbitrary points P_1 and P_2 takes the zero value[8]:

$$\delta\int_{P_1}^{P_2} n ds = 0. \qquad (10)$$

The integral in Eq. (10) is also referred to as the reduced ray length.[9] Therefore, below we will use this term to avoid confusion with the term "optical path" for τ. Inside every layer, the Fermat principle leads to the law of rectilinear light propagation. As the ray intersects the interface between layers with different n values, this principle gives the Snell law of light refraction[9]:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2, \tag{11}$$

where θ_1 and θ_2 are, respectively, the radiation incidence and refraction angles, n_1 and n_2 are the refractive indices in the first and the second layers along the beam path. The relative refractive index $n_{21} = n_2/n_1$ takes the values ≥ 1 and ≤ 1.

From the law of conservation for the light ray as a ray tube,[10] taking into account the law of refraction (11), we can easily find the equation of a ray invariant

$$I^{\text{o}} = \frac{I_{\text{t}}}{n_2^2} = \frac{I_{\text{i}} - I_{\text{r}}}{n_1^2} = \frac{I_{\text{i}}}{n_1^2}(1 - R) = \text{const}, \tag{12}$$

that determines the distribution of the reduced intensity $I^{\text{o}} = \dfrac{I}{n^2}$ over the ray trajectory in a transparent layered medium. The subscripts i, r, and t denote, respectively, the intensity of the incident, reflected, and refracted light; $R = R(n_{21}, \theta_1)$ is the coefficient of Fresnel reflection at the layer interface.[9] Thus, in the layered model medium, according to Eq. (12), the reduced intensity I^{o} decreases not only due to the effects of light absorption and scattering inside the layers, but also due to the reflection loss, because $1 - R < 1$ at $n_{21} \neq 1$ at the layer interface. This factor will be referred to as Fresnel extinction in the medium. The effect of Fresnel extinction manifests itself at spatially separated points. The Fresnel extinction coefficient has a singularity at these points and it is not defined in the classical meaning. However, it can be defined from the viewpoint of the theory of generalized functions[7] in the Sobolev–Shvarts meaning

$$(f', g) = -(f, g'). \tag{13}$$

Here the functional (f', g) is a derivative of the singular generalized function f defined at the main function g. The function f' is called the distribution density.

The ray direction $\mathbf{s}$ is described by the spherical coordinates (ϑ, φ), where ϑ is the polar angle of the direction vector and φ is the azimuth angle. According to Eq. (13), determine the optical thickness $\tau_{\text{F}} = \tau_{\text{F}}(\mathbf{r})$ due to Fresnel extinction along the ray path $\breve{L}$ in the medium from the atmospheric UB to the point $P(\mathbf{r})$ as a functional

$$\tau_{\text{F}}(\mathbf{r}) = \int_{\breve{L}} \varepsilon_{\text{F}}\, \text{d}s = \int_{\breve{L}} \sum_{j=1}^{N} \tau_{\text{F}j}\delta(r - r_j)\, \delta(\vartheta' - \vartheta'_j)\text{d}s = \sum_{j=1}^{N} \tau_{\text{F}j}(r_j, \vartheta'_j), \tag{14}$$

where

$$\varepsilon_{\text{F}}(r, \vartheta') = \sum_{j=1}^{N} \tau_{\text{F}j}\delta(r - r_j)\, \delta(\vartheta' - \vartheta'_j) \tag{15}$$

is used as a singular distribution density. This parameter will be called the index of Fresnel extinction. The subscript $j = 1, 2, \ldots, N$ here is the serial number of the layer interfaces ($j - 1$th and jth ones) along the path of a solar ray to the point $P(\mathbf{r})$; r_j is the radius of the jth layer interface in the spherical coordinate system with the origin at the planet's center; ϑ'_j is the polar angle of the direction vector of the ray incident onto the jth layer interface; $\tau_{\text{F}j} = -\ln(1 - R_j)$ is the Fresnel optical thickness corresponding to intersection of the jth interface; $R_j = R_j(n_{21}, \theta_1^j)$ is the Fresnel coefficient of reflection from the jth interface. When calculating $R_j(n_{21}, \theta_1^j)$ at every interface, the angle of ray incidence θ_1^j can be found as $\theta_1^j = \arccos|\cos\vartheta'_j|$.

With the allowance for the layer interfaces according to Eqs. (14) and (15), determination of the total optical distance $\tau^{\text{o}}(\mathbf{r})$ along the ray path $\breve{L}$ can be reduced to simple summation $\tau^{\text{o}}(\mathbf{r}) = \tau_{\text{F}}(\mathbf{r}) + \tau(\mathbf{r})$ of the Fresnel optical thickness $\tau_{\text{F}}(\mathbf{r})$ at the layer interfaces and the optical thickness $\tau(\mathbf{r})$ due to absorption and scattering inside every layer.

Let us pass now to determination of the direction $\mathbf{s}'_0$ of the nonscattered solar ray reaching the point $P(\mathbf{r})$. In the spherical AOS model, the allowance for refraction does not change the azimuth angle φ_0 of the direction vector, that is, $\varphi'_0 = \varphi_0$ (Ref. 8). The polar angle ϑ'_0 of the ray direction at the point $P(\mathbf{r})$ differs from its initial value ϑ_0 by the refraction angle β, namely, $\vartheta'_0 = \vartheta_0 - \beta$. In the layered AOS model, the ray deflects from its initial direction at discrete trajectory points located at the layer interfaces. Therefore, the refraction angle $\beta = \beta(\mathbf{r})$ along the beam path $\breve{L}$ can be determined, similarly to the Fresnel optical thickness (14), in the sense (13) as a functional:

$$\beta(\mathbf{r}) = \int_{\breve{L}} \beta' \text{d}s = \int_{\breve{L}} \sum_{j=1}^{N} \beta_j\, \delta(r - r_j)\, \delta(\vartheta' - \vartheta'_j)\text{d}s = \sum_{j=1}^{N} \beta_j(r_j, \vartheta'_j). \tag{16}$$

Here the ray refraction angle $\beta_j(r_j, \vartheta'_j)$ at the jth layer interface is determined from the law of light refraction (11) by the following equation:

$$\beta_j(r_j, \vartheta'_j) = \vartheta' - \arcsin(\sin\vartheta'/n_{21})\,. \tag{17}$$

The cosine of the scattering angle $\cos\gamma'_0 = \mathbf{s}\cdot\mathbf{s}'_0$ for the scattering phase function at the point $P(\mathbf{r})$ with the allowance for refraction can be determined from the equation of spherical trigonometry

$$\cos\gamma_0' = \cos\vartheta \cos\vartheta_0' + \sin\vartheta \sin\vartheta_0' \cos(\varphi - \varphi_0'), \quad (18)$$

where ϑ_0' and φ_0' are the polar and azimuth angles of the direction vector $\mathbf{s}_0'$ of the ray incident onto the elementary volume; ϑ and φ are the corresponding angles for the ray $\mathbf{s}$ scattered by the medium.

The singular component due to Fresnel reflection at the layer interfaces is taken into account as an extra intensity in the angular scattering pattern. Let the points $\mathbf{r}_{jv}$ belonging to the interfaces be included in the elementary volume v. The density of the angular distribution of the scattered radiation $\frac{\sigma_F}{4\pi} x_F(\gamma)$ due to the reflected component is obviously the following:

$$\frac{\sigma_F}{4\pi} x_F(\gamma) = \sum_{jv} R_j\, \delta(r - r_j)\, \delta(\mu' - \mu_j')\delta(\varphi' - \varphi_j'),$$
$$\mu = \cos\vartheta, \quad (19)$$

where the summation is performed over the points within the elementary volume. Here σ_F and $x_F(\gamma)$ are, respectively, the index and the phase function of Fresnel scattering. They will be defined in the next section.

The functional

$$\varepsilon^o B_1^o(\mathbf{r}, \mathbf{s}) = \frac{\sigma^o}{4\pi} x^o(\mathbf{r}, \mathbf{s}, \mathbf{s}_0')\pi F \exp(-\tau^o) \quad (20)$$

follows from Eq. (9) with the allowance for Eqs. (14), (16), and (19). In Eq. (20), $\sigma^o = \sigma + \sigma_F$ is the total scattering index, and the scattering phase function $x^o(\mathbf{r}, \mathbf{s}, \mathbf{s}_0')$ is a weighted mean function of the form $x^o = \sum_i q_i x_i$ that is determined through the weighting coefficients q_i of the scattering components ($\sum_i q_i = 1$).

3. Consideration of the interface in two-medium spherical AOS model

The two-medium spherical AOS model follows from the multilayer model considered above as a result of limiting transition to the infinitely large number of system layers. Consider how the equations for τ^o, ϑ_0', and $\frac{\sigma^o}{4\pi} x^o(\gamma)$ determining the source function change at such a limiting transition. Thus, the trajectory of the solar ray transforms from the broken line to the bending one with only one break at the air–water interface. At the levels corresponding to air and water shells, the relative refractive index n_{21} becomes equal to unity, the reflection coefficient is $R = 0$, the ray invariant (12) takes the form $I^o = I/n^2 = \text{const}$, and, consequently, breaks at the ray trajectory disappear. At the air–water interface, to the contrary, the Fresnel reflection coefficient keeps its value $R \neq 0$ and beam refraction occurs, because $n_{21} \neq 1$.

Let us find the optical length τ^o for the point P located, for example, at the level r in the water medium. Toward this end, let us first derive the condition, every point of the actual trajectory of the solar ray should meet. From Eq. (4) in the spherical coordinates (see Refs. 6 and 12), the component of the differential operator including the refraction term can be presented as

$$\left(\frac{\partial I}{\partial s}\right)\bigg|_\vartheta = -\sin\vartheta\left(\frac{1}{r} - \frac{1}{r_c}\right)\frac{\partial I}{\partial \vartheta} = \frac{\partial I}{\partial \vartheta}\frac{\partial \vartheta}{\partial r}\frac{\partial r}{\partial s}. \quad (21)$$

From Eq. (21) we can obtain the differential equation

$$\frac{\partial\vartheta}{\partial r} = \tan\vartheta\left(\frac{1}{r} - \frac{1}{r_c}\right) = -\tan\vartheta \frac{\partial}{\partial r}\ln(nr), \quad (22)$$

including the refraction curvature of the ray

$$\frac{1}{r_c} = -\frac{\partial}{\partial r}\ln n. \quad (23)$$

Here r_c is the length of curvature. Separate the variables in Eq. (22)

$$-\frac{\partial(\sin\vartheta)}{\sin\vartheta} = \frac{\partial(nr)}{nr}, \quad (24)$$

and integrate. As a result, we derive the equation of the invariant

$$\breve{C} = nr\sin\vartheta = n_P r_P \sin\vartheta_P = \text{const}, \quad (25)$$

the actual ray trajectory in the AOS from the atmospheric UB to the fixed point P should obey. Here $n_P r_P \sin\vartheta_P$ is the invariant value at the point P, which can be at an arbitrary level r, above or below the air–water interface. Equation (25) is obviously applicable to the air–water interface as well. Just at $r = r_P$, the law of refraction (11) follows, as a particular case, from Eq. (25). In a homogeneous medium at $n = n_P$, the particular equation for the invariant

$$\bar{C} = r\sin\vartheta = r_P \sin\vartheta_P = \text{const} \quad (26)$$

for the straight ray trajectory follows from Eq. (25). Take the invariant (25) into account in the equation for the element ds of the curvilinear ray trajectory $\breve{L}$ and, as a result of substitutions, we obtain

$$\tau(\mathbf{r}) = \int_{\breve{L}} \varepsilon\, \mathrm{d}s =$$
$$= \int_{\breve{L}} \frac{\varepsilon\, \mathrm{d}r}{\sqrt{1 - \sin^2\vartheta}} = \int_{\breve{L}} \frac{\varepsilon\, nr\, \mathrm{d}r}{\sqrt{(nr)^2 - (n_P r_P \sin\vartheta_P)^2}}. \quad (27)$$

It should be emphasized that the optical thickness $\tau(\mathbf{r})$, as an integral over the curvilinear ray trajectory, can be found by Eq. (27) regardless of where the point P is located: in the air or water medium. In accordance with the requirements of the Fermat principle, following Ref. 12, let us change the curvature of the coordinate lines and pass to the reduced radius $r^{\mathrm{o}} = nr$. Equation (27) in this case is reduced to the form[11]:

$$\tau(\mathbf{r}) = \int_{\breve{L}} \frac{\varepsilon r \, \mathrm{d}r}{\sqrt{r^2 - \breve{C}^2}} \tag{28}$$

characteristic of a straight ray trajectory, that is, in the absence of the air–water interface. It can be concluded from here that the nontraditional method[12] of accounting for refraction through deformation of the system of spherical coordinates in accordance with the refractive index profile can be generalized to the air–water interface. When using such a coordinate system, the trajectories of rays in the AOS become straight lines.

Now consider derivation of the equation for ϑ_0' at the point P. It was already mentioned that this problem can be reduced to determination of the refraction angle $\beta(\mathbf{r})$. It should be noted that the refraction component of the differential operator, after deduction of the geometric curvature of the coordinate line, follows from Eq. (22). Expressing then $\tan\vartheta$ through $\sin\vartheta$ and taking into account the invariant (25), we can find that

$$\beta(\mathbf{r}) = \int_{\breve{L}} \beta' \mathrm{d}s = \int_{\breve{L}} -\frac{\partial}{\partial r}(\ln n) \tan\vartheta \, \mathrm{d}r =$$

$$= \int_{\breve{L}} -\frac{\partial}{\partial r}(\ln n) \frac{\mathrm{d}r}{\sqrt{(nr/\breve{C})^2 - 1}} \, . \tag{29}$$

At the parts of the curvilinear trajectory $\breve{L}$ in the air and water shells, the parameter $\frac{\partial}{\partial r}(\ln n)$ is a regular smooth function of radius and therefore its integration over levels in Eq. (29) leads to gradual increase of ϑ_0' determining the direction of the solar ray. At the air–water interface at the point of ray refraction, the refraction curvature $\frac{\partial}{\partial r}(\ln n)$ in Eq. (29) has a singularity in the form of the delta function. Therefore, integration of Eq. (29) at the interface, representing the property of the invariant (25) mentioned above, leads to the jump change of the ray direction by the angle following from the refraction law. Thus

$$\vartheta_0' = \vartheta_0 + \int_{\breve{L}} \frac{\partial}{\partial r}(\ln n) \frac{\mathrm{d}r}{\sqrt{(nr/\breve{C})^2 - 1}} \, . \tag{30}$$

It is obvious that for a homogenous medium at $n = \mathrm{const}$ the second term in Eq. (30) is zero and $\vartheta_0' = \vartheta_0$.

The angular diagram $\frac{\sigma^{\mathrm{o}}}{4\pi} x^{\mathrm{o}}(\gamma)$ of radiation scattered by the elementary volume can be expressed in the ordinary way, if the point P is in the air or water shell. If the point P lies at the level r_0 of the air–water interface, then the angular radiation density is determined, as known,[6] by the singular brightness coefficient

$$\rho(\mu, \varphi, \mu') = \frac{\pi}{\mu} R(n_{21}, \mu') \, \delta(\mu - \mu')\delta(\varphi' - \varphi_0'),$$

$$\mu' = \mu_r = \cos\vartheta_r. \tag{31}$$

The delta function in Eq. (31) has the meaning of the Fresnel scattering phase function $x_{\mathrm{F}}(\mathbf{r}, \mathbf{s}{\cdot}\mathbf{s}')/4\pi$. It is determined in accordance with the normalization condition

$$\frac{1}{4\pi} \int_{4\pi} x_{\mathrm{F}} \, \mathrm{d}\Omega = \int_{4\pi} \delta(\mu - \mu')\delta(\varphi' - \varphi_0')\mathrm{d}\Omega = 1. \tag{32}$$

The reflection coefficient R of the air–water interface can be expressed through the brightness coefficient $\rho(\mu, \varphi, \mu')$ as a functional

$$R = \frac{1}{\pi} \int_{2\pi} \rho(\mu, \varphi, \mu) \, \mu \mathrm{d}\mu =$$

$$= \int_{4\pi} R(n_{21}, \mu')\delta(\mu - \mu')\delta(\varphi' - \varphi_0')\mathrm{d}\Omega = R(n_{21}, \mu_r, \varphi_r) \tag{33}$$

at the selected values $\mu_r = \mu'$ and $\varphi_r = \varphi_0'$.

The Fresnel scattering index σ_{F} is determined according to Eq. (13)

$$R = \int_L \sigma_{\mathrm{F}} \mathrm{d}s = \int_L R(n_{21}, \mu_r, \varphi_r) \, \delta(r - r_0) \mathrm{d}s =$$

$$= R(n_{21}^0, \mu_r, \varphi_r) \tag{34}$$

as a linear density of the reflection coefficient. Here $n_{21}^0 = n_{21}(r = r_0)$. The path element in Eq. (34) is selected along the direction of the reflected ray. It is obvious that at σ_{F} determined in such a way the total scattering index is $\sigma^{\mathrm{o}} = \sigma + \sigma_{\mathrm{F}}$.

The scattering phase function $x^{\mathrm{o}} = x^{\mathrm{o}}(\mathbf{r}, \mathbf{s}{\cdot}\mathbf{s}_0')$ for the elementary volume can be presented as a sum of the regular and singular components by use of the following integral equality:

$$\frac{1}{4\pi} \iint_{4\pi v} \sigma^{\mathrm{o}} x^{\mathrm{o}} I(\mathbf{s}_0') \, \mathrm{d}\Omega \, \mathrm{d}s = \frac{1}{4\pi} \iint_{4\pi v} \sigma x I(\mathbf{s}_0') \, \mathrm{d}\Omega \, \mathrm{d}s +$$

$$+ \frac{1}{4\pi} \iint_{4\pi v} \sigma_{\mathrm{F}} x_{\mathrm{F}} I(\mathbf{s}_0') \, \mathrm{d}\Omega \, \mathrm{d}s. \tag{35}$$

Integration in Eq. (35) is made over the elements $\mathrm{d}\Omega$ of the solid angle (in the directions of scattered radiation) and the elements $\mathrm{d}s$ of the elementary volume thickness. Upon integration at $I(\mathbf{s}_0') = \mathrm{const}$, from Eq. (35) we have

$$x^{\mathrm{o}} = x^{\mathrm{o}}(\mathbf{r}, \mathbf{s}{\cdot}\mathbf{s}_0') = q x(\mathbf{r}, \mathbf{s}{\cdot}\mathbf{s}_0') + q_{\mathrm{F}} x_{\mathrm{F}}(\mathbf{r}, \mathbf{s}{\cdot}\mathbf{s}_0'). \tag{36}$$

The weighting factors here

$$q = \frac{\tau_{\sigma v}}{\tau_{\sigma v} + R}, \quad q_{\mathrm{F}} = \frac{R}{\tau_{\sigma v} + R} \tag{37}$$

are expressed through the optical thickness of the elementary volume $\tau_{\sigma v}$ due to scattering and through the reflection coefficient R of the air–water interface. For the regular functions, from Eq. (37) we have the well-known particular representation of the weighting factors in the form of the ratio $q_i = \sigma_i / \sum_i \sigma_i$.

It follows from the above-said that within the accepted definitions and designations used of optical characteristics of the elementary volume, the equation for the total source function in a two-medium spherical AOS model, as in the multilayer spherical model, can be reduced to the following form:

$$\varepsilon B^{\text{o}} = \frac{\sigma^{\text{o}}}{4\pi} \int_{4\pi} x^{\text{o}}(\mathbf{r},\ \mathbf{s}\cdot\mathbf{s}')\ I_{\text{d}}^{\text{o}}(\mathbf{r})\text{d}\Omega + $$

$$+ \frac{\sigma^{\text{o}}}{4\pi} x^{\text{o}}(\mathbf{r},\ \mathbf{s}\cdot\mathbf{s}_0')\pi F \exp(-\tau^{\text{o}}) \quad (38)$$

corresponding to the model of the medium at n = const (Ref. 11), when there are no interface between media and the ray trajectory is straight.

Conclusion

The spherical model of the atmosphere–ocean system is reduced to the mathematically equivalent model, in which reflection, transmission, and refraction of light at the interface between the media are expressed through optical parameters of the elementary volume in the radiative transfer equation. These parameters are derived using the theory of generalized functions and Fermat principle. The parameters determining the source function for the radiative transfer equation in such a model are represented in the form corresponding to the homogenous medium without interface, where ray trajectories are straight lines.

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