OPTIMIZATION OF THE ADAPTIVE OPTICAL SYSTEMS WITH MULTICHANNEL PHASE MODULATION AND ANALYSIS OF THEIR EFFICIENCY

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An optimal algorithm is synthesized to control the adaptive optical systems, in which the wavefront phase distortions are estimated based on multichannel phase modulation of the received signal. The algorithm takes into account both the atmospheric background noise and the photodetection noise. The efficiency of optimal and suboptimal algorithms is compared under conditions of the turbulent atmosphere.

It is well known that the adaptive optical systems (AOS's) have large possibilities of compensating for distortions induced by the turbulent atmosphere in the optical devices implemented for laser communication, detection and ranging, and navigation.[1] Along with the AOS's operating on the basis of the phase conjugation (PC) principle, considerable attention is devoted to the systems with multichannel phase modulation[2] (MPM) in which the data on the wavefront phase distortions are derived from the received optical signal that was phase—modulated at a stage of emission or reception of the optical signal. From the viewpoint of statistical optimization of information systems and of the theory of statistical solutions,[3,4] the AOS and the MPM devices can be treated as the systems with partially specified structure, since the idea of phase modulation is fundamental for them. When implementing the algorithm for measuring the distortions in the AOS's, it is possible to use one of the two ways: either to use a standard photodetector or to synthesize this unit of the system as well. Both variants of optimization are considered in the present paper.

1. MODEL OF A RECEIVED SIGNAL

We shall now investigate the pulsed multistep regime of operation of the AOS. Let us consider the arbitrary mth ($m = 1, 2, \ldots$) sensing step. The field reflected from a point target with coordinates (z_m, θ_m) can be written in the form

$$(Y_m(\mathbf{r}, t) = \mathrm{Re}\, u(t - t_m)X_m(\mathbf{r}, t)V_m(\mathbf{r}) + n(\mathbf{r}, t)\ ,$$
$$\mathbf{r} \in X_a,\ (m-1)T_0 \le t \le mT_0\ , \tag{1}$$

where Z_m is the distance to the target, θ_m are the angular coordinates of the target, Ω_a is the area of the receiving aperture, T_0 is the sensing period, $u(t)$ is the function describing the sensing pulse shape, $t_m = (m - 1)\,T_0 + 2z_m/c$ is the moment of signal reception. The functions $X_m(\boldsymbol{r}, t) = \exp\{-i\omega t + ik\boldsymbol{r}^2/2z_m - ik\theta_m\boldsymbol{r}\}$ and $V_m(\boldsymbol{r}) = E\exp\{i\varphi_m(\boldsymbol{r})\}$ describe the run—on of the phase accumulated along the propagation path from the target to the receiving aperture through a homogeneous medium and the random wavefront phase distortions caused by the turbulent atmosphere, in addition, in the phase approximation E_m is the real deterministic amplitude of the field while $\varphi(\boldsymbol{r}) = \varphi(0, \boldsymbol{r};\ z_m, \theta_m;\ t_m)$ is the random run—on of the phase, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, $\omega = k/c$ is the carrier frequency of the desired signal, and $n(\boldsymbol{r}, t)$ is the background noise, which is usually a Gaussian process with a uniform spectral density N_0.

In order to measure the wavefront phase distortions of the signal at the receiving aperture Ω_a collocated with the transmitting aperture, the signal was phase—modulated with the frequencies ω_j ($j = 1, \ldots, N$) within the corresponding areas (subapertures) $\Omega_j \subset \Omega_a$. The received field was then focused onto the photodetector (PD). For the "point" PD, whose sensitivity maximum was located at the point with the coordinates $\xi = -\,d_m\theta_m$, where $d_m = z_mF/(z_m - F)$ (F is the focal length of the receiving optics), the condition $n = S_aS/\lambda^2d_m^2 < 1$ is satisfied, where S_a is the area of the receiving aperture and S is the area of the sensitive surface of the PD. At the same time, in accordance with Eq. (1), the field at the PD takes the form

$$Y_m(\xi, t) = \mathrm{Re}\,E_mX_m(\xi, t)\sum_{j=1}^{N}\frac{1}{\Delta}\int_{\Omega_j}\exp\{i[a\sin\omega_j t + \varphi_m(\mathbf{r})]\}\mathrm{d}^2r +$$
$$+\ \mathrm{Re}\,\frac{1}{i\lambda d_m}\sum_{j=1}^{N}\int_{X_j} n(\mathbf{r}, t)\exp\Big\{ia\sin\omega_j t +$$
$$+\ i\kappa d_m - \frac{i\kappa}{2F}r^2 + \frac{i\kappa}{2d_m}|\mathbf{r} - \xi|^2\Big\}\mathrm{d}^2r\ , \tag{2}$$

where $X_m(\xi, t) = \dfrac{\Delta}{i\lambda d_m}u(t - t_m)\exp\{-i\omega t + ikd_m + ik\xi^2/2d_m\}$, $\Delta = S_a/N$ is the area of the subaperture Ω_j($j = 1, \ldots, N$), the function $a\sin\omega_j t$ describes phase modulation within the jth subaperture, with amplitude a and frequency ω_j.

2. OPTIMAL POSTDETECTOR SIGNAL PROCESSING

The PD output signal, from which the information on phase distortions is derived, has the form

$$i_m(t) = \sum_{k=1}^{K_m}\delta(t - t_{km}), \tag{3}$$

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where t_{km} and K_m are the random moments of photoelectron emission and the number of electrons recorded during the mth sensing step. In most of the cases,[5] when the experimental phase $\varphi_m(r)$ was fixed, the PD output signal obeyed the Poisson distribution.

The statistical approach allows one to implement the processing algorithm in the form of a tracking system of automatic control. Indeed, we can write the formula for logarithm of the functional of the likelihood ratio during the mth sensing step in the form

$$L_m[t, \nu_{sm}(t)] = \int\limits_{(m-1)T_0}^{t} l\,[\tau, \nu_{sm}(\tau)]\,\mathrm{d}\tau,$$

$$(m-1)T_0 \le t \le mT_0. \tag{4}$$

where $l[t, \nu_{sm}(t)] = i_m(t)\ln[1 + \nu_{sm}(t)/\nu_n] - \nu_{sm}(t)$, $\nu_{sm}(t)$ and ν_n are the signal and noise components of the intensity of the Poisson flux of electrons (3). Taking into account Eq. (2) for the field incident on the PD, we have

$$\nu_{sm}(t) = \mu_m(t)\left|\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\Delta}\mathrm{B}_j\right|^2, \tag{5}$$

$$\mathrm{B}_j = \int\limits_{\Omega_j}\exp\{i[a\sin\omega_j t + \varepsilon_{jm}(t)]\}\mathrm{d}^2r,$$

where $\mu_m(t) = \eta|E_m|^2 S_a n|u(t - t_m)|^2/2\hbar\omega$ is the signal component of the intensity of flux (3) when the radiation propagates through a homogeneous medium, η is the quantum efficiency of the PD, $\hbar\omega$ is the quantum of energy at the carrier frequency ω, $\varepsilon_{jm}(\boldsymbol{r},t) = \varphi_m(\boldsymbol{r}) - {}_{jm}(t)$ is the error signal representing the difference between the phase distribution $\varphi_m(\boldsymbol{r})$ measured within the subaperture $\boldsymbol{r} \in \Omega_j$ and the control signal ${}_{jm}(t)$ (the average established position of the jth subaperture) during the mth sensing step. The intensity of noise component of flux is apparently equal to

$$\nu_n = \eta N_o n/\hbar\omega + \nu_d, \tag{6}$$

where ν_d is the intensity of the flux of the PD dark electrons.

When the error signals in the channels of the tracking AOS are small for the errors averaged over the subapertures $\varepsilon_{jm}(t) = \frac{1}{\Delta}\int\limits_{\Omega_j}\varepsilon_{jm}(\mathbf{r}, t)\mathrm{d}^2r$, the following system of differential equations, which follows from Eqs. (4) and (5) is valid:

$$\frac{\mathrm{d}\varepsilon_{jm}(t)}{\mathrm{d}t} = \sum_{k=1}^{N} C_{jkm}(t)B_{km}(t)), \tag{7}$$

$$j = 1, \ldots, N, \quad (m-1)T_0 \le t \le mT_0$$

$$B_{km}(t) = \frac{\partial}{\partial\varepsilon_{km}} l[t, \nu_{sm}(t)] =$$

$$= -\frac{2\mu_m(t)}{N^2}\left[i_m(t)/(\nu_{sm}(t) + \nu_n) - 1\right] \times$$

$$\times \sum_{p=1}^{N}\left[a(\sin\omega_k t - \sin\omega_p t) + \varepsilon_{km}(t) - \varepsilon_{pm}(t)\right]; \tag{8}$$

where $C_{jkm}(t)$ are the coefficients of the matrix inverse the matrix

$$A_{jkm}(t) = -\frac{\partial^2}{\partial\varepsilon_{jm}\partial\varepsilon_{km}} L_m[t, \nu_{sm}(t)] = A_{jkm}((m-1)T_0) +$$

$$+\frac{4}{N^4}\int\limits_{(m-1)T_0}^{t}\mu_m^2(\tau)i_m(\tau)/(\nu_{sm}(\tau) + \nu_n)^2 \times$$

$$\times \sum_{p,\,q=1}^{N}\left[a(\sin\omega_j\tau - \sin\omega_p\tau) + \varepsilon_{jm}(\tau) - \varepsilon_{pm}(\tau)\right] \times$$

$$\times \left[a(\sin\omega_k\tau - \sin\omega_q\tau) + \varepsilon_{km}(\tau) - \varepsilon_{qm}(\tau)\right]\mathrm{d}\tau +$$

$$+\frac{2}{N^2}\int\limits_{(m-1)T_0}^{t}\mu_m(\tau)\left[i_m(\tau)/(\nu_{sm}(\tau) + \nu_n) - 1\right]\mathrm{d}\tau(N\delta_{jk} - 1). \tag{9}$$

Equations (7)–(9) describe the optimal algorithm of control. However, this algorithm requires the inversion of matrix $A_{jkm}(t)$ $(j, k = 1, \ldots, N)$ during each sensing step. To overcome this difficulty, Eq. (9) may be averaged preliminariely over the ensemble of realization of the shot photodetection noise and the matrix coefficients $A_{jk}(t)$ may be taken not at the point corresponding to the estimate of maximum likelihood ${}_{jm}(t)$ $(j = 1, \ldots, N)$, but rather at the point where these parameters reach their actual values, namely, ${}_{jm}(t) = \frac{1}{\Delta}\int\limits_{\Omega_j}\varphi_m(\mathbf{r})\mathrm{d}^2r$. Formula (9) then acquires the form of the Fisher information matrix [3]

$$\overline{A_{jkm}(t)} = \overline{A_{jkm}((m-1)T_0)} +$$

$$+\frac{2a^2}{N^3}\int\limits_{(m-1)T_0}^{t}\mu_m(\tau)q_m(\tau)/(1 + q_m(\tau))\mathrm{d}\tau\,(N\delta_{ik} - 1),$$

$$j, k = 1, \ldots, N, \quad (m-1)T_0 \le t \le mT_0, \tag{10}$$

where $q_m(t) = \mu_m(t)/\nu_n$ is the signal–to–noise ratio of radiation propagating through a homogeneous medium during the mth sensing step. A horizontal bar denotes averaging over the ensemble of the shot photodetection noise of the receiver.

However, the Fisher matrix given by Eq. (10) is degenerate. The rank of the matrix is a unity smaller than the matrix dimensions, so that the system of N equations (7) includes the equation that can be expressed in the form of linear combination of the rest of the linearly independent equations. These linearly independent equations determine $N - 1$ unknown quantities in the form of the linear functions of a single unknown quantity that may be chosen arbitrarily. This situation reflects the fact that phase measurements actually yield only phase differences rather than the absolute values of the phases. Let us take the phase within the Nth subaperture as the reference phase for all of the rest of the phases. If we set $\varepsilon_{Nm}(t) = 0$ and neglect the Nth differential equation of system (7) we obtain the control algorithm in the following form:

$$\frac{d\varepsilon_{jm}(t)}{dt} = \sum_{k=1}^{N-1} C_{jkm}(t) B_{km}(t) ,$$

$$j = 1, \ldots, N-1, \ (m-1)T_0 \le t \le mT_0, \tag{11}$$

The explicit analytical form for the matrix of coefficients $C_{jkm}(t)$ can be found in two limiting cases. Indeed, if the estimated wavefront phase distortions remain unchanged over the period of observation, that is $\varphi_m(\boldsymbol{r}) = \varphi(\boldsymbol{r})$ for any m, then, obviously

$$C_{jkm}(t) \simeq k_{0m}(t)(1 + \delta_{jk}), \tag{12}$$

where

$$k_{0m}(t) = N^2 \Big/ \left(2a^2 \left[\sum_{l=1}^{m-1} \int\limits_{(l-1)T_0}^{lT_0} \frac{\mu_l(\tau) q_l(\tau)}{(1 + q_l(\tau))} d\tau + \right.\right.$$

$$\left.\left. + \int\limits_{(m-1)T_0}^{t} \frac{\mu_m(\tau) q_m(\tau)}{(1 + q_m(\tau))} d\tau \right] \right).$$

The opposite situation happens more often in practice, in particular, when we observe rapidly moving targets, for which the phase distortions during the successive sensing steps are uncorrelated. In this case the information on the wavefront phase (WFP) distortions is not available at the start of the mth successive step of reception of the sensing signal, that is, $\overline{A_{jkm}((m-1)T_0)} = 0$. The matrix of coefficients $C_{jkm}(t)$ can than be determined once again from Eq. (12) provided that

$$k_{0m}(t) = N^2 \Big/ \left(2a^2 \int\limits_{(m-1)T_0}^{t} \frac{\mu_m(\tau) q_m(\tau)}{(1 + q_m(\tau))} d\tau \right). \tag{13}$$

By substituting Eq. (13) into Eq. (11), we obtain the control algorithm in a simpler form

$$\frac{d\varepsilon_{jm}(t)}{dt} = \mu_m(\tau) N \Big/ \left(a^2 \int\limits_{(m-1)T_0}^{t} \frac{\mu_m(\tau) q_m(\tau)}{(1 + q_m(\tau))} d\tau \right) \times$$

$$\times \frac{1 - i_m(t)}{\nu_{sm}(t) + \nu_n} \left[a(\sin\omega_j t - \sin\omega_N t) + \varepsilon_{im}(t) \right]$$

$$j = 1, \ldots, N-1, \ (m-1)T_0 \le t \le mT_0 . \tag{14}$$

It follows from Eq. (14) that the N–channel control system has transformed into a system with $N-1$ independent channels plus a reference channel. Therefore, the algorithm described by Eqs. (14) can be implemented using the simplest analog devices.

The aperture averaged variance of the residual error of compensation for the WFP distortions taken at the moment of emission of the signal during the current sensing step may be conveniently chosen as the accuracy characteristic of the AOS:

$$D_m = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\Delta} \int\limits_{\Omega_j} < \overline{[Q - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\Delta} Z]^2} > d^2 r,$$

$$Q = \delta_j(\mathbf{r}, mT_0), \ Z = \int\limits_{\Omega_k} \delta_k(\mathbf{r}_1, mT_0) d^2 r_1 , \tag{15}$$

where $\delta_j(\boldsymbol{r}, mT_0) = \varphi(\boldsymbol{r}, mT_0) - {}_{jm}(mT_0)$ is the error of compensation within the jth subaperture $\boldsymbol{r} \in \Omega_j$ at the moment of successive sensing mT_0, and the angular brackets denote an averaging over the ensemble of realizations of the WFP of the derived signal. It is obvious that the error $\delta_j(\boldsymbol{r}, mT_0)$ can be represented in the form

$$\delta_j(\mathbf{r}, mT_0) = \varphi(\mathbf{r}, mT_0) - {}_{jm} + \varepsilon_{jm}(mT_0) ,$$

where $\varphi_{jm} = \frac{1}{\Delta} \int\limits_{\Omega_j} \varphi_m(\mathbf{r}) d^2 r$ is the phase of the signal averaged over the jth subaperture. As follows from the form of control algorithm (14), the mean control error is equal to zero: $\overline{\varepsilon_{jm}(t)} = 0$, $j = 1, \ldots, N$. As for the correlation matrix of errors $< \overline{\varepsilon_{jm}(mT_0)\, \varepsilon_{km}(mT_0)} >$, $j = 1, \ldots, N-1$, $k = 1, \ldots, N-1$, it can be determined easily on the basis of the Kramer–Rao theorem[3]

$$< \overline{\varepsilon_{jm}(mT_0)\, \varepsilon_{km}(mT_0)} > = C_{jkm}(mT_0) ,$$

$$j = 1, \ldots, N-1, \ k = 1, \ldots, N-1 \tag{16}$$

For this reason when approximating the structure function of WFP by the function by $D_\varphi(\boldsymbol{r}) = 2(|\mathbf{r}|/\rho_c)^{5/3}$, where ρ_c is the coherence radius of the desired received signal, accepting the hypothesis of frozen atmospheric turbulence, and assuming that the time delay of the sensing signal Δt_m (relative to the moment of signal reception) is smaller than the coherence time of the received signal τ_c, we shall have

$$D_m \simeq (N-1) k_{0m}(mT_0)/N + \Gamma(11/6)(N_B/N)^{5/6} +$$

$$\frac{25}{18} \Gamma(5/6)(\Delta t_m/\tau_c)^2 (N^{1/6} - 1)/N_B^{1/6}. \tag{17}$$

where $\Gamma(x) = \int\limits_0^\infty \varepsilon^{-t} t^{x-1}\, dt$ – is the gamma function, $N_B = S_a \pi \rho_c^2$ is the number of coherent "spots" of the signal field within the receiving aperture, $\Delta t_m = mT_0 - t_m = T_0 - 2z_m/c$; $\tau_c = \rho_c/|\mathbf{v}_\perp|$, and $\mathbf{v}_\perp$ is the velocity of the transverse (relative to the direction of beam propagation) movement of the refractive index inhomogeneities in the turbulent atmosphere.

As follows from Eq. (17), the variance of compensation errors includes three components: noise, "dynamic", and extrapolation errors. The latter results from the extrapolation of the phase front measured at the moment t_m of signal reception to the moment of sensing mT_0. When the wavefront phase distortions during the successive sensing steps are uncorrelated and sensing pulse shape is close to rectangular, the noise component has the form

$$D_{nm} \simeq (1 + q_m)N(N - 1)/\ 2a^2 q_m \overline{K}_{sm} n, \tag{18}$$

where

$$\bar{K}_{sm} = \int\limits_{(m-1)T_0}^{mT_0} \mu_m(t)\mathrm{d}t = \eta\,|E_m|^2 S_a T_{eff}/2\bar{\eta}\omega$$

is the average number of photoelectrons recorded by the detector during the mth step, $T_{eff} = \int\limits_{(m-1)T_0}^{mT_0} |u(t - t_m)|^2 \mathrm{d}t$ is the effective duration of the received pulse, $q_m = n\overline{K}_{sm}/\overline{n}_n$ is the signal–to–noise ratio, and $\overline{n}_n = \nu_n T_{eff}$ is the average number of noise electrons recorded during the period of reception of the signal pulse. Thus, the noise component of error is proportional to the squared number of channels in the receiving device N (when N is sufficiently large), since after detection the signals in the channels are added incoherently. It is natural that the error increases as the signal–to–noise ratio q_m decreases, while for fixed q_m it increases as the signal "energetics" $\overline{K}_{sm}$ decreases (when the effect of the photodetection noise increases). Reducing the degree of signal modulation a makes distinguishing between the modulated signals in the detector more difficult and, consequently, increases the error.

The "dynamic" component of error $\Gamma(11/6)(N_s/N)^{5/6}$ results from the finite size of individual subapertures in comparison with the size of coherent "spot" of the received field and is determined by the number of such spots within each subaperture. The extrapolation error $\frac{25}{18}\Gamma(5/6)(\Delta t_m \tau_c)^2 (N^{1/6} - 1)/N_s^{1/6}$ depends but weakly on the "rate" of the WFP change within the aperture and is mainly determined by the rate at which the realizations of the phase front change within the characteristic size of the coherent "spot" of the signal. It is natural that the noise and the extrapolated components of error increase for larger number of channels N (when the aperture size is fixed) while the "dynamic" component simultaneously decreases. It is obvious that there exists the optimal number of channels of the AOS, which can be determined by minimizing Eq. (17). Formula (17) derived for the variance of error of compensation makes it also possible to calculate the Strehl factor St_m during the arbitrary sensing step. Indeed, in accordance with Ref. 6, we have

$$\mathrm{St}_m \simeq \exp(-D_m) + (1 - \exp(-D_m/2)^2/[1 +$$

$$+ N_p(1 - \exp(-D_m/2))]\ . \tag{19}$$

It is interesting to compare the efficiency of the synthesized algorithm with that of the scheme with one synchronous detector and corresponding filter in each channel. This scheme is widely used in practice (see, for example, Ref. 7). The signal from the filter output is used directly to control and to move the controllable subaperture in each channel. It is evident that different algorithms may lead only to the change in the noise component of error. We will now compare these components for a simple case when the WFP of the signal remains unchanged over the period of observation $\varphi_m(\boldsymbol{r}) = \varphi(\boldsymbol{r})$, and the sounding pulses are short enough to control the subapertures during the time intervals between the signal pulses. Then the control signal for the jth subaperture is written in the form

$$_{jm} = {}_{jm-1} + \gamma_m \cdot \int\limits_{(m-1)T_0}^{mT_0} i_m(t)\sin\omega_j t\mathrm{d}t,\ j = 1, \ldots, N, \tag{20}$$

where γ_m is the coefficient, which determines the steepness of the discrimination characteristic of the measuring device,[8] $i_m(t)$ is the flux of photoelectrons from the output of the PD determined by Eq. (3).

When we control the system according to Eq. (20), the compensation error is equal to

$$\varepsilon_{jm} = \varphi_j - \hat{\varphi}_{jm} = \varepsilon_{jm-1} - P_m[\varepsilon_p,\ p = 1, \ldots, N] + \xi_{jm}\ , \tag{21}$$

where $P_m[\varepsilon_p,\ p = 1, \ldots, N] = \gamma_m \cdot \int\limits_{(m-1)T_0}^{mT_0} \overline{i_m(t)}\ \sin\omega_j t\mathrm{d}t$ and

$$\xi_{jm} = -\gamma_m \cdot \int\limits_{(m-1)T_0}^{mT_0} \left[i_m(t) - \overline{i_m(t)}\right]\sin\omega_j t\mathrm{d}t$$

are the slowly and last varying components of the error of the tracking measuring device. An averaging was carried out over the ensemble of photodetection noise of the given sensing step. According to Eq. (5), the slowly varying component may be written in the form

$$P_m[\varepsilon_p,\ p = 1, \ldots, N] = \varepsilon_{jm-1} - \frac{1}{N}\sum_{p=1}^{N}\varepsilon_{pm-1}\ . \tag{22}$$

The steepness of the discrimination characteristic is equal to $\gamma_m = -N/an\overline{K}_{sm}$, while the correlation matrix of the fast varying components of error signal of controlling N subapertures has the simple form

$$k_{jkm} = \overset{\circ\circ}{\xi}_{jm}\overset{\circ\circ}{\xi}_{km} = (1 + q_m)N^2\delta_{jk}/2\,a^2 n q_m \overline{K}_{sm}\ ,$$

$$j = 1, \ldots, N,\ k = 1, \ldots, N\ . \tag{23}$$

Taking Eq. (22) into account, we have

$$\varepsilon_{jm} = \frac{1}{N}\sum_{p=1}^{N}\varepsilon_{pm-1} + \xi_{jm},\ j = 1, \ldots, N,$$

instead of Eq. (21), therefore, the noise component of the variance of compensation error during the mth sensing step exactly coincides with the component of the synthesized algorithm given by Eq. (18).

The latter result requires a physical explanation. There are $N - 1$ controllable subapertures in the synthesized algorithm (one subaperture is immovable and the phase measured within it is used as the reference phase), and N such subapertures in the scheme under study, therefore, the variance of the noise error of the synthesized algorithm must be smaller. However, in the case examined the error of the synthesized algorithm is formed in the following way:

$$\varepsilon_{jm} - \varepsilon_{Nm} = \frac{1}{N}\sum_{p=1}^{N} \varepsilon_{pm-1} + \varepsilon_{jm} -$$

$$-\left(\frac{1}{N}\sum_{p=1}^{N} \varepsilon_{pm-1} + \xi_{Nm}\right) = \xi_{jm} - \xi_{Nm},\ j = 1, \ldots, N-1 .$$

The correlation matrix of this error is equal to

$$k_{jkm} = \overline{(\varepsilon_{jm} - \varepsilon_{Nm})(\varepsilon_{km} - \varepsilon_{Nm})} = k_{0m}(1 + \delta_{jk}) , \qquad (24)$$

where $k_{0m} = (1 + q_m)N^2/(2a^2 q_m n \overline{K}_{sm})$, from which it follows that the diagonal coefficients of the error matrix of the synthesized algorithm are twice as many as the corresponding coefficients of matrix (23). This can be explained by summation of the phase measurement errors within the given and the reference subapertures. In contrast to Eq. (23), the nondiagonal elements in Eq. (24) are nonzero, instead they are determined by the variance of noise error within the reference aperture.

3. OPTIMAL ALGORITHM FOR THE MULTICHANNEL PHASE MODULATION OF RECEIVED SIGNAL

In contrast to the foregoing section, we will not prescribe here the operation of photodetection of the received signal, but will make use of the theory of statistical solutions to obtain the optimal algorithm for processing the field with the MPM.

According to formula (2) for the field in the image plane of the target during the mth step, logarithm of the likelihood functional is written in the form

$$L_m\Big[Y_m(\xi, t)/V_{jm},\ j = 1, \ldots , N\Big] \sim \int\limits_{(m-1)T_0}^{mT_0} \mathrm{d}t \int\limits_{\Omega} \mathrm{d}^2\xi \times$$

$$\times\left[Y_m(\xi, t) - \mathrm{Re}X_m(\xi, t)\sum_{j=1}^{N} V_{jm}\exp(ia\sin\omega_j t)\right]^2 , \qquad (25)$$

where Ω is the observation area in the image plane and $V_{jm} = \dfrac{E_m}{\Delta}\displaystyle\int_{\Omega_j}\exp(i\varphi_m(\mathbf{r}))\mathrm{d}^2 r$. The estimate of the WF of the signal V_{jm} maximizing functional (25) and corresponding to the jth subaperture, is equal to

$$\hat{V}_{jm} \sim a_{jm} - \frac{\alpha}{1 + \alpha(N-2)}\sum_{p\neq j}^{N} a_{pm},\ N \geq 2 , \qquad (26)$$

where

$$a_{jm} = \frac{1}{T_{eff}}\int\limits_{(m-1)T_0}^{mT_0}\mathrm{d}t\int\limits_{\Omega}\frac{1}{s}\,\mathrm{d}^2\xi\ Y_m(\xi, t)x^*_m(\xi, t)\exp(-i\, a\sin\omega_j t);$$

$$\alpha = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{a}{2}\right)^{2n} C_n^{2n} .$$

In its turn, Eq. (26) gives the optimal spatial distribution of the complex amplitude of the sensing signal over the transmitting aperture $\Omega_\mathrm{a} = \bigcup_{j=1}^{N}\Omega_\mathrm{j}$

$$A_m(\mathbf{r}) \sim \hat{V}^*_{jm},\ \ \mathbf{r}\in\Omega_j,\ \ j = 1, \ldots, N. \qquad (27)$$

It follows from Eqs. (26) and (27) that the optimal distribution of the sensing signal field can be formed in two steps. During the first step, the distorted WF "encoded" in the received signal is measured. In order to measure the WF within each of N subapertures, an N–channel heterodyne receiver must be employed with output signals from the channels being proportional to the quantities a_{jm}, $j = 1, \ldots, N$. In accordance with the expressions for a_{jm}, the received field is spatially and temporally filtered in each channel of the receiver. The reference radiation of heterodyne in the jth channel is given by the function $x_{jm}(\xi, t)\exp(-ia\sin\omega_j t)$. During the second step, a "weighted" summation of output signals from the receiver is performed following Eq. (26). After conversion of the signal into the optical frequency, phase conjugation in each channel, and amplification, the sensing signal is emitted into the atmosphere.

The physical meaning of the procedure of "weighted" summation described by Eq. (26) is of interest. Since the parameter α, which enters in Eq. (26), is a normalized correlation coefficient of noise in different channels, the second term in Eq. (26) gives an estimate of noise in the jth channel. Therefore, procedure (26) consists in compensating for the correlated measurement noise. In particular, if the noise in different channels is completely correlated ($a \to 0$ and $\alpha \to 1$), the "weight" for summation ascribed to the measurements in each channel approaches unity: $\hat{V}_{jm} \sim a_{jm} - \dfrac{1}{N-1}\displaystyle\sum_{p\neq j}^{N} a_{pm}$. However, if the noise is uncorrelated ($a \sim \pi$ and $\alpha \to 0$), the estimate of WF only in the jth channel is used to form the radiation of this channel:

$$\hat{V}_{jm} \sim a_{jm} . \qquad (28)$$

Similar to the foregoing section, the variance of the WF estimation error averaged over the aperture at the moment of signal emission

$$\delta V_m = \sum_{j=1}^{N}\int\limits_{\Omega_j}\frac{\overline{<|V(\mathbf{r}, mT_0) - \hat{V}_{jm}|^2>\ \mathrm{d}^2 r}}{\int\limits_{\Omega_\mathrm{a}}<|V(\mathbf{r}, mT_0)|^2>\ \mathrm{d}^2 r} =$$

$$= 1 - R_N(0) + 2[R_N(0) - R_N(\Delta t_m)] + N^2\beta/Q_m . \qquad (29)$$

and the Strehl factor

$$\mathrm{St}_m = \frac{<\left|\frac{1}{N}\sum\limits_{j=1}^{N}\hat{V}^*_{jm}\frac{1}{\Delta}\int\limits_{\Omega_j}\exp(i\varphi_m(\mathbf{r},\, mT_0))\mathrm{d}^2r\right|^2>}{\frac{1}{N}\sum\limits_{j=1}^{N}<\left|\overline{\hat{V}_{jm}}\right|^2>}$$

$$\simeq \frac{\left(R_N^2(\Delta t_m) + \frac{N\beta}{Q_m}\right)\left[A - \frac{\alpha}{1+\alpha(N-2)}\left(NR_1(0) - A\right)\right]}{\left(R_N(0) + N^2\beta/Q_m\right)}. \quad (30)$$

$A = R_N(0)$

may provide the simplest criteria for the performance of algorithm. The following notations are used in Eqs. (29) and (30):

$$R_N(\Delta t_m) = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{\Delta^2}\int\limits_{\Omega_j}\mathrm{d}^2r_1\int\limits_{\Omega_j}\mathrm{d}^2r_2 \times$$

$$\times \exp\left(-\frac{1}{2}D_\varphi(\mathbf{r}_1 - \mathbf{r}_2 + \overline{\nu}_\perp\cdot(\Delta t_m))\right);$$

$$\beta = \frac{1}{1-\alpha}\left[1 - \frac{\alpha}{1+\alpha(N-1)}\right]; \quad Q_m = \frac{\bar{K}_{sm}n}{\left(1 + \frac{\nu_T}{\nu_h} + \frac{\eta N_0}{\hbar\omega}\right)}$$

is the signal–to–noise ratio in the heterodyne receiver, and ν_h is the intensity of the flux of photoelectrons resulted from the heterodyne emission. The inequality $\nu_T/\nu_h + \eta N_0/\hbar\omega \ll 1$ holds usually for the heterodyne receiver, therefore $Q_m \simeq \overline{K}_{sm}n$.

From Eq. (29) it follows that the error of the algorithm consists, as previously, of three components, namely, the "dynamic" error $1 - R_N(0)$, the extrapolation error $2[R_N(0) - R_N(\Delta t_m)]$, and the noise error, $N^2\beta/Q_m$. It is not difficult to verify that when the errors are small (then $R_N(\Delta t_m) \simeq \simeq 1 - \Gamma(11/6)(N_n/N)^{5/6} - (25/36)\,\Gamma(5/6)(\Delta t_m/\tau_c)^2$) the "dynamic" and the extrapolation components agree well with the corresponding characteristics of the algorithm of direct signal detection considered in the foregoing section. As for the noise component, it depends strongly on the value of the parameter α. In particular, when the degree of phase modulation is low, $a \to 0$, that is, $\alpha \to 1 - a$, and the noise component is equal to

$$\delta V_{\mathrm{nm}} \simeq N(N-1)/an\bar{K}_{sm}\,. \quad (31)$$

The comparison of Eqs. (18) and (31) shows that the error variance is smaller by a factor of $(1 + q_m)/2aq_m$ in the second case. Thus, for instance, when $q_m = S$ and $a = 0.1$, $D_{nm}/\delta V_{nm} = 6$. When the degree of phase modulation in the optimal algorithm is high $a_{opt} \sim \pi$, that is, $\alpha \to 0$, we have

$$\delta V_{\mathrm{nm}} \simeq N^2/\bar{K}_{sm}n. \quad (32)$$

The noise error of the optimal algorithm differs here from the error of algorithm (14) already by a factor of $(1 + q_m)/2a^2q_m$, that is, $D_{nm}/\delta V_{nm} = 60$ for the same values of the parameters. In both cases if the errors are small ($\delta V_{nm} \ll 1$), the Strehl factor is equal to $St_m \simeq 1 - \delta V_{\mathrm{nm}}$. In the opposite case when the errors are large ($N^2\beta/Q_m \gg R_N(0)$), $St_m \simeq R_N(0)/(N-1)$, $N \gg 2$.

The comparative efficiency of the optimal algorithm and algorithm (14) is illustrated by Table I, in which the values of the Strehl factors are listed as functions of the "energetics" of the received signal $Q_m = \overline{K}_{sm}n$.

TABLE I.

Q_m	10^5	$3\cdot10^4$	10^4
$\mathrm{St}_{\mathrm{opt}}$	0,78	0,76	0,70
St_m	0,51	0,23	0,16

The following values of the parameters were used in calculations: $N = 30$, $N_n = 5$, $a_{opt} = 3\pi/4$ (see Ref. 9), $a = 0.1$, and $\Delta t_m/\tau_c = 0$. From the given results it follows that the efficiency of the optimal algorithm is significantly higher. It is most vivid for low energetics ($Q_m = 10^4$) when the compensation effect is absent for algorithm (14), while the optimal algorithm still keeps reasonably high performance characteristic.

However, in practice the implementation of the optimal algorithm with the "weighted" summation of signals (26) may be rather difficult. It is therefore interesting to study the efficiency of simpler algorithm for estimation of the wavefront phase distortions, for example, described by Eq. (28). It is easy to show that the variance of the WFP estimation error for this algorithm (corresponding to Eq. (29) for the optimal case) also consists of three components, namely, the "dynamic" error $1 - (1 - \alpha^2)R_N(0) + \alpha^2N(N-2)R_I(0)$, the extrapolation error $2\{(1-\alpha)[R_N(0) - R_N(\Delta t_m)] + \alpha N[R_1(0) - R_1(\Delta t_m)]\}$, and the noise error N^2/Q_m. However, even when the conditions $N_n/N \to 0$, $N^2/Q_m \to 0$, and $\Delta t_m/\tau_c \to 0$, sufficient to provide high efficiency of the optimal algorithm, are satisfied, the "dynamic" and extrapolation components are nonzero: $\delta V_m \simeq a^2[1 + N(N-2)R_I(0)]$. It follows from this formula that for low degree of modulation ($\alpha \to 1$) the error is unacceptably large.

The Strehl factor for this algorithm is equal to

$$\mathrm{St}_m \simeq \Big\{(1-\alpha)\left[R_N^2(\Delta t_m) + NR_N(0)/Q_m\right] +$$

$$+ \alpha N\left[\alpha NR_1^2(\Delta t_m) + (1-\alpha)R_N(\Delta t_m)R_1(\Delta t_m) +\right.$$

$$\left.+ NR_1(0)/Q_m\right]\Big\}\Big/\Big\{(1-\alpha)^2R_N(0) +$$

$$+ \alpha NR_1(0)[2 + (N-2)\alpha] + N^2/Q_m\Big\}\,. \quad (33)$$

Hence, for low degree of modulation depths we obtain

$$\mathrm{St}_m \simeq \frac{R_1^2(\Delta t_m) + R_1(0)/Q_m}{R_1(0) + 1/Q_m} \simeq$$

$$\simeq \begin{cases} R_1^2(\Delta t_m)/R_1(0) & \text{for } Q_m/N^2 \gg 1; \\ R_1(0) & \text{for } Q_m/N \ll 1; \end{cases}$$

that is, the algorithm of compensation is absolutely inefficient since the Strehl factor does not exceed its value in the absence of compensation, i.e., $R_1(0) \ge R_1(\Delta t_m)$. In the case of high degree of modulation ($\alpha \to 0$), the efficiency of the algorithm

$$\mathrm{St}_m \simeq \frac{R_N^2(\Delta t_m) + NR_N(0)/Q_m}{R_N(0) + N^2/Q_m} \simeq$$

$$\simeq \begin{cases} R_N^2(\Delta t_m)/R_N(0) & \text{for } Q_m/N^2 \gg 1; \\ R_N(0)/N & \text{for } Q_m/N \ll 1; \end{cases}$$

is close to that of the optimal scheme.

Thus, when the degree of modulation is high, the optimal algorithm can provide highly efficient operation of the AOS's without the complicated procedure of "weighted" summation of high–frequency signals.

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