Iterative method of the wave front reconstruction using the intensities of defocused images and spots of a Shack−Hartmann sensor

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A method is proposed for calculation of the field function in the plane of the exit pupil of an optical system on the basis of point source images, formed by adaptive optical systems in several parallel planes, and the intensity distributions over Shack−Hartmann spots.

Introduction

One of the ways to reconstruct wave front (WF) distortions is the method that uses images, formed by an optical system (OS). Any image contains information on the wave front distortions introduced by the OS, and the task is to retrieve this information from an image.

Let Ω be he exit pupil domain and

$$G = A(\xi,\eta)\exp\left[k\Phi(\xi,\eta)\right], \quad (\xi,\eta) \in \Omega,$$

be the OS pupil function[1]; A defines the amplitude distribution; Φ is the distribution of wave aberrations over the pupil; and $k = 2\pi/\lambda$ is the wave number. In the focal plane oxy, the wave field, generated by a point source, is described, accurate to an insignificant factor, by the function $g(x, y)$, determined by the Fourier transform:

$$g(x,y) = F\left(G; x/\lambda R, y/\lambda R\right)/R =$$

$$= 1/R \int\!\!\int_{-\infty}^{+\infty} G(\xi,\eta)\exp\left[-i2\pi(x\xi + y\eta)/\lambda R\right]\mathrm{d}\xi\mathrm{d}\eta, \quad (1)$$

where R is the focal length.

Gerchberg and Saxton[2] proposed the well proven in practice iterative method of wave front reconstruction from the field amplitude A in the pupil and the field amplitude $\left|g(x, y)\right|$ within a given area, ω, of the focal plane. Formally, the WF recovery problem can be reduced to determination of the function G from two restrictions imposed on it: from the preset amplitude in the spatial and frequency domains.

The measurements in the focal plane are always carried out, since they enable obtaining information about the object observed with an OS. The measurements of the amplitude distribution over the pupil are mostly needed because of the requirements to the method. If no such measurements have been carried out, then it is natural to substitute the requirement that the amplitude is known in the pupil by the requirement that the *a priori* information

about this amplitude is available. Thus, the support, Ω, of the function G is always known. The restriction of this kind was proposed by Fienup.[3]

The decrease in the information about the function G in the pupil can be compensated for by additional measurements of the amplitude in the image space. In Ref. 4, it was proposed to take into account the amplitude distributions over several planes parallel to the focal plane. If a measurement is carried out in a non-focal plane $z \neq 0$, then, formally, within the framework of the diffraction image theory[1] this is equivalent to the measurement in the focal plane under the condition that the pupil function changed by the phase factor and became equal to $GG_0(z)$, where $G_0(z) = \exp[-ikz(\xi^2 + \eta^2)/2R^2]$. This can be also interpreted as a measurement in the focal plane at various transformations of the light beam in the pupil region with the aid of transmission-type phase screen $G_0(z_s)$, $s = \overline{1,S}$, where S is the number of defocusings introduced.

The Shack−Hartmann sensor is widely used in OSs to measure local WF tilts. In an OS with such a sensor, the input beam is split into two beams. One beam forms the image of a point source in the OS focal plane, while the other is split by the lenslet array of the sensor, and each part forms the image of the same source in the focal plane of the corresponding lenslet. The number of lenslets is determined by the spatial spectrum of the WF distortions. At large dimensions of the lenslet array, a problem of lenslet alignment arises and the light flux passing through lenslets decreases. It is possible to increase the dimensions of the lenslet array and to decrease the number of lenslets, if the information about WF is represented not only by the single characteristic of the spot in the sensor lenslets, namely, the spot displacement, but by the intensity distributions over each spot instead. The method of WF reconstruction from the intensity distribution in the focal plane of OS and lenslets with the known support of the function G was proposed in Ref. 5.

The lenslets, as well as OS, perform the Fourier transform. The optical system performs the Fourier

transform of the wave function G, while the lenslets transform the $\chi_s G$, where χ_s is the characteristic function of the aperture Ω_s of the sth lenslet. Therefore, the wave field in the focal plane of the sth lenslet with the center (ξ_s, η_s) and the focal length r is determined, accurate to an insignificant factor, by the function

$$g_s(x,y) = c/r \times$$

$$\times \int\int\limits_{-\infty}^{+\infty} \chi_s(\xi',\eta') G(\xi',\eta') \exp\left[-i2\pi(x\xi' + y\eta')/\lambda r\right] d\xi' d\eta', \quad (2)$$

where $\xi' = \xi - \xi_s$; $\eta' = \eta - \eta_s$; c is the coefficient, characterizing the fraction of the light amplitude directed to the sensor.

The intensity distributions measured in light spots of the sensor allow us to determine the amplitude of the Fourier transform of the wave field in the plane of the exit pupil at various transformations of this field with the aid of transmission-type amplitude screens, determined by functions χ_s. Therefore, the method of wave front reconstruction proposed in Ref. 5 can be interpreted as a problem of determination of the function G from its support and the amplitude of Fourier transforms of its various transformations (constriction) by the transmission-type amplitude screens, $\chi_s G$.

The analysis carried out allows us to conclude that the methods of WF reconstruction used in Refs. 4 and 5 are particular cases of the general approach, which reduces the task to determination of the pupil function G from its support and the amplitudes of Fourier transforms of its various transformations, in the general case, by amplitude-phase spatial transmission-type screens.

1. Mathematical formulation of the problem

Let us pass in equality (1), to the relative coordinates

$$(\xi,\eta) = a(\tilde{\xi},\tilde{\eta}), \quad (x,y) = (\lambda R/a)(\tilde{x},\tilde{y}), \quad z = R^2 \tilde{z}/(ka^2),$$

where a is the radius of the exit pupil, $\tilde{\xi}^2 + \tilde{\eta}^2 \leq 1$ in Ω, and introduce the functions

$$g(x,y) = (a^2/R)\tilde{g}(\tilde{x},\tilde{y}),$$

$$G_0(\xi,\eta,z_s) G(\xi,\eta) = \tilde{G}_0(\tilde{\xi},\tilde{\eta},\tilde{z}_s) \tilde{G}(\tilde{\xi},\tilde{\eta}) = \tilde{G}_s(\tilde{\xi},\tilde{\eta}).$$

Then, with the allowance made for defocusing, equality (1) takes the form

$$\tilde{g}_s(\tilde{x},\tilde{y}) = \int\int\limits_{-\infty}^{+\infty} \tilde{G}_s(\tilde{\xi},\tilde{\eta}) \exp\left[-i2\pi(\tilde{x}\tilde{\xi} + \tilde{y}\tilde{\eta})\right] d\tilde{\xi} d\tilde{\eta},$$

$$s = \overline{1,S}. \quad (3)$$

In equality (2), we also pass to relative coordinates

$$(\xi_s,\eta_s) = a(\tilde{\xi}_s,\tilde{\eta}_s), \quad (\xi',\eta') = b(\tilde{\xi}',\tilde{\eta}'), \quad \tilde{\xi}'^2 + \tilde{\eta}'^2 \leq 1;$$

$(x,y) = (\lambda r/b)(\tilde{x},\tilde{y})$, b is the lenslet radius, and introduce the functions

$$g_s(x,y) = (b^2/r)\tilde{g}_s(\tilde{x},\tilde{y});$$

$$\chi_s(\xi_s + \xi', \eta_s + \eta') = \tilde{\chi}_s\left[\tilde{\xi}_s + (b/a)\tilde{\xi}', \tilde{\eta}_s + (b/a)\tilde{\eta}'\right];$$

$$G_s(\xi_s + \xi', \eta_s + \eta') = \tilde{G}_s\left[\tilde{\xi}_s + (b/a)\tilde{\xi}', \tilde{\eta}_s + (b/a)\tilde{\eta}'\right];$$

$$\tilde{\chi}_s\left[\tilde{\xi}_s + (b/a)\tilde{\xi}', \tilde{\eta}_s + (b/a)\tilde{\eta}'\right] \times$$

$$\times \tilde{G}\left[\tilde{\xi}_s + (b/a)\tilde{\xi}', \tilde{\eta}_s + (b/a)\tilde{\eta}'\right] = \tilde{G}_s(\tilde{\xi}',\tilde{\eta}'),$$

$$s = \overline{S+1, S+S_1}.$$

Then equality (2) takes the form

$$\tilde{g}_s(\tilde{x},\tilde{y}) = \int\int\limits_{-\infty}^{+\infty} \tilde{G}_s(\tilde{\xi}',\tilde{\eta}') \exp\left[-i2\pi(\tilde{x}\tilde{\xi}' + \tilde{y}\tilde{\eta}')\right] d\tilde{\xi}' d\tilde{\eta}',$$

$$s = \overline{S+1, S+S_1}. \quad (4)$$

In the relative coordinates, equalities (1) and (2) transform into equalities (3) and (4), which have identical forms. Therefore, we write them as a single equality and, for simplicity, omit the tilde «~»:

$$g_s(x,y) =$$

$$= \int\int\limits_{-\infty}^{+\infty} G_s(\xi,\eta) \exp\left[-i2\pi(x\xi + y\eta)\right] d\xi d\eta = F(G_s; x, y),$$

$$s = \overline{1, S+S_1}, \quad (5)$$

where

$$G_s(\xi,\eta) = G_0(\xi,\eta,z_s) G(\xi,\eta), \quad s = \overline{1,S}, \quad (6)$$

$$G_s(\xi,\eta) = \chi_s\left[\xi_s + (b/a)\xi, \eta_s + (b/a)\eta\right] \times$$

$$\times G\left[\xi_s + (b/a)\xi, \eta_s + (b/a)\eta\right], \quad (7)$$

$$s = \overline{S+1, S+S_1}.$$

All the $G_s(\xi,\eta)$ functions have identical supports in the form of a circle of a unit radius, and they are determined through the same function $G(\xi,\eta)$ by equalities (6) and (7). The Fourier transforms of the functions $G_s(\xi,\eta)$ form a set, which we designate as V_1, that is,

$$\left[g_1 = F(G_1), \ g_2 = F(G_2),..., g_{S+S_1} = F(G_{S+S_1})\right] \in V_1.$$

Thus, the right-hand sides of equalities (5) specify points of the set V_1.

As to the left-hand sides of equalities (5), it is known that they have a preset absolute value $a_s(x, y)$ on the sets ω_s of the corresponding planes of intensity measurements, that is

$$\left| g_s(x,y) \right| = a_s(x,y), \quad (x,y) \in \omega_s, \quad s = \overline{1, S + S_1} \quad (8)$$

In writing equalities (1) and (2), we mentioned that they are specified accurate to insignificant factors. Actually, these are the phase factors, which do not change equalities (8). The sets of the functions $\left[g_1(\xi,\eta),\dots,g_{S+S_1}(\xi,\eta) \right]$ satisfying equalities (8) form the set V_2.

The problem of WF reconstruction from the intensities in the defocused OS images and in the sensor spots is reduced to the determination of the phase of such a G function, which generates a point from V_1, belonging also to the set V_2. The geometric formulation of the problem on WF reconstruction as a problem of determination of a common point of the given sets is known, and it can be solved by an iterative method in an appropriate Gilbert space, if the projections onto these sets have been found.

2. Projections onto the sets V_1 and V_2

Let us introduce, in the plane oxy, the space L of the complex-valued functions with summable squares and the direct product $H = L^{S+S_1}$. Then the vector function $g(x, y) = \left[g_1(x,y),\dots,g_{S+S_1}(x,y) \right] \in H$ if at any s its coordinates $g_s(x, y) \in L$. Now specify the scalar product and the norm in H,

$$(g,\varphi) = \sum_{s=1}^{S+S_1} (g_s,\varphi_s)_L, \quad \|g\| = (g,g)^{1/2}.$$

The projection of the point $g \in H$ onto the set $V \subset H$ is the point $g_0 \in V$, determined by the condition

$$\|g - g_0\| = \inf_{g' \in V} \|g - g'\|.$$

If the set V is convex and closed, then the projection exists and it is unique. Let us designate the projection as $g_0 = P_V g$.

According to the conditions of the problem, the pupil function G is limited in the absolute value $|G| \le C$ and has the restricted support Ω, and therefore the set V_1 is convex, closed, and norm-limited. The set V_2 is closed.

Let the sets $\Omega_0, \Omega_1, \dots, \Omega_{S_1}$ form a partition Ω, $\{\chi_s\}$ is a set of characteristic functions of these sets and $g' \in V_1$. Using the Parseval's equality for the Fourier transform, we find that

$$\|g - g'\|^2 = \left\| F^{-1}g - F^{-1}g' \right\|^2 = \sum_{s=1}^{S} \left\| F^{-1}g_s - G_0(z_s)G \right\|^2 +$$

$$+ \sum_{s_1=1}^{S_1} \left\| F^{-1}g_{S+s_1} - \chi_{s_1}G \right\|^2 = \sum_{s=1}^{S} \left\| G_0^*(z_s)F^{-1}g_s - G \right\|^2 +$$

$$+ \sum_{s_1=1}^{S_1} \left\| \chi_{s_1}\left(F^{-1}g_{S+s_1} - G \right) \right\|^2 + \text{const},$$

where const joins the terms, independent of G; the asterisk «*» denotes the complex conjugate. Then, taking into account that the sets $\Omega_0, \Omega_1, \dots, \Omega_{S_1}$ do not overlap we have:

$$\|g - g'\|^2 = \sum_{s=1}^{S} \left\| \chi_0\left(G_0^*(z_s)F^{-1}g_s - G \right) \right\|^2 +$$

$$+ \sum_{s_1=1}^{S_1} \left[\left\| \chi_{s_1}\left(F^{-1}g_{S+s_1} - G \right) \right\|^2 + \right.$$

$$\left. + \sum_{s=1}^{S} \left\| \chi_{s_1}\left(G_0^*(z_s)F^{-1}g_s - G \right) \right\|^2 \right] + \text{const}.$$

Introducing additional terms, independent of G, it is possible to make summation under the norm sign:

$$\|g - g'\|^2 = S^2 \left\| \chi_0 \left(\frac{1}{S}\sum_{s=1}^{S} G_0^*(z_s)F^{-1}g_s - G \right) \right\|^2 +$$

$$+ (S + 1)^2 \sum_{s_1=1}^{S_1} \left\| \chi_{s_1} \left[\frac{\left(F^{-1}g_{S+s_1} + \sum_{s=1}^{S} F^{-1}g_s \right)}{(S + 1) - G} \right] \right\|^2 + \text{const}.$$

Introduce the functions

$$\varphi_0 = \sum_{s=1}^{S} G_0^*(z_s)F^{-1}g_s \quad \text{and} \quad \varphi_{s_1} = F^{-1}g_{S+s_1} + \varphi_0,$$

then

$$\|g - g'\|^2 = S^2 \left\| \chi_0 \left(\frac{\varphi_0}{S} - G \right) \right\|^2 +$$

$$+ (S + 1)^2 \sum_{s_1=1}^{S_1} \left\| \chi_{s_1} \left(\frac{\varphi_{s_1}}{(S + 1) - G} \right) \right\|^2.$$

It follows from this equality that the projection $g_0 \in V_1$ is determined by the function

$$G_0 = \begin{cases} \varphi_0(\xi,\eta)/S, & (\xi,\eta) \in \text{int}\,\Omega_0; \\ \varphi_{s_1}(\xi,\eta)/(S+1), & (\xi,\eta) \in \Omega_{s_1}, \quad s_1 \in \overline{1, S_1}. \end{cases}$$

The projection onto the set V_2 has been described in different papers, for example in Ref. 6:

$$P_{V_2}g = (g_{01},\dots,g_{0S+S_1}) =$$

$$= \begin{cases} g_{0s} = \dfrac{a_s(x,y)g_s(x,y)}{|g_s(x,y)|}, & (x,y) \in \omega_s, \quad g(x,y) \ne 0, \\ g_{0s}(x,y) = g_s(x,y). \end{cases}$$

The second row in the last equality corresponds to the rest (x, y) points.

3. Iterative method
of WF reconstruction

In the geometric treatment, the problem of WF reconstruction is reduced to the problem on determination of a common point for the sets V_1 and V_2. There are several algorithms of searching for this point, in particular, the Gerchberg—Saxton algorithm,[2] Yule algorithm,[6] and the method of increasing the dimensionality.[7] Let us apply the algorithm from Ref. 7. Introduce the approaching functional, defined on $H \times H \times H$:

$$J(g, g_1, g_2) = \alpha_1 \|g - g_1\|^2 + \alpha_2 \|g - g_2\|^2,$$

$$\alpha_1, \alpha_2 > 0, \quad \alpha_1 + \alpha_2 = 1.$$

The functional achieves a minimum on the set $H \times V_1 \times V_2$ at the point (g, g_1, g_2), satisfying the condition

$$g = g_1 = g_2 \in V_1 V_2, \quad J(g, g_1, g_2) = 0.$$

Any converging minimizing sequence has a limit, determining the point from $V_1 V_2$. Since the functional is convex, and the sets V_1 and V_2 are limited, any minimizing sequence slightly converges to the point of the minimum.

The simplest algorithm of constructing the minimizing sequence is based on the coordinate descent. Let g_0 be the zero approximation to the point of the minimum and $g_{10} = P_{V_1} g_0$, $g_{20} = P_{V_2} g_0$. The following approximations are constructed by the scheme:

$$g_{1n} = P_{V_1} g_n, \quad g_{2n} = P_{V_2} g_n;$$

$$g_{n+1} = \alpha_1 g_{1n} + \alpha_2 g_{2n}, \quad n = 1, 2, \dots .$$

The condition of termination of the iterations is determined by the closeness of the functional to the minimum value.

Conclusions

The WF reconstruction method proposed assumes the generalization, which takes into account the intensity distribution in defocused OS images and defocused spots of the Shack—Hartmann sensor.

References

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