RECONSTRUCTION OF THE LASER BEAM PARAMETERS FROM THE TEMPERATURE FIELD OF A HEATED SURFACE

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The problem of reconstructing the laser beam intensity distribution from temperature distribution over the front surface of the target heated by the beam has been solved analytically. The dynamic relations have been derived for the instantaneous position of the intensity distribution centroid, size, and the functional of focusing with arbitrary boundary conditions on the back surface of the target.

In the previous paper[1] we proposed to reconstruct the beam intensity from the measurements of the temperature field $T(\rho, t)$ of a heated surface to solve the problem of measuring the intensity distribution $I(\rho, t)$ over the cross section of high–power laser beams. In general the multidimensional spatiotemporal inverse problem for thermal conductivity, to the solution of which the reconstruction problem was reduced in Ref. 1, was transformed to the one–dimensional problem by means of a choice of a one–dimensional target of a special configuration. In this paper we have obtained the analytical solutions of the three–dimensional spatiotemporal inverse problem for thermal conductivity, which is the problem of conversion of the boundary conditions. Unlike Ref. 1 the uniform plate was chosen as a target. The dynamic relations have been derived for reconstruction of the spatiotemporal intensity distribution, instantaneous position of the intensity distribution centroid, effective size, and functional of laser beam focusing for the arbitrary boundary conditions on the back side of the plate.

Assuming the plate to be infinite in the transverse direction, we describe the process of heat transfer through the plate by the three–dimensional thermal conductivity equation

$$\frac{\partial}{\partial t}T(\rho,\ z,\ t) = a^2\Delta T(\rho,\ z,\ t)\ ,\quad t > 0,\quad 0 \le z \le L\ ,$$

$$-\infty < x,\ y < \infty\ . \tag{1}$$

The boundary conditions on the front

$$q(\rho,\ t) = k\frac{\partial}{\partial z}T(\rho,\ z,\ t)\big|_{z=L} =$$

$$= (1 - R)\, I(\rho,\ t) + \vartheta\, T(\rho,\ t) - \sigma b T^4(\rho,\ t)\ ; \tag{2}$$

$$T(\rho,\ z;\ t)\big|_{z=L} = T(\rho,\ t) \tag{3}$$

and back sides of the plate

$$T(\rho,\ z;\ t)\big|_{z=0} = 0\ ; \tag{4}$$

$$k\ \frac{\partial T(\rho,\ z;\ t)}{\partial z}\bigg|_{z=0} = 0 \tag{5}$$

are used in different combinations depending on the regime maintained on the back surface. Moreover, the relations

$$T(\pm\infty,\ y,\ z;\ t) = T(x,\ \pm\infty,\ z;\ t) = 0\ , \tag{6}$$

$$\frac{\partial}{\partial y}T(x,\ \pm\infty,\ z;\ t) = \frac{\partial}{\partial x}T(\pm\infty,\ y,\ z;\ t) = 0 \tag{7}$$

should be added to the boundary conditions.

For simplicity, the initial distribution of temperature is assumed to be constant and equal to zero

$$T(\rho,\ z;\ 0) = 0\ . \tag{8}$$

In formulas (1) and (2) $\Delta = \Delta_\perp + \dfrac{\partial^2}{\partial z^2}$, where $\Delta_\perp = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the transverse Laplacian operator; v, k, and a^2 are the coefficients of the convective heat transfer, thermal conductivity, and thermal diffusivity, respectively; σ is the Stefan–Boltzmann constant; b is the emittance; and, R is the reflectance of the plate surface.

Let us apply to Eq. (1) the Fourier transform

$$\mathbf{F}[T(\rho,\ z;\ t)] = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty} d^2\rho T(\rho,\ z;\ t)e^{-i\kappa\rho} \tag{9}$$

and the Laplacian transform

$$\mathbf{L}[T(\rho,\ z;\ t)] = \int\limits_{0}^{\infty} dt T(\rho,\ z;\ t)e^{-st}\ . \tag{10}$$

Using Eqs. (6)–(8) we obtain for

$$\mathbf{LF}[T(\rho,\ z;\ t)] = \tilde{T}(\kappa,\ z;\ s)$$

the following equation:

$$(s + a^2\kappa^2)\ \tilde{T}(\kappa,\ z;\ s) = a^2\,\frac{d^2\tilde{T}(\mathbf{k},\ z;\ s)}{dz^2} \tag{11}$$

with the boundary conditions

$$\tilde{q}(\kappa,\ s) = k\frac{\partial}{\partial z}\tilde{T}(\kappa,\ z;\ s)\big|_{z=L}\ , \tag{12}$$

$$\tilde{T}(\kappa,\ z;\ s)\big|_{z=L} = \tilde{T}(\kappa,\ s)\ , \tag{13}$$

$$\tilde{T}(\kappa,\ z;\ s)\big|_{z=0} = 0\ , \tag{14}$$

$$k\ \frac{\partial T(\mathbf{k},\ z;\ s)}{\partial z}\bigg|_{z=0} = 0\ . \tag{15}$$

At first we find the heat flux through the plate whose back surface is at a constant temperature. In this case we use boundary conditions (12) and (13). Let us introduce the new functions $V(\kappa, z; s)$ and $U(\kappa, z; s)$ so that to satisfy the relations

$$V(\kappa, z; s)\,\tilde{q}(\kappa, s) = k\frac{\partial}{\partial z}\tilde{T}(\kappa, z; s) ; \quad (16)$$

$$U(\kappa, z; s)\,\tilde{q}(\kappa, s) = k\,\tilde{T}(\kappa, z; s) . \quad (17)$$

Then instead of Eqs. (11), (12), and (14) we obtain the system of ordinary differential equations

$$(s + a^2\kappa^2)\, U(\kappa, z; s) = a^2\frac{\mathrm{d}V(\mathbf{k}, z; s)}{\mathrm{d}z} ;$$

$$\frac{\mathrm{d}U(\mathbf{k}, z; s)}{\mathrm{d}z} = V(\kappa, z; s) ; \quad (18)$$

with the boundary conditions

$$V(\kappa, L; s) = 1 ,$$

$$U(\kappa, 0; s) = 0 . \quad (19)$$

Let us reduce boundary problem (18)–(19) to the problem with the initial conditions. To this end, we use the embedding method[2] assuming the parameter L to be an argument of the functions V and U and differentiate Eqs. (18) and (19) with respect to L.

Taking into account that

$$\frac{\mathrm{d}}{\mathrm{d}L}V(\kappa, L, L; s) = \frac{\partial}{\partial L}V(\kappa, z, L; s)\Big|_{z=L} + \frac{\partial}{\partial z}V(\kappa, z, L; s)\Big|_{z=L}, \quad (20)$$

where

$$\frac{\partial}{\partial z}V(\kappa, z, L; s)\Big|_{z=L} = \frac{s + a^2\mathrm{k}^2}{a^2}\, U(\kappa, L, L; s) ,$$

we derive from Eqs. (18) and (19)

$$(s + a^2\kappa^2)\frac{\partial}{\partial L}U(\kappa, z, L; s) = a^2\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial L}V(\kappa, z, L; s) ,$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial}{\partial L}U(\kappa, z, L; z) = \frac{\partial}{\partial L}V(\kappa, z, L; s) ; \quad (21)$$

$$\frac{\partial}{\partial L}U(\kappa, 0, L; s) = 0 ;$$

$$\frac{\partial}{\partial L}V(\kappa, z, L; s)\Big|_{z=L} = -\frac{s + a^2\mathrm{k}^2}{a^2}\, U(\kappa, L, L; s) . \quad (22)$$

Comparing Eqs. (18) and (19) with Eqs. (21) and (22) and assuming that the solution of the problem is unique, we find

$$\frac{\partial}{\partial L}U(\kappa, z, L; s) = -\frac{s + a^2\mathrm{k}^2}{a^2}\, U(\kappa, L, L; s)U(\kappa, z, L; s) , \quad (23)$$

$$\frac{\partial}{\partial L}V(\kappa, z, L; s) = -\frac{s + a^2\mathrm{k}^2}{a^2}\, U(\kappa, L, L; s)\, V(\kappa, z, L; s) . \quad (24)$$

Because

$$\frac{\mathrm{d}}{\mathrm{d}L}U(\kappa, L, L; s) = \frac{\partial U(\mathbf{k}, z, L; s)}{\partial L}\Bigg|_{z=L} + \frac{\partial U(\mathbf{k}, z, L; s)}{\partial z}\Bigg|_{z=L}, \quad (25)$$

according to Eqs. (18), (19), (23), and (25) we obtain the Riccati equation for the quantity $U(\kappa, L, L; S)$

$$\frac{\mathrm{d}}{\mathrm{d}L}U(\kappa, L, L; s) = -\frac{s + a^2\mathrm{k}^2}{a^2}\, U^2(\kappa, L, L; s) + 1 \quad (26)$$

with the initial condition

$$U(\kappa, 0, 0; s) = 0 . \quad (27)$$

By integrating Eq. (26)

$$\int\limits_0^U \frac{\mathrm{d}U}{\dfrac{a^2}{S + a^2\mathrm{k}^2} - U^2} = \frac{s + a^2\mathrm{k}^2}{a^2}\int\limits_0^L \mathrm{d}L , \quad (28)$$

we derive

$$U(\kappa, L, L; s) = \frac{a}{\sqrt{s + a^2\mathrm{k}^2}}\,\mathrm{th}\left(\frac{L\sqrt{s + a^2\mathrm{k}^2}}{a}\right). \quad (29)$$

Taking into account Eq. (17), we obtain

$$\tilde{q}(\kappa, s) = k\frac{\sqrt{s + a^2\mathrm{k}^2}}{a}\,\tilde{T}(\kappa, s)\,\coth\left(\frac{L\sqrt{s + a^2\mathrm{k}^2}}{a}\right). \quad (30)$$

To invert Eq. (30), we make sequential use of the operational calculus theorems on the transform product, similarity, translation, differentiation of the original function as well as the theorem on the Fourier transform of convolution.[3] Taking into account that

$$\mathbf{L}[\Theta_3(1, t)] = \frac{\coth\sqrt{s}}{\sqrt{s}} ,$$

where

$$\Theta_3(\vartheta, t) = 1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi^2 t}\cos 2\pi n\vartheta$$

is the Jacobi function,[3] we finally obtain

$$q(\rho, t) = -\frac{k}{4\pi a^2 L}\int\limits_0^t \mathrm{d}\tau\int\limits_{-\infty}^{\infty} \mathrm{d}^2\rho'\,\frac{T(\rho', \tau)}{(t - \tau)} \times$$

$$\times \exp\left(-\frac{(\rho - \rho')^2}{4a^2(t - \tau)}\right)\frac{\mathrm{d}}{\mathrm{d}\tau}\,\Theta_3\left(1 , \frac{a^2}{L^2}(t - \tau)\right) . \quad (31)$$

Formula (31) is valid for the arbitrary target parameters a^2, k and L as well as ρ and t. However, Eq. (31) can be simplified for the specific relations between these parameters. Assuming that $L/a \ll 1$ we make use of the power series expansion of the function $\coth(x)$:

$$\coth(x) = \frac{1}{x} + \frac{x}{3} + \mathrm{O}(x^3) , \quad x < 1 .$$

Then instead of Eq. (30) we obtain

$$\tilde{q}(\kappa, s) = \frac{k}{L}\tilde{T}(\kappa, s) + \frac{kL}{3a^2}(s + a^2\kappa^2)\tilde{T}(\kappa, s) + $$

$$+ O\left(\frac{k}{L}\left(\frac{k}{a}\right)^4 (s + a^2k^2)^2 \tilde{T}(\mathbf{k}, s)\right).$$

Taking into account the first two terms of this expansion and making inversion, we have

$$q(\rho, t) = \frac{k}{L}T(\rho, t) + \frac{kL}{3a^2}\left[-a^2\Delta_\perp T(\rho, t) + \frac{\partial}{\partial t}T(\rho, t)\right]. \quad (32)$$

Eq. (32) describes the process of heat transfer through the thin two–dimensional plate. It can be rewritten in the form of the heat–conduction equation with the sources

$$\frac{\partial}{\partial t}T(\rho, t) = a^2\Delta_\perp T(\rho, t) + \frac{3a^2}{kL}q(\rho, t) - \frac{3a^2}{L^2}T(\rho, t). \quad (33)$$

For the other limiting case in which $L/a \gg 1$ we make use of the asymptotic expansion

$$\coth x = 1 + 2e^{-x} + O(e^{-2x}), \quad x \gg 1$$

and consider the first term only. Inverting the relation

$$\tilde{q}(\kappa, s) = k\frac{\sqrt{s + a^2k^2}}{a}\tilde{T}(\kappa, s),$$

we obtain the solution of the problem for the semi–infinite body

$$q(\rho, t) = -\frac{k}{8\sqrt{\pi^3 a^3}}\int_0^t d\tau \int_{-\infty}^{\infty} d^2\rho' \frac{T(\rho', \tau)}{\sqrt{(t-\tau)^5}} \exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)}\right). \quad (34)$$

Eqs. (33) and (34) can be directly derived from Eq. (31). This can be made by introducing the generalized thermophysical parameter $F_0 = \frac{a^2}{L^2}t$ into Eq. (31), by application of the Laplace method for $F_0 \gg 1$, and by reducing the infinite summation in Eq. (31) to integration[1] for the other limiting case ($F_0 \ll 1$).

When the back surface of the plate is thermally insulated, then to solve the inverse problem we must use Eq. (11) with boundary conditions (13) and (15). Setting up and solving the embedding equations we find that

$$\tilde{q}(\kappa, s) = k\tilde{T}(\kappa, s)\frac{\sqrt{s + a^2k^2}}{a}\,\mathrm{th}\left(\frac{L\sqrt{s + a^2k^2}}{a}\right). \quad (35)$$

Inversion of Eq. (35) yields the heat flux on the front side of the target with thermally insulated back surface

$$q(\rho, t) = -\frac{k}{4\pi a^2 L}\int_0^t d\tau \int_{-\infty}^{\infty} d^2\rho' \frac{T(\rho', \tau)}{(t-\tau)} \times$$

$$\times \exp\left(-\frac{(\rho-\rho')^2}{4a^2(t-\tau)}\right)\frac{d}{d\tau}\Theta_1\left(\frac{1}{2}, \frac{a^2}{L^2}(t-\tau)\right), \quad (36)$$

where

$$\Theta_1(\vartheta, t) = 2\sum_{n=0}^{\infty}(-1)^n \exp\left(-\pi^2\left(n + \frac{1}{2}\right)^2 t\right)\sin\pi(2n+1)\vartheta.$$

For $F_0 \gg 1$ the asymptotic formula

$$q(\rho, t) = \frac{Lk}{a^2}\left[\frac{\partial}{\partial t}T(\rho, t) - a^2\Delta_\perp T(\rho, t)\right]$$

can be derived while for $F_0 \ll 1$ Eq. (34) can be derived for the semi–infinite body.

Having obtained the integral representation for reconstruction of the heat flux it is easy to derive the corresponding relations for the integral moments of the flux: the position of the intensity distribution centroid, effective size, and functional of focusing. As the heat losses associated with radiation and heat exchange on the front side of the target are known, the integral moments of the flux are uniquely related with the integral moments of the laser beam intensity. The heat losses are assumed to be negligible. Then for the total intensity flux we obtain

$$P_0(t) = \int_{-\infty}^{\infty} d^2\rho\, I(\rho, t)$$

and for the semi–infinite body we obtain

$$P_0(t) = -\frac{k}{2\sqrt{\pi}a(1-R)}\int_0^t d\tau \frac{M_0(\tau)}{\sqrt{(t-\tau)^3}},$$

while for the cooled and thermally insulated targets we have

$$P_0(t) = -\frac{k}{L(1-R)}\int_0^t d\tau M_0(\tau)\frac{d}{d\tau}\Theta_3\left(1, \frac{a^2}{L^2}(t-\tau)\right),$$

$$P_0(t) = -\frac{k}{L(1-R)}\int_0^t d\tau M_0(\tau)\frac{d}{d\tau}\Theta_1\left(\frac{1}{2}, \frac{a^2}{L^2}(t-\tau)\right),$$

respectively, where

$$M_0(t) = \int_{-\infty}^{\infty} T(\rho; t)\, d^2\rho.$$

The vector specifying the coordinates of the intensity distribution centroid

$$\rho_c\{\rho_{cx}, \rho_{cy}\} = \mathbf{i}\,\rho_{cx} + \mathbf{j}\,\rho_{cy} = \frac{1}{P_0(t)}\int_{-\infty}^{\infty} I(\rho; t)\,\rho d^2\rho$$

is given by the formulas

$$\rho_c = -\frac{k}{2\sqrt{\pi}a(1-R)P_0(t)}\int_0^t \mathrm{d}\tau \frac{\mathbf{M}_T(\tau)}{\sqrt{(t-\tau)^3}},$$

$$\rho_c = -\frac{k}{L(1-R)P_0(t)}\int_0^t \mathrm{d}\tau \mathbf{M}_T(\tau)\frac{\mathrm{d}}{\mathrm{dt}}\Theta_3\left(1, \frac{a^2}{L^2}(t-\tau)\right),$$

and

$$\rho_c = -\frac{k}{L(1-R)P_0(t)}\int_0^t \mathrm{d}\tau \mathbf{M}_T(t)\frac{\mathrm{d}}{\mathrm{dt}}\Theta_1\left(\frac{1}{2}, \frac{a^2}{L^2}(t-\tau)\right),$$

for the semi–infinite, cooled, and thermally insulated targets, respectively. Here

$$\mathbf{M}_T(t) = \int_{-\infty}^{\infty} T(\rho'; t)\, \rho' \mathrm{d}^2\rho .$$

The effective beam radius ρ^2_{eff} is given by the relation

$$\rho^2_{\mathrm{eff}} = \rho^2_{x\mathrm{eff}} + \rho^2_{y\mathrm{eff}} = P_0^{-1}(t)\int_{-\infty}^{\infty} I(\rho; t)\, \rho^2 \mathrm{d}^2\rho .$$

The representations of ρ^2_{eff} in terms of the temperature distribution over the surface for the three types of boundary conditions take the forms

$$\rho^2_{\mathrm{eff}} = -\frac{k}{2\sqrt{\pi}a(1-R)P_0(t)}\int_0^t \frac{\mathrm{d}\tau}{\sqrt{(t-\tau)^3}}\left[M_{T2}(\tau) + 4a^2(t-\tau)M_0(\tau)\right];$$

$$\rho^2_{\mathrm{eff}} = -\frac{k}{L(1-R)P_0(t)}\int_0^t \mathrm{d}\tau\left[M_{T2}(\tau) + 4a^2(t-\tau)M_0(\tau)\right] \times$$

$$\times \frac{\mathrm{d}}{\mathrm{d}\tau}\Theta_3\left(1, \frac{a^2}{L^2}(t-\tau)\right),$$

$$\rho^2_{\mathrm{eff}} = -\frac{k}{L(1-R)P_0(t)}\int_0^t \mathrm{d}\tau\left[M_{T2}(\tau) + 4a^2(t-\tau)M_0(\tau)\right] \times$$

$$\times \frac{\mathrm{d}}{\mathrm{d}\tau}\Theta_1\left(\frac{1}{2}, \frac{a^2}{L^2}(t-\tau)\right),$$

where

$$M_{T2}(t) = \int_{-\infty}^{\infty} T(\rho'; t)\, \rho'^2 \mathrm{d}^2\rho'$$

is the moment of the thermal inertia.

Thus, for determination of the integral moments of the intensity distribution it is necessary to measure the moments of the temperature distribution. The functional of focusing

$$S_2(t) = \int_{-\infty}^{\infty} I(\rho'; t)\, K(\rho')\, \mathrm{d}^2\rho' ,$$

being equal to the laser beam power inside the aperture prescribed by the function $K(\rho)$ is related to the temperature distribution by the more complicated dependence. In particular, when focusing the radiation onto the circle of radius α ($K(\rho) = 1$ for $\rho \le \alpha$ and $K(\rho) = 0$ for $\rho > \alpha$) using Eqs. (34), (31), and (36), we obtain

$$S_2(t) = -\frac{k}{2\sqrt{\pi}a(1-R)}\int_0^t d\tau \int_{-\infty}^{\infty} \mathrm{d}^2\rho' \frac{T(\rho', \tau)}{\sqrt{(t-\tau)^3}} \times$$

$$\times\left[1 - J\left(\frac{\alpha^2}{4a^2(t-\tau)}, \frac{\rho'^2}{4a^2(t-\tau)}\right)\right];$$

$$S_2(t) = -\frac{k}{L(1-R)}\int_0^t d\tau \int_{-\infty}^{\infty} \mathrm{d}^2\rho'\, T(\rho', \tau) \times$$

$$\times\left[1 - J\left(\frac{\alpha^2}{4a^2(t-\tau)}, \frac{\rho'^2}{4a^2(t-\tau)}\right)\right]\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta_3\left(1, \frac{a^2}{L^2}(t-\tau)\right),$$

$$S_2(t) = -\frac{k}{L(1-R)}\int_0^t d\tau \int_{-\infty}^{\infty} \mathrm{d}^2\rho'\, T(\rho', \tau) \times$$

$$\times\left[1 - J\left(\frac{\alpha^2}{4a^2(t-\tau)}, \frac{\rho'^2}{4a^2(t-\tau)}\right)\right]\frac{\mathrm{d}}{\mathrm{d}\tau}\Theta_1\left(\frac{1}{2}, \frac{a^2}{L^2}(t-\tau)\right).$$

Here

$$J(x, y) = 1 - \mathrm{e}^{y}\int_0^x \mathrm{e}^{-t} I_0(2\sqrt{yt})\, \mathrm{d}t ,$$

where $I_0(\xi)$ is the modified Bessel function.[4] It is obvious that the relations for the integral moments and energy functionals of radiation are simpler for the case of the thin two–dimensional targets.

Thus, we derived the calculational formulas for reconstruction of the intensity distribution, integral moments, and energy functionals of a laser beam from the temperature distribution over the surface of the heated target. The examined problem belongs to the inverse problems of the thermal conductivity and is called the problem of conversion of the boundary conditions.[5]

As far as we know, the analytical solution of this problem in the multidimensional formulation has been obtained by us for the first time. The problem is classified among the ill–posed problems that is associated with the occurence of the characteristic Abel kernel $(\sqrt{t-\tau})^{-n}$ in integral representations (31), (34), and (36). The regularization of the solution of such a problem in one–dimensional formulation was considered in detail in Ref. 1. It is obvious that the practical implementation of the above–derived relations calls for the development of an effective engineering algorithms stable with respect to the noise in the initial data in the form of the measured temperature.

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