RESTORATION OF THE PHASE OF THE FOURIER SPECTRUM IN THE KNOX-THOMPSON AND TRIPLE-CORRELATION METHODS

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Received December 1, 1989

Expressions are derived for the accuracy of phase restoration in the processing of a series of short-exposure images, distorted by the atmosphere, of an object. It is shown that in the practically important two-dimensional case the accuracies of the methods studied are the same and close to the potential accuracy.

In the last ten years the comparatively new method of triple correlations (MTC) has been under active discussions in scientific circles concerned with the development of methods of speckle interferometry — postdetector restoration of the image of a remote object from a series of short-exposure realizations distorted by the atmosphere and detection noise.[1–6] This method, which supplements Labeyrie's method,[7] for restoring the modulus of the Fourier spectrum of an image makes it possible to obtain the Fourier phase. From this standpoint it can compete with the previously proposed[8] Knox-Thompson method (MKT), studied in Refs. 9–11. At the first stage many investigators believed that the MTC is much more accurate (though more difficult to implement) than the MKT and will replace the latter. The results of further in-depth theoretical investigations, supported by statistical modeling on a computer,[3–5] cast serious doubt on this. In this paper we study this question for the case when the detection noise is weak and atmospheric distortions of the images play the main role (the case of a bright object).

FORMULATION OF THE METHODS

The Fourier spectrum $J(\vec{f})$ of a short-exposure image (SI) is given by the expression

$$J(f) = O(f)\cdot H(f), \tag{1}$$

where $O(f)$ is the Fourier spectrum of the image of the object and $H(f)$ is the optical transfer function (OTF) of the atmosphere-telescope system. Under the usual assumption $D \gg r_0$, where D is the diameter of the aperture of the telescope and r_0 is Fried's parameter, which characterizes the average size of the region in which the atmospheric distortions of the optical radiation field are correlated, the OTF is a normal random quantity[12] with the average

$$\langle H(\vec{f})\rangle = H_0(\vec{f})\cdot\exp\left\{-3.44\cdot\left(\lambda\cdot f/r_0\right)^{5/3}\right\}, \tag{2}$$

where

$$H_0(\vec{f}) = S_a^{-1}\int d\vec{v}\; W(\vec{v})\; W(\vec{v}-\lambda\vec{f}) \tag{3}$$

is the OTF of the telescope, $W(\bar{v})$ is the aperture function and is equal to unity within the aperture and zero outside it; $S_a = \int d\bar{v}W(\bar{v})$ is the area of the aperture, and λ is the wavelength.

Eq. (2) shows that in the region of high spatial frequencies $f > f_a = r_0/\lambda$, which is of greatest practical interest, the average spectrum approaches zero. Because of this the methods employed to extract information in this region must be based on the formation of correlation functions of second and higher orders. Thus in Labeyrie's method the Fourier modulus is estimated from the expression

$$\langle|J(\vec{f})|^2\rangle = |O(\vec{f})|^2\langle|H(\vec{f})|^2\rangle. \tag{4}$$

The transfer function of the method $\left\langle\left|H(\vec{f})\right|^2\right\rangle$ differs appreciably from zero in the entire diffraction frequency range, and for $f > f_a$ it is given by[12]

$$\langle|H(\vec{f})|^2\rangle = \frac{1}{n}\cdot H_0(\vec{f}), \tag{5}$$

where $n = 2^{6/5}(D/r_0)^2$ is the effective number of regions of correlation of the atmospheric distortions within the aperture. The accuracy q_0 of the method, which is characterized by the ratio of $\left\langle\left|J(\vec{f})\right|^2\right\rangle$ to the rms error in estimating it from M recorded images, is given by the expression[10]

$$q_0 = \sqrt{M}. \tag{6}$$

To obtain information about the Fourier phase a correlation function of the form

$$E(\vec{f},\cdot\Delta\vec{f}) = \langle J(\vec{f})\cdot J^*(\vec{f}+\Delta\vec{f})\rangle, \tag{7}$$

is formed in the MKT and a function of the form

0235-6880/90/05 483-07 $02.00

$$T(\vec{f}_1, \vec{f}_2) = \langle J(\vec{f}_1) \cdot J(\vec{f}_2) \cdot J^*(\vec{f}_1 + \vec{f}_2) \rangle. \tag{8}$$

is formed in the MTC. The transfer functions of these methods in the region of high spatial frequencies are approximated as[3–5]

$$(f,\ f_1,\ |\vec{f}_1 + \vec{f}_2| > f_a)$$

$$\langle H(\vec{f}) \cdot H^*(\vec{f} + \Delta\vec{f}) \rangle = \langle |H(\vec{f})|^2 \rangle \cdot G(\Delta f), \tag{9}$$

$$\langle H(\vec{f}_1) H(\vec{f}_2) H^*(\vec{f}_1 + \vec{f}_2) \rangle = \begin{cases} \langle H(\vec{f}_1) H^*(\vec{f}_1 + \vec{f}_2) \rangle \langle H(\vec{f}_2) \rangle & \text{for } f_2 < f_a \\ n^{-3/2} \cdot \left[H_1(\vec{f}_1, \vec{f}_2) + H_1(\vec{f}_2, \vec{f}_1) \right] & \text{for } f_2 \geq f_a, \end{cases} \tag{10}$$

where

$$G(\Delta f) = \exp\left\{ -1.72 \cdot \left[\Delta f / f_a \right]^2 \right\}, \tag{11}$$

$$H_1(\vec{f}_1, \vec{f}_2) = S_a^{-1} \cdot \int d\vec{v}\ W(\vec{v})\ W(\vec{v} - \lambda\vec{f}_1)\ W(\vec{v} + \lambda\vec{f}_2). \tag{12}$$

Because atmospheric distortions are isotropic their phase is zero. As a result we obtain for the phase $\varphi(\vec{f}) = \arg O(\vec{f})$ difference equations of the form

$$\varphi(\vec{f}) - \varphi(\vec{f} + \Delta\vec{f}) = \Psi(\vec{f}, \Delta\vec{f}); \tag{13}$$

$$\varphi(\vec{f}_1) + \varphi(\vec{f}_2) - \varphi(\vec{f}_1 + \vec{f}_2) = \Theta(\vec{f}_1, \vec{f}_2), \tag{14}$$

where

$$\Psi = \arg E, \quad \Theta = \arg T.$$

The phase φ is reconstructed by solving these equations and by supplementing the phase with the modulus. The distorted image of the object is obtained by taking the inverse Fourier transform.

Before we estimate the accuracy of the restoration we must make some preliminary remarks.

1. In the absence of measurement error φ is restored from the system (13) uniquely ($\varphi(0) = 0$) and from the system (14) to within an arbitrary linear term of the form $\vec{a} \cdot \vec{f}$. However since this term corresponds to a shift of the image without a change in its structure this arbitrariness is unimportant. At the same time, an important preliminary operation in the MKT is combining the short-exposure images being averaged, for example, with respect to their center of gravity. The errors that are admissible in combining the short-exposure images are determined as one tenth of the average size of the images. The MTC is not sensitive to displacements of the short-exposure images.

2. The values of ψ and Θ are measured with an accuracy of up to 2π. To eliminate any possible uncertainty when restoring φ in practice it is best to work with $\exp\{i \cdot \varphi\}$ instead of φ. In what follows, however, to simplify the mathematical calculations we shall ignore this uncertainty.

3. The signal-to-noise ratios (SNR's) and q_E (measurements of the functions (7) and (8) from M recorded short-exposure images) are determined by the expressions[3,5]

$$q_E(\Delta f) = \sqrt{M}\ G(\Delta f), \tag{15}$$

$$q_T(\vec{f}_1, \vec{f}_2) = \begin{cases} q_E(f_2) & \text{for } f_2 < f_a \\ n^{-1/2} \cdot \sqrt{M} \cdot \dfrac{H_1(\vec{f}_1, \vec{f}_2) + H_1(\vec{f}_2, \vec{f}_1)}{\left[H_0(\vec{f}_1) H_0(\vec{f}_2) H_0(\vec{f}_1 + \vec{f}_2) \right]^{1/2}} & \\ \quad \text{for } f_2 \geq f_a. \end{cases} \tag{16}$$

The correlation functions of the errors in the measurements of the phase differences are approximated as

$$\langle \delta\Psi(\vec{f}_1, \Delta\vec{f})\ \delta\Psi(\vec{f}_2, \Delta\vec{f}) \rangle = \sigma_\Psi^2(\Delta\vec{f}) \cdot G^2(\vec{f}_1 - \vec{f}_2); \tag{17}$$

$$\langle \delta\Theta(\vec{f}_1, \vec{f}_2) \cdot \delta\Theta(\vec{f}_1 + \Delta\vec{f}_1, \vec{f}_2 + \Delta\vec{f}_2) \rangle = \sigma_\Theta^2(\vec{f}_1, \vec{f}_2) \cdot G(\Delta\vec{f}_1) \cdot G(\Delta\vec{f}_2) \cdot G(\Delta\vec{f}_1 + \Delta\vec{f}_2), \tag{18}$$

where the variances σ_ψ^2 and σ_Θ^2 are related with the SNR's q_E and q_T the relations[3,9]

$$\sigma_\Psi^2(\Delta\vec{f}) = \frac{1 - G^2(\Delta\vec{f})}{2q_E^2}; \tag{19}$$

$$\sigma_\Theta^2(\vec{f}_1, \vec{f}_2) = \begin{cases} \sigma_\Psi^2(\vec{f}_2) & \text{for } f_2 < f_a, \\ \dfrac{1}{2 \cdot q_T^2(\vec{f}_1, \vec{f}_2)} & \text{for } f_2 \geq f_a. \end{cases} \tag{20}$$

4. Because the methods of speckle interferometry are complicated and nonlinear the actual processing of the short-exposure images is performed, as a rule, digitally on a computer. The starting images are digitized, and their Fourier spectra are obtained with the help of the discrete Fourier transform algorithm. As a result the Fourier values are given at a finite number of points with some constant spacing $\Delta \ll f_a$.

5. For a fixed, small value of the frequency $\vec{f}_2 (f_2 < f_a)$ the MTC transforms into the MKT. In the usual formulation of the MTC, however, the frequency f_2 runs through all possible values right up to the diffraction cutoff frequency $f_d = D/\lambda$.

To simplify the analysis we shall begin with the one-dimensional case.

ESTIMATION OF THE ACCURACY IN THE ONE-DIMENSIONAL CASE

When the phase φ_m is given at the discrete points $f_m = m\Delta$ the difference equations (13) and (14) assume the form

$$\varphi_{m+1} - \varphi_m = \Psi_{m+1}; \tag{21}$$

$$\varphi_m + \varphi_l - \varphi_{m+l} = \Theta_{m,l}. \tag{22}$$

The MKT algorithm for restoring the phase follows from Eq. (21) in the form of the sum

$$\varphi_p = \sum_{m=1}^{p} \Psi_m. \tag{23}$$

In this case, the variance of the error for $f_p > f_a$ satisfies the following relations:[9]

$$\langle(\delta\varphi_p)^2\rangle = \sum_{m_1, m_2=1}^{p} \langle\delta\Psi_{m_1}\delta\Psi_{m_2}\rangle \approx \sum_{m=1}^{p} \langle\delta^2\Psi_m\rangle \sum_{q=-\infty}^{\infty} G^2(q\Delta) = \frac{f_p}{f_a}\cdot q_0^{-2} = \frac{\lambda\cdot f_p}{M\cdot r_0}. \tag{24}$$

The linear increase in the error corresponds to its obvious accumulation in the summation process (23).

In the MTC the system of equations (22) is overdetermined: the number of equations is greater than the number of unknowns. Because the errors in the measurements of Θ are independent (on different correlation intervals), as a rule, the system does not have an exact solution. In this case, the following recurrence scheme is employed to construct an approximate solution, as proposed by Weigelt.[1,2] Because the choice of the linear term is arbitrary, the phase φ_1 is set equal to zero. Then the values of φ_2 and φ_3 are found uniquely from Eq. (22) with $m = l = 1$ and $m = 1$, $l = 2$, respectively. The values of φ_4 and φ_5 are obtained by two methods: 1) ($m = 1$, $l = 3$) and ($m = l = 2$) and 2) ($m = 1$, $l = 4$), ($m = 2$, $l = 3$). Then the results are averaged. The values of φ_6 and φ_7 are found by averaging over the three corresponding variants of the estimates, etc. The mathematical experiment based on statistical modeling on a computer revealed an important drawback of this approach: the accuracies of the phases obtained at the middle frequencies ($f_a < f \le 10f_a$) is appreciably lower than at high frequencies ($10f_a < f < f_d - 5f_a$), where a large number of independent variants is averages. Figure 1 shows as an illustration an experimental plot of the error $\delta\varphi_m$ of the Fourier spectrum of a point source with $D/r_0 = 128$ and $M = 30$ short-exposure images.

FIG. 1. Experimental plot of the error $\delta\varphi_m$ in the Fourier spectrum of a point source obtained by Weigelt's method with 30 images as a function of the normalized spatial frequency f/f_d (f_d is the diffraction frequency).

The solution of the equation

$$\sum_{l,m=1}^{L_d} q_T^2(f_l, f_m)\cdot\left[\Theta_{m,l} - \varphi_m - \varphi_l + \varphi_{m+l}\right]^2 = \min \tag{25}$$

or the equivalent system of equations

$$\varphi_m = \sum_{l=1}^{L_d} \left[-\Theta_{l,m-l} + \varphi_{m-l} + \varphi_l\right]\cdot q_T^2(f_l\cdot f_{m-l})/Q_m, \tag{26}$$

where

$$Q_m = \sum_{l=1}^{L_d} q_T^2(f_l, f_{m-l}), \tag{27}$$

$L_d = D/\lambda\Delta$ is the number of the reading corresponding to the diffraction cutoff frequency, is free of this drawback. We note that the equations (26) can be used as a basis for an iteration algorithm for searching for an optimal solution in the sense (25). This algorithm usually converges within 10 iterations. The experiment shows that the average error of the estimate obtained right up to $f \le f_d - 5f_a$ is virtually independent of the frequency and is appreciably smaller than the errors in Weigelt's method (Fig. 2).

It should be noted that the ultrahigh frequency range $f > f_d - 5f_a$ is not important, since here the OTF drops to zero. Physically the estimate (26) is the average of different variants of the estimate obtained from all existing couplings. We shall estimate the magnitude of this error.

FIG. 2. Experimental plot of the error in estimating the phase $\delta\varphi_m$ using the proposed iteration algorithm with 30 images as a function of the normalized spatial frequency f/f_d (f_d is the diffraction frequency).

Let us assume that the error $\delta\varphi_m$ is much smaller than the errors $\delta\Theta_{m,1}$ made in measuring the phases of triple correlations (8). Then we obtain from Eq. (26)

$$\delta\varphi_m \approx -\sum_{l=1}^{L_d} q_T^2\cdot\delta\Theta_{l,m-1}/Q_m. \qquad (28)$$

From here, using Eqs. (16)—(20) and the fact that the OTF $H_0(\vec{f})$ does not change much on intervals of width f_a (this is valid if $D/r_0 \gg 1$), we obtain the following expression for the variance $\delta\varphi_m$ for $f_m > f_a$:

$$\langle(\delta\varphi_m)^2\rangle = \frac{1}{2}\sum_{l=-\infty}^{\infty} G^2(l\Delta)/Q_m. \qquad (29)$$

Now, since

$$\sum_l G^2(l\Delta) \approx f_a/\Delta, \qquad (30)$$

$$Q_m \approx \frac{M}{n} f_d/\Delta, \qquad (31)$$

where for a one-dimensional aperture the effective number of regions of correlation n is estimated as $2(D/r_0)$, we obtain finally

$$\langle(\delta\varphi_m)^2\rangle \approx \frac{n}{2M}\frac{f_a}{f_d} = \frac{1}{M} = q_0^{-2}. \qquad (32)$$

Thus the previous assumption that $\langle(\delta\varphi_m)^2\rangle \ll \langle(\delta\Theta)^2\rangle$ has been justified, and as a result we have found that in the one-dimensional case the MTC is much more accurate than the MKT. Moreover the following relation is satisfied:

$$\langle(\delta\varphi_m)^2\rangle^{1/2} = q_0^{-1}. \qquad (33)$$

This relation reflects the fact that the accuracies of the estimates of the modulus and phase (in the MTC) are comparable in magnitude. Here it must be emphasized that the last property is of more general significance. Dividing the restoration problem into the subproblems of estimating the modulus and phase of the spectrum is an artificial mathematical device that makes it easier to construct a practical algorithm. When the processing is done correctly the accuracy of both estimates should be the same. In our case the subproblem of restoring the modulus has an obvious solution, and its accuracy q_0 can serve as a criterion (in the sense of Eq. (33)) for optimality of a specific phase restoration algorithm.

ESTIMATION OF THE ACCURACY IN THE TWO-DIMENSIONAL CASE

The difference equations of the MKT are now

$$\begin{cases} \varphi_{m+1,l} - \varphi_{m,l} = \Psi^1_{m+1,l}, \\ \varphi_{m,l+1} - \varphi_{m,l} = \Psi^2_{m,l+1}. \end{cases} \qquad (34)$$

The system (34) is overdetermined, and its construction is optimal based on the least-squares criterion:

$$\sum_{l,m}\left\{\left[\Psi^1_{m+1,l} - \varphi_{m+1,l} + \varphi_{m,l}\right]^2 + \left[\Psi^2_{m,l+1} - \varphi_{m,l+1} + \varphi_{m,l}\right]^2\right\} = \min. \qquad (35)$$

The solution is constructed based on equations of the form (10):

$$\varphi_{m,l} = \frac{1}{4}\left\{\left(\varphi_{m-1,l} + \Psi^1_{m,l}\right) + \left(\varphi_{m+1,l} - \Psi^1_{m+1,l}\right) + \left(\varphi_{m,l-1} + \Psi^2_{m,l}\right) + \left(\varphi_{m,l+1} - \Psi^2_{m,l+1}\right)\right\}. \qquad (36)$$

This method of reconstruction can be interpreted essentially as a collection of two operations: a) determination of a set of variants of estimates of the phase $\varphi_{m,1}$ by "joining" the measured phase differences Ψ on different one-dimensional contours, connecting the frequencies studied $f_{m,1}$ with the reference frequency $f = 1$, where $\varphi_{0,0} = 0$, and b) averaging these estimates. Since the variants of their errors, in accordance with Eq. (24), are determined approximately as $\Delta(m+n)q_0^{-2} / f_a$ and number of quasi-independent variants is equal to $((m + n)\Delta / f_a)^2$, the variance of the error in the resulting estimate should be equal to q_0^{-2}. An exact mathematical analysis of the accuracy of restoration from Eq. (34) based on Eq. (36) confirms this result.[10,11]

As regards the MTC, the accuracy of this method remains the same: for each phase $\varphi_{m,n}$ the number of equations of the form (22) as well as the variance of the error in the corresponding measurement Θ increase by a factor of (D/r_0). To obtain a more rigorous proof it is sufficient to perform calculations analogous to the above-presented equtions (25)–(32).

Thus we arrive at the conclusion that in the two-dimensional case the MKT and MTC have the sane accuracies, satisfying the criterion (33). At the same time the MKT is much simpler to implement, since it does not require working with four-dimensional arrays. Attempts to eliminate this drawback of the MTC by using triple correlations $T(\vec{f}_1, \vec{f}_2)$, formulated independently only for the one-dimensional sections in the frequency plane,[14] lead to sharply lower accuracy. This is explained by the fact that for each value of the phase the number of accessible equations of the form (22) decreases by a factor of (D/r_0), while the accuracy of each of them remains the same.

Nevertheless, in the MTC the invariance to displacements of the short-exposure images is attractive. This is especially important when processing so-called photocount images,[13] whose distributions are collections of small numbers of separate photopulses. Since the integration errors can be significant for them, the accuracy of the MKT is appreciably lower. In this connection the hybrid algorithm, based on the use of triple correlations of the form (22), but with a fixed value of $\vec{f}_2 (f_2 < f_a)$, will apparently be more efficient. For it the phase restoration is similar to that described for the MKT.

In conclusion we note that this analysis of the accuracy was performed without any restrictions on the size of the object. At the same time, in processing short-exposure images of an object, when the angular size R is much less than the average atmospheric resolution $R_a = \lambda/r_0$ and the size of the region of characteristic variation of the Fourier spectrum λ/R is much larger than the size f_a of the region of correlation of the errors, the accuracy q_0 and correspondingly $\left\langle (\delta\varphi)^2 \right\rangle^{-1/2}$ can be R_a/R times higher owing to the smoothing of the estimates. As a result, for astronomical objects at the limit of resolution of the telescope accuracy of the order of $\sqrt{M} \cdot D / r_0$ can b achieved. This reflects the fact that for such small objects a satisfactory estimate of the image can be reconstructed even from a single short-exposure image.

REFERENCES

1. A.W. Lohman, G.R. Weigelt, and B. Wirnitzer, Appl. Opt. **22**, No. 24, 4028 (1983).
2. M.J. Northcott, G.R. Ayers, and J.C. Dainty, JOSA A **5**, No. 7, 986 (1988).
3. G.R. Ayers, M.J. Northcott, and J.C. Dainty, JOSA A **5**, No. 7, 963 (1988).
4. T. Nakajima, JOSA A **5**, No. 9, 1477 (1988).
5. I.P. Plotnikov, "*Development and study of image processing algorithm for multi-aperture optical systems*". Author's Abstact of Dissertat., Moscow Physico-Technical Institute, Moscow (1988).
6. P.A. Bakut, I.P. Plotnikov, and I.A. Rozhkov, et al., Opt. Atmos. **1**, No. 1, 55 (1988).
7. A. Labeyrie, Astron. Astrophys. **6**, 85 (1970).
8. K.T. Knox and B.J. Thompson, Astrophys. J. **193**, P.L. 45 (1974).
9. P.A. Bakut, E.N. Kuklin, and A.D. Ryakhin, Opt. Spekrtosk. **58**, No. 6, 1314 (1985).
10. P.A. Bakut, A.D. Ryakhin, and K.N. Sviridov, et al., Opt. Spektrosk. **59**, No. 1, 137 (1985).
11. P.A. Bakut, A.D. Ryakhin, and K.N. Sviridov, et al., Radiotekh.Electron. **33**, No. 7, 1446 (1988).
12. P.A. Bakut, A.D. Ryakhln, and K.N. Sviridov, et al., Izv. Vyssh.Uchebn. Zaved. SSSR, Ser. Radiofiz. **29**, No. 3, 274 (1986).
13. A.P. Bakut, V.V. Kolin, and I.A. Rozhkov, et al., Zarub. Radioelektron., No. 9, 14 (1989).
14. G.R. Ayers, J.C. Dainty, and M.J. Northcott Proc. of SPIE **808**, 19 (1987).