NUMERICAL SIMULATION OF THE SPATIOTEMPORAL STRUCTURE OF THE SEA SWELL SURFACE IN OPTICAL PROBLEMS

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In this paper we present a spatiotemporal model of the sea swell surface constructed based on the spectral methods of numerical modeling of random fields. This model is then used for studying the statistical properties of optical radiation reflected from the sea surface.

1. DESCRIPTION OF THE SPECTRAL METHODS FOR NUMERICAL SIMULATION OF UNIFORM RANDOM SPATIOTEMPORAL FIELDS

Let us consider the problem of simulation of a uniform Gaussian field $w(x, y, t)$ with zero mean value $M\,w(x, y, t) \equiv 0$, where the variable t is time. So at a fixed t the function $w(x, y, t)$ is a uniform random field, while at a fixed point (x, y) the function $w(x, y, t)$ is a stationary random process.

The uniform Gaussian field is uniquely described by its correlation function $K(x, y, t) = M\,w(0, 0, 0)w(x, y, t)$ or by the spectral measure $F(\mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu) = f(\lambda, \nu, \mu)\mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu$ with the density $f(\lambda, \nu, \mu)$. (The density can be generalized one.)

$$f(\lambda, \nu, \mu) = \frac{2}{(2\pi)^3} \int\limits_{R^3} \cos(\lambda x + \nu y + \mu t)\, K(x, y, t)\, \mathrm{d}x\, dy\, dt$$

or

$$K(x, y, t) = \int\limits_P \cos(\lambda x + \nu y + \mu t)\, f(\lambda, \nu, \mu)\, \mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu\ ,$$

here P is a half–space in R^3, $P \cup (-P) = R^3$, $P \cap (-P) = \{0\}$, in which the spectral measure is defined. Symbols λ, ν, and μ denote the Cartesian coordinates in the half–space P. We can write the spectral representation for an uniform random field[3] in the form

$$w(x, y, t) = \int\limits_P \cos(\lambda x + \nu y + \mu t)\, \xi(\mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu) -$$

$$- \int\limits_P \sin(\lambda x + \nu y + \mu t)\, \eta(\mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu)\ ,$$

where ξ and η are the Gaussian orthogonal stochastic measures on P, such that the following conditions are fulfilled in $A, B \subset P$:

a. $M\,\xi(A) = M\,\eta(B) = 0$;

b. $M\,\xi^2(\mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu) = M\,\eta^2(\mathrm{d}\lambda\, \mathrm{d}\nu\, \mathrm{d}\mu) = f(\lambda, \nu, \mu)\, d\lambda\, d\nu\, d\mu$;

c. $M\,\xi(A)\,\eta(B) = 0$;

d. $M\,\xi(A)\,\xi(B) = M\,\eta(A)\,\eta(B) = 0$ for $A \cap B = \varnothing$.

Based on the spectral representation of the field, approximate models for $w(x, y, t)$ of the form

$$w_n(x, y, t) = \sum_{i=1}^{n} a_i\, [\xi_i \cos(\lambda_i x + \nu_i y + \mu_i t) +$$

$$+ \eta_i \sin(\lambda_i x + \nu_i y + \mu_i t)]$$

will be reffered to as the spectral ones. Here $a_i > 0$, (ξ_i, η_i) are independent random vectors, equally distributed over plane, such that $M\xi_i = M\eta_i = M\xi_i\eta_i = 0$, $D\xi_i = D\eta_i = 1$.

Mikhailov[4] has proposed the general principle for approximate simulation of the uniform Gaussian fields, which allows exact reconstruction of the field correlation function based on the splitting and randomization of the spectrum, when vectors $(\lambda_i, \nu_i, \mu_i)$ are randomly selected in nonoverlapping regions with the distributions induced by the spectral measure $F(d\lambda\, d\nu\, d\mu)$. Randomized spectral models of the Gaussian fields are used for solving a wide range of problems by statistical simulations.

The fields $w_n(x, y, t)$ are the uniform ones. In the case of strong uniformity (when there exists an invariance of finite measure distributions relative to shifts), the isotropy of vectors (ξ_i, η_i) on a plane is both the necessary condition and the sufficient one. It may happen so that the representation

$$w_n(x, y, t) = \sum_{i=1}^{n} a_i\, r_i \cos(\lambda_i x + \nu_i y + \mu_i t + \varphi_i)$$

is more efficient for making simulations. Here $r_i = \sqrt{\xi_i^2 + \eta_i^2}$ and φ_i are random variables uniformly distributed over the interval $[0, 2\pi]$ and independent on r_i. Let us then assume that vectors (ξ_i, η_i) are Gaussian. Let the spectral space be divided into sets of zero intersection $P = \sum\limits_{i=1}^{n} \Lambda_i$, $\Lambda_i \cap \Lambda_j = \varnothing$ for $i \neq j$. Consider now the following model: $a_i^2 = F(\Lambda_i)$, $(\lambda_i, \nu_i, \mu_i)$ are random vectors, whose distributions are grouped in corresponding regions Λ_i and induced by the spectral measure F. This is a randomized model having correlation function, which coincides with the correlation function of the field $w(x, y, t)$ under simulation. In Ref. 5 the convergence of finite measure distributions under condition that $\max\limits_{i \le n} F(\Lambda_i) \to 0$ for $n \to \infty$ has been proved for the given model. The problems of weak convergence of spectral models of uniform Gaussian fields have been studied in Ref. 6.

2. STATISTICAL MODEL OF THE WIND–DRIVEN SEA WAVES

2.1. The experimental data on statistical properties of the wind–driven sea waves show that one can describe the sea swell surface with a uniform Gaussian random field of

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heights of the elevations $w(x, y, t)$ above the mean sea level to a high degree of accuracy.[7]

Let the coordinates x and y be fixed and sea waves be considered as a random process

$u(t) = w(x, y, t)$.

In this case the process $u(t)$ is the Gaussian and stationary one with the correlation function

$K_t(t) = K(0, 0, t)$

and spectral power density

$$f_\mu(\mu) = \int_{R^2} f(\lambda, \nu, \mu)\, d\lambda\, d\nu .$$

Let now the variable t be fixed and sea waves be considered as a random field $v(x, y)$. In this context the field $v(x, y) = w(x, y, t)$ is the uniform Gaussian one with the correlation function

$K_{xy}(x, y) = K(x, y, 0)$

and spectral power density

$$f_{\lambda\nu}(\lambda, \nu) = \int_R f(\lambda, \nu, \mu)\, d\mu .$$

The parameters θ and ρ are quite convenient for use as the spectral ones in parallel with the parameters λ and ν

$$\lambda = \rho \cos\theta \ , \ \nu = \rho \sin\theta \ , \ \rho^2 = \lambda^2 + \nu^2 \ , \ \theta = \arg(\lambda + i\,\nu) .$$

Corresponding spectral power densities are interrelated by the following expressions:

$$f_{\lambda\nu}(\lambda, \nu) = \frac{1}{r(l, n)} f_{\rho\theta}(\rho(\lambda, \nu), \theta(\lambda, \nu)) ;$$

$$f_{\rho\theta}(\rho, \theta) = \rho\, f_{\lambda\nu}(\lambda(\rho, \theta), \nu(\rho, \theta)) .$$

Provided that f_ρ is the unconditional spectral power density on ρ, i.e.,

$$f_{\rho\theta}(\rho, \theta) = f_\rho(\rho)\, f_{\theta|\rho}(\theta|\rho) ;$$

$$\int_0^{2\pi} f_{\theta|\rho}(\theta|\rho)\, d\theta = 1 ; \qquad f_\rho(\rho) = \int_0^{2\pi} f_{\rho\theta}(\rho, \theta)\, d\theta , \qquad (1)$$

the spectral power densities f_ρ and f_μ are related by the following equations:

$$f_\rho(\rho) = 0.5\sqrt{\frac{g}{\rho}}\, f_\mu(\mu(\rho)) , \quad \mu(\rho) = \sqrt{g\,\rho} ,$$
$$f_\mu(\mu) = \frac{2\,\mu}{g}\, f_\rho(\rho(\mu)) , \qquad \rho(\mu) = \mu^2 / g , \qquad (2)$$

where g is the acceleration due to gravity.

Thus, in order to describe the spatial structure of the random field of sea waves it is sufficient to set the "frequency" spectrum f_μ and "angular" spectrum $f_{\theta|\rho}$.

We shall describe the stochastic structure of the sea surface considering its temporal variations. In the case of monochromatic wave when the spectrum is concentrated at a single point (λ, ν, μ) with the weight A^2, the spectral representation of a Gaussian spatiotemporal field has the form

$$w(x, y, t) = A\,[\xi \cos(\lambda\, x + \nu\, y + \mu\, t) - \eta \sin(\lambda\, x + \nu\, y + \mu\, t)] ,$$

where ξ and η are independent standard normal variables, A^2 is the field variance. This representation may also be written in the form

$$w(x, y, t) = A\, r \cos(\lambda\, x + \nu\, y + \mu\, t + \varphi) =$$
$$= A\, r \cos[\rho\,(x \cos\theta + y \sin\theta) + \mu\, t + \varphi] ,$$

where $r = \sqrt{x^2 + h^2}$ is the random variable obeying the Rayleigh distribution, $\varphi = \arg(\xi + i\eta)$ is the random variable uniformly distributed over the interval $[0, 2\pi]$. The variables r and φ are independent. Such a field is a sinusoidal wave of length $2\pi/\rho$ and amplitude $A\, r$, running along the direction $-\,\mathrm{sgn}(\mu)\theta$. In accordance with the spectral model an arbitrary uniform Gaussian field can be presented by superposition of such fields

$$w(x, y, t) = \sum_k A_k\, [\xi_k \cos(\lambda_k x + \nu_k y + \mu_k t) -$$
$$- \eta_k \sin(\lambda_k x + \nu_k y + \mu_k t)] .$$

As known from hydrodynamic theory $\mu^2 = g\,\rho \tanh(\rho H)$ for a monochromatic wave. For the case of a deep water when $\rho H \gg 1$ (H is the reservoir depth), we have $\mu^2 = g\,\rho$.

2.2. Let us describe, following Ref. 8, a number of approximations for the sea waves spectrum. The frequency spectrum of a mildly sloping swell closely follows the function

$$f_\mu = 6\, d_0(\mu_{max} / \mu)^5\, \mu^{-1} \exp[-\,1.2(\mu_{max} / \mu)^5]$$

in a wide frequency range. Here d_0 is the variance of the field of heights of elevations, and μ_{max} is the frequency of the spectrum maximum. (Here and further it is assumed that the frequency spectrum is specified on the positive semi–axis, i.e. $f_\mu(\mu) = 0$ for $\mu < 0$). In the case of wind–driven waves the high–frequency spectrum is more rich. One may separate out from the spectrum the gravitational (the frequency up to 5 Hz, wavelength longer than 7 cm), gravitational–capillary (the frequency from 5 to 50 Hz, wavelengths from 7 to 0.7 cm), capillary (the frequency up to 10^4–10^6 Hz), and viscous ranges. The gravitional–capillary and the capillary ranges of spectrum are approximated with the functions $f_\mu(\mu) = a\mu^{-4}$ and $f_\mu(\mu) = b\mu^{-7/3}$, respectively. In its turn the portion near the spectrum principal maximum, the transitional, and equilibrium intervals are usually separated out from the gravitational range.

The approximation of the gravitational range of the frequency spectrum, which takes into the consideration the separation of the range into three intervals, has the form

$$f_\mu(\mu) = \begin{cases} (n+1) d_0(\mu_1)(\mu_{max}/\mu)^n \mu^{-1} \times \\ \times \exp\left\{-\dfrac{(n+1)}{n}\left[\left(\dfrac{\mu_{max}}{\mu}\right)^n - \left(\dfrac{\mu_{max}}{\mu_1}\right)^n\right]\right\}, & \mu < \mu_1, \\ f_\mu(\mu_1) + \dfrac{f_\mu(\mu_2) - f_\mu(\mu_1)}{\mu_2 - \mu_1}(\mu_2 - \mu_1), & \mu \in (\mu_1, \mu_2), \\ 7.8 \cdot 10^{-3} g^2 \mu^{-5}, & \mu \in (\mu_2, \mu_3), \end{cases}$$

(3)

there μ_{max}, n, μ_1, μ_2, $d_0(\mu_1)$ are the parameters of wind–driven sea waves ($\mu_s \approx 30\ s^{-1}$ is the maximum frequency of the spectrum of gravitational waves, $g = 9.8\ m/s^2$).

There exists a number of approximations of the gravitational range, which do not take into consideration the separation into intervals. Typically it is an approximation of the form

$$f_\mu(\mu) = A\,\mu^{-k}\exp(B\,\mu^{-n})\ . \qquad (4)$$

In this case the variance of the field of elevations d_0 and the frequency of maximum of the spectrum μ_{max} are obtained from the following expressions:

$$d_0 = A\,B^{(n-k)/n}\frac{1}{n}\,\Gamma\left(\frac{k-1}{n}\right),$$

$$B = \frac{k}{n}\,\mu^{n}_{max}\ .$$

The expression (4) may be written as

$$f_\mu(\mu) = n\left(\frac{k}{n}\right)^{(k-1)/n} d_0\,\mu^{k-1}_{max}\,\mu^{-k}\times$$

$$\times\exp\left[-\frac{k}{n}\left(\frac{\mu_{max}}{\mu}\right)^{n}\right]\Big/\,\Gamma\left(\frac{k-1}{n}\right).$$

Let us present, as an example, the Pearson–Moscowitz approximation

$$f_\mu(\mu) = 8.1\cdot 10^{-3}\,g^2\,\mu^{-5}\exp\left[-\,0.74\left(\frac{g}{V\mu}\right)^4\right],$$

where V is the wind velocity (m/s) at an altitude 10 m above the sea level.

The frequency spectrum (4) corresponds to the spectrum

$$f_\rho(\rho) = 0.5\,g^{-(k-1)/2}\,A\,\rho^{-(k+1)/2}\exp\left(-\,B(g\,\rho)^{-n/2}\right).$$

As to the angular spectrum $f_{\theta|\rho}$, the following its form is most frequently used:

$$f_{\theta|\rho}(\theta\,|\,\rho) = 2^m\,\frac{\Gamma^2\left(\dfrac{m+1}{2}\right)}{\Gamma(m+1)}\cos^m(\theta)\ ,$$

where m depends on ρ, and $\theta \in [-\pi/2,\ \pi/2]$. Below we use a simplified approximation

$$f_{\theta|\rho}(\theta\,|\,\rho) = \frac{2}{\pi}\cos^2\theta\ . \qquad (5)$$

For numerical simulation we employed the spectra (3) and (5) with the following parameters (see Ref. 8):

$$n = 5;\ \mu_1 = 1.8\,\mu_{max}\,\tilde{\mu}^{-0.7}\ ;\ \ \mu_2 = 2.0\,\mu_{max}\,\tilde{\mu}^{-0.7}\ ;$$

$$\tilde{\mu} = V\,\mu_{max}/g\ ;\ \ d_0 = 0.001\,27\,g^{-2}V^4\,\tilde{\mu}^{-3.19}\ ;$$

where μ_{max} is the frequency of maximum of the spectrum f_μ, V is the wind velocity (m/s) at the altitude 10 m above the sea level. In the context of the accepted approach, statistical characteristics of the sea waves are determined by the wind velocity V and the frequency of spectral maximum μ_{max}.

3. ALGORITHM FOR NUMERICAL SIMULATION OF THE SEA SWELL SURFACE

To make modeling of the sea swell surface, we employed the method of splitting and randomization of spectrum.[4] The approximate model $w^*(x, y, t)$ with the spectrum (2), (3), and (5) we used is as follows:

$$w^*(x,\ y,\ t) = \sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}\,[r_{ij}\cos(x\,\rho_i\cos\theta_i\ y\,\rho_i\sin\theta_i\ \mu_i\,t+\varphi_{ij})+r'_{ij}\,c$$

Here ρ_i are random variables with the probability density function proportional to f_ρ in the corresponding sets A_i,

$A_i = [\rho^*(i-1)\,/\,(M-1),\ \rho^* i\,/\,(M-1)]$,

$i = 1,\ \dots\ ,\ M-1\ ,\ \ A_m = [\rho^*,\ \infty]$;

θ_i are random variables distributed over the intervals

$B_i = [\pi(j-1)\,/\,(2\,N),\ \pi\,j\,/\,(2\,N)]$

with the probability density proportional to $f_{\theta|\rho}$ from Eq. (5); r_{ij} and r'_{ij} are random variables obeying the Rayleigh distribution; φ_{ij} and φ'_{ij} are uniformly distributed over the interval $[0,\ 2\pi]$; $\mu_i = \sqrt{\rho_i\,g}$; and, $a^2_{ij} = \int\limits_{A_i}\int\limits_{B_j} f_{\rho\theta}(\rho,\ \theta)\,d\rho\,d\theta$.

Random variables ρ_i were simulated by the method of the inverse distribution function, and θ_i was simulated by the rejection method (with a linear majorant for $j > N/2$ and a constant majorant for $j \le N/2$, where N is even number). The results of simulation of the spatiotemporal structure of the sea swell surface are shown in Fig. 1. The modeling algorithm uses three parameters, M, N, and ρ^* which determine the accuracy of the approximation. Field w^* possesses required density and becomes asymptotically Gaussian at

$$\max(M,\ N) \to \infty,\ \ \rho^* \to \infty\ .$$

These conditions are sufficient for w^* to be weakly converging in the space of differentiable functions.[6]

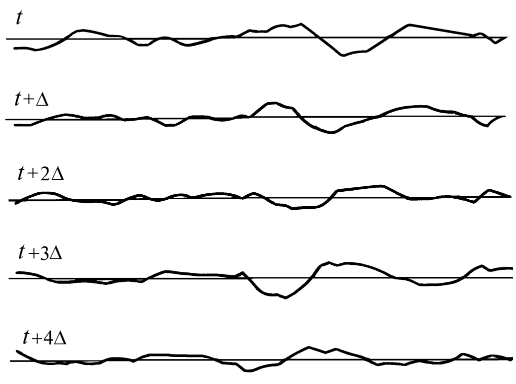


FIG. 1. Examples of the model sea surface relief in a time order sequence.

4. APPLICATION OF THE NUMERICAL MODEL TO STUDYING THE STATISTICAL PROPERTIES OF OPTICAL RADIATION REFLECTED FROM THE SEA SURFACE

Let us consider the following modeling problem as an example for application of the statistical spatiotemporal simulation of the wind–driven sea waves. Let the variations of heights of elevations above the mean sea level be described by the function $z = w(x,\ y,\ t)$, where $w(x,\ y,\ t)$ is the uniform Gaussian field with zero mean value and spectrum (1)–(3), and (5) given in Section 2. Let also a light beam be propagating from an arbitrary point (x_0, y_0, z_0) above the sea surface normally toward the plane XY. Let a detector, recording the intensity of radiation reflected from the sea surface $J(t)$ be also at this point. The "divergence" of the light beam incident on the surface is taken into account by the weighting function[9]

$$p(x,\, y) = \text{const}\, \exp\{-\,\alpha[(x - x_0)^2 + (y - y_0)^2]\}\ .$$

It is also assumed that a water surface element reflects incident radiation toward the detector if this surface element is horizontal accurate to a small value ε

$$\left(1 + \left[\frac{\partial w}{\partial x}(x,\, y,\, t)\right]^2 + \left[\frac{\partial w}{\partial y}(x,\, y,\, t)\right]^2\right)^{-1/2} \le 1 - \varepsilon\ . \quad (6)$$

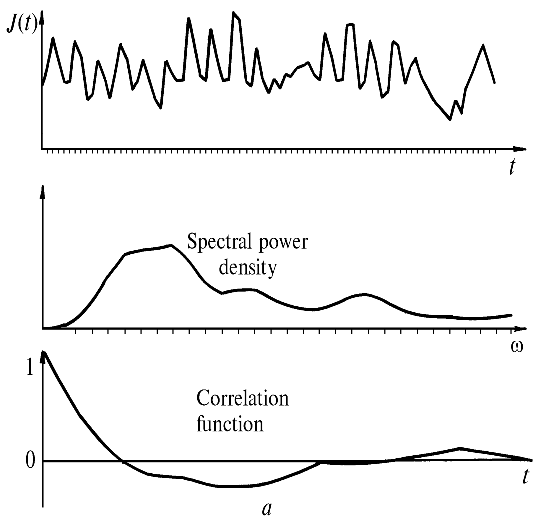
Thus, the intensity of radiation recorded with the detector is determined by the following expression:

$$J(t) = \int\limits_{R^2} \varphi_\varepsilon(x,\, y,\, t)\, p(x,\, y)\, \mathrm{d}x\, \mathrm{d}y\ , \quad (7)$$

where $\varphi_\varepsilon(x, y, t)$ is equal to unity under condition (6) otherwise being zero. Note that in our model we do not consider the interaction of light with the atmosphere and do not take into account the radiation scattered by water medium coming to the detector.

In numerical experiments we used the model of sea surface described in Section 3, the integral (7) was calculated by approximate methods. From a calculational series of the random process $J(t)$ we estimated such its statistical characteristics as correlation function and spectral power density.

Figures 2 and 3 present examples of the random process $J(t)$ and calculational results (for $\alpha = 0.05$, $\varepsilon = 0.01$) demonstrating qualitatively dependence of the statistical properties of $J(t)$ process on wind velocity above the sea surface. This results allow the assumption to be made that the approach presented in this paper may be useful for solving optical problems in active and passive sounding of water as well as a number of other applied problems, spatiotemporal stochastic structure of the sea swell has to be taken into account.



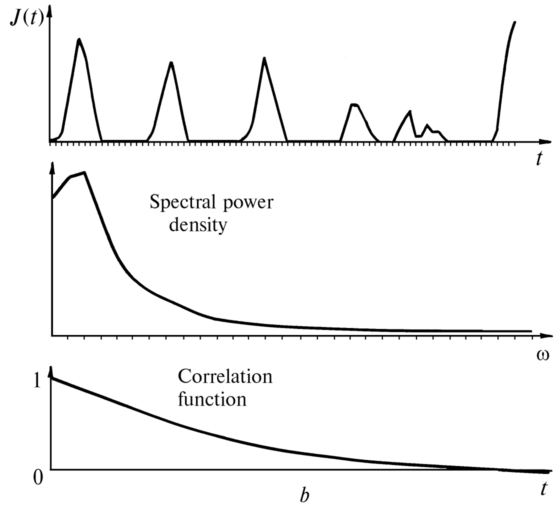


FIG. 2. Intensity, spectral power density, and correlation function of signal reflected from sea swell surface for wind velocity of 1 *(a) and* 6 m/s *(b).*

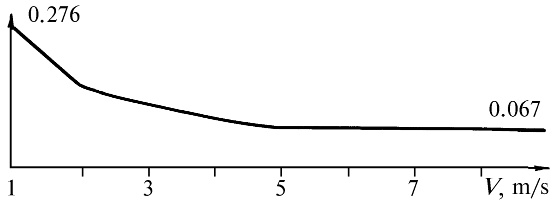


FIG. 3. Dependence of the variance of spectral power density of signal reflected from the sea swell surface on wind speed (m/s).

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