Method for simulating random perturbations of the wave front within a wide range of fluctuation scales

T.A. Sheremet'eva and G.N. Filippov

Scientific–Research Institute of Examination of Optical–Electronic Devices and Systems at S.I. Vavilov State Optical Institute, Sosnovyi Bor, Leningrad Region

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We propose a method for numerically simulating stochastic fields with the spatial inhomogeneities varying in a wide range. The method allows one to combine the modal approach based on expansion of random homogeneous isotropic field over Karhunen–Loeve–Obukhov functions with a spectral approach or the sliding average approach. Advantages of the method over the other known methods of random field generation manifest themselves, in particular, in the case when a random field with a wide range of inhomogeneities in relatively small areas is to be induced. This problem arises, for example, in simulating the laser beam wave front propagated through the atmosphere.

Theoretical investigations of wave propagation through randomly inhomogeneous media use the approaches that are based on solving the stochastic wave equation or the equations for the statistical moments of the field. However, solution of the corresponding differential equations for the statistical moments of fourth order is a very difficult mathematical problem, for which only some asymptotic solutions have been obtained so far. Therefore, the method of statistical tests based on the phase screen model[1] is an important instrument, in nonlinear optics, for investigation of the wave propagation through randomly inhomogeneous media.

Numerical formation of the phase screens – random fields with prescribed statistics – is the key issue in constructing such an investigation method. To form a homogeneous isotropic random field, one usually uses the spectral method, and more rarely the method of sliding average. The spectral method makes it possible to form the fields, the maximum scale of fluctuations in which does not exceed a half size of the domain the calculations are being performed in. On the other hand, there are limitations on the minimum fluctuation scale, i.e., the step of the calculation grid should be less than the sixth part of the minimum scale. The combination of these conditions significantly limits the range of spatial fluctuations that can be reconstructed by this method. The method of moving summing is used in simulating small-scale fluctuations.[2] The method of enclosed grids[3] and the method of subharmonics[4] were proposed for extending the range of the spectral method applicability to the region of the large-scale fluctuations. The modification of these methods[5] improves the accuracy of reconstruction of the low-frequency spatial fluctuations.[6] When applying these methods, one should check the number of iterations sufficient for reconstructing fields with a preset correlation function. The larger the preset outer scale, the larger is the number of iterations needed.[6]

The modal approach is known, in which the random field is formed as a superposition of basis functions. In practice, a limited number of basis functions impose a limitation on the spatial scales that can be reconstructed. The modal approach adequately reconstructs the large-scale fluctuations.[6]

In this paper, we propose the method for the formation of a homogeneous isotropic random Gaussian field with a wide range of the fluctuation scales that combines the spectral and modal approaches. The stochastic field S is presented, in the calculation domain, as a sum of two (or more) statistically independent fields[7]:

$$S(\boldsymbol{\rho}) = S_{H1}(\boldsymbol{\rho}) + S_{H2}(\boldsymbol{\rho}) + \ldots + S_{HN}(\boldsymbol{\rho}) + S_B(\boldsymbol{\rho}) . \quad (1)$$

Each field is characterized by its own correlation function $B_{H1}(\boldsymbol{\rho} - \boldsymbol{\rho}')$, and the set of this fields forms the field with the correlation function $B(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ corresponding to the preset spectrum. In this case the relation

$$B(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = B_{H1}(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) + \\ + B_{H2}(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) + \ldots + B_{HN}(|\boldsymbol{\rho} - \boldsymbol{\rho}'|) + B_B(|\boldsymbol{\rho} - \boldsymbol{\rho}'|).$$

is true.

The stochastic fields S_{Hi}, corresponding to the first terms of this series, are formed on the basis of the modal approach, and they correspond to the low-frequency and medium-frequency components of the simulated field. The last field S_B is formed by the method of sliding average or by the spectral method, and it corresponds to high-frequency component. The formation starts with the first field of this series according to the following algorithm:

Eigenfunctions $\psi_k(\boldsymbol{\rho})$ (Karhunen–Loeve–Obukhov functions) and eigenvalues Λ_k of the integral operator are determined. The kernel of the operator is the correlation function $B(\boldsymbol{\rho}, \boldsymbol{\rho}') = B(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ of the stochastic field $S(\boldsymbol{\rho})$ simulated:

0235-6880/00/05 492-05 $02.00

$$\iint w(\boldsymbol{\rho}')\, B(\boldsymbol{\rho}, \boldsymbol{\rho}')\, \psi_k(\boldsymbol{\rho}')\, \mathrm{d}^2\boldsymbol{\rho}' = \Lambda_k\, \psi_k(\boldsymbol{\rho})\ ; \quad (2)$$

$$B(\boldsymbol{\rho}, \boldsymbol{\rho}') = \langle |S(\boldsymbol{\rho}) - \langle S(\boldsymbol{\rho})\rangle| \cdot |S(\boldsymbol{\rho}') - \langle S(\boldsymbol{\rho}')\rangle| \rangle,$$

where angular brackets mean averaging over an ensemble;

$$w(\boldsymbol{\rho}) = \begin{cases} 1, & |\boldsymbol{\rho}| \le R, \\ 0, & |\boldsymbol{\rho}| \ge R. \end{cases}$$

In Ref. 8 the way was proposed for determining the Karhunen–Loeve–Obukhov functions $\psi_k(\boldsymbol{\rho})$ through the expansion over Bessel functions $J_e(x)$. The functions $\psi_k(\boldsymbol{\rho})$ are obtained in the following form:

$$\psi_k(\boldsymbol{\rho}) = K_j^e(\rho) \exp(ie\theta) = \sum_{p=0}^{P} d_e^j J_e\left(\frac{\mu_p \rho}{2R}\right) \exp(ie\theta), \quad (3)$$

$$\boldsymbol{\rho} = (\rho, \theta).$$

Here P is the number, which presets an approximation order; d_e^j are the components of the eigenvectors of the matrix C with the elements:

$$c_{pp'}^e = -\pi\, a_p \int_0^R J_e\left(\mu_p \frac{\rho}{2R}\right) J_e\left(\mu_{p'} \frac{\rho}{2R}\right) \rho\, \mathrm{d}\rho\ ,$$

where μ_p are the roots of the equation $J_0(x) = 0$; where $J_0(x)$ is the zero order Bessel function;

$$a_p = \frac{2}{R^2\, [J_0(\mu_p)]^2} \int_0^R \rho\, D_s(\rho)\, J_0\left[\mu_p \frac{\rho}{2R}\right] \mathrm{d}\rho.$$

The eigenvalues λ_e^j are arranged in the order of decreasing values $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_k$, where Λ_1 is determined as follows: $\Lambda_1 = \max \lambda_e^j$. Then the eigenfunctions are sorted according to the obtained order.

The obtained sequence $\psi_k(\boldsymbol{\rho})$, $k = 1, 2, \ldots, K$ is the sequence of functions that are close to the Karhunen–Loeve–Obukhov functions with the accuracy determined by P value.

The field is set in the form of K functions $\psi_k(\boldsymbol{\rho})$ corresponding to K greatest eigenvalues as follows:

$$S_{H1}(\boldsymbol{\rho}) = \sum_{k=0}^{K} b_k\, \psi_k(\boldsymbol{\rho})\ , \quad (4)$$

where b_k are the independent normally distributed random values with zero mathematical expectation and the variance Λ_k; Λ_k are the eigenvalues and $\psi_k(\boldsymbol{\rho})$ are the eigenfunctions of the integral operator.

It is known that the larger the number of eigenfunctions are taken, the higher is the accuracy with which the field $S_{H1}(\boldsymbol{\rho})$ represents the field $S(\boldsymbol{\rho})$ determined by the correlation function $B(\boldsymbol{\rho} - \boldsymbol{\rho}')$. In practice, one is forced to use a finite number of eigenfunctions. The field determined by such a finite series has the correlation function close, but not equal, to the preset correlation function. To form the preset field, one can add the statistically independent field $S_{H2}(\boldsymbol{\rho})$ to the field $S_{H1}(\boldsymbol{\rho})$. The correlation function $B_2(\boldsymbol{\rho} - \boldsymbol{\rho}')$ of the field $S_{H2}(\boldsymbol{\rho})$ should be equal to the difference between the correlation functions $B(\boldsymbol{\rho} - \boldsymbol{\rho}')$ of the initial field $S(\boldsymbol{\rho})$ and $B_{H1}(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ of the first approximation field $S_{H1}(\boldsymbol{\rho})$.

As the modal method reconstructs practically any of the large-scale fluctuations, the correlation function $B_2(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ is different from zero in the area of a smaller size than in the case with the correlation function $B(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$. If the size of this area is yet big for using the method of sliding average to form the field corresponding to the function $B_2(|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$, one should again apply the modal method, but on a shorter range $R_2 < R$.

Such a process can be continued until the spectral range of the correlation function of the residue becomes convenient for application of the spectral method or the method of sliding average. Once such a situation is reached, the spectral method or the method of sliding average can be applied to form the last field S_B, and the process of formation of the field $S(\boldsymbol{\rho})$ stops. The number of fields formed by the modal method should be selected based on the idea of optimization of the calculation process. If the size of the domain where it is necessary to set the field is significantly smaller than the region in which the initial correlation function is different from zero, it is expedient, from the standpoint of organization of the calculation process, to form several fields by the modal method, while narrowing in succession the domain of their definition. The advantage of the method proposed over other known methods of numerical generation of random fields manifests itself at generation of random fields with a wide range of inhomogeneities in relatively small regions. Such a case appears, for example, in simulating wave fronts of laser beams propagating through the atmosphere.

Thus, the process of simulating random fields involves the following stages:

1. To form the field by modal technique the eigenfunctions $\psi_k(\boldsymbol{\rho})$ and the eigenvalues Λ_k of the correlation function $B(\boldsymbol{\rho} - \boldsymbol{\rho}')$ are determined.

2. The field $S_{H1}(\boldsymbol{\rho})$ is formed using the algorithm (4).

3. The difference is determined between the correlation function $B(\boldsymbol{\rho} - \boldsymbol{\rho}')$ and the correlation function $B_{H1}(\boldsymbol{\rho} - \boldsymbol{\rho}')$ obtained:

$$B(\boldsymbol{\rho} - \boldsymbol{\rho}') - B_{H1}(\boldsymbol{\rho} - \boldsymbol{\rho}') = \Delta B(\boldsymbol{\rho} - \boldsymbol{\rho}').$$

4. The possibility is analyzed of applying the spectral method or the method of sliding average to form the field with the correlation function $\Delta B(\boldsymbol{\rho} - \boldsymbol{\rho}')$ and then apply these methods, otherwise, the stages 1–4 are repeated for the field $\Delta S(\boldsymbol{\rho})$ with the correlation functions $\Delta B(\boldsymbol{\rho} - \boldsymbol{\rho}')$.

The stages of random field formation are shown in the left-hand panel of Fig. 1 as the stages of approximation of the correlation function of the field.

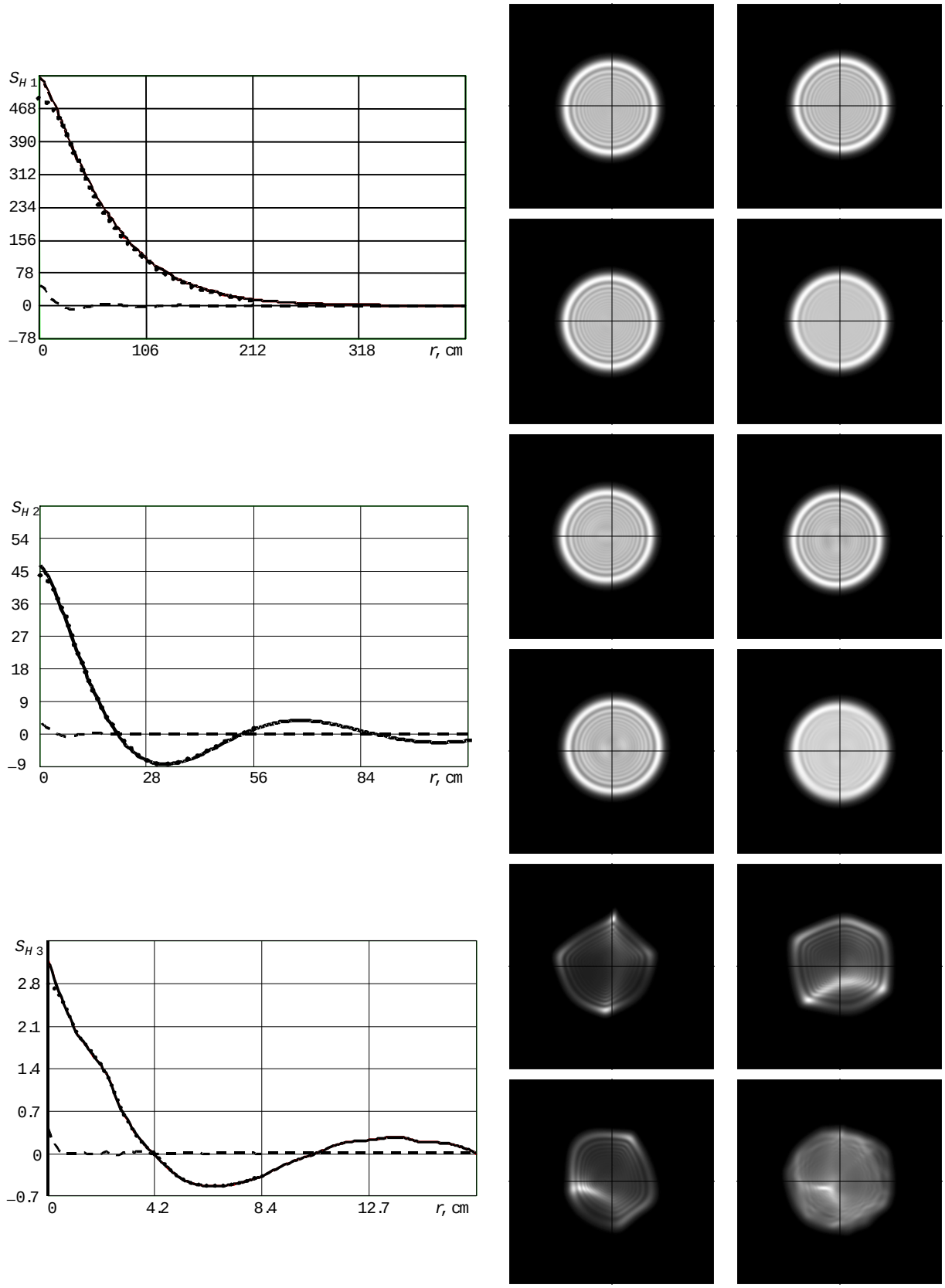
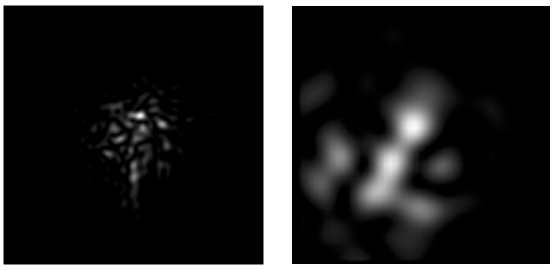


Fig. 1. Stages of the approximation of the correlation function and the effect of the degree of approximation on the intensity distribution of the beam at $\lambda = 1.06$ μm with the diameter of 3 cm in the near zone (690 cm); —— – initial correlation function; ····· – approximate correlation function; - - - - - – residue of the approximation.

The way of representing the random field in the form of a set of random fields corresponding to contributions from different in scale inhomogeneities is useful for studying the effects of these contributions on the processes simulated. In particular, one can visually demonstrate the effect of each stage of the correlation function approximation on the model of laser beam propagation through a medium with random fluctuations of the refractive index.

The distributions of the laser beam intensity in the observation plane after passing through the phase screen of fluctuations of the refractive index are shown in the right-hand panel of Fig. 1. The phase screen at each stage is the combined random field with the correlation function corresponding to this stage of approximation. Three samples of the intensity distribution are shown for each stage, as well as the sum of these three samples, which demonstrate the displacement of the beam as a whole. Besides, this method of formation of the random field makes it possible to complicate the model of the random field by introducing the anisotropy of the field in selected scales of the inhomogeneities. For example, it can be useful in the study of atmospheric turbulence, which may have the anisotropy of inhomogeneities of the refractive index at low spatial frequencies.

Typical distributions of the intensity of laser beams of different diameters at propagating through a 1-m thick phase screen are shown in Fig. 2 for the far zone of observations.



a *b*

Fig. 2. Distributions of the laser beam intensity at propagation through a random phase screen with the spectrum (5) and the parameters: $\alpha = 11/3$, $C_n^2 = 1.74 \cdot 10^{-9}$ m$^{-2/3}$, $l_0 = 0.15$ cm, $L_0 = 100$ cm, $dZ = 1$ m, $\lambda = 1.06$ μm; beam diameter is 30 mm (*a*) and 10 mm (*b*).

The correctness of algorithm operation was examined by means of statistical estimations of the correlation function of the field on the set of the field realizations. Figure 3 shows the correlation function corresponding to the spectrum

$$\Phi 1_n^u(\mathbf{p}) = A(\alpha)\, C_n^2 L^{(3+\alpha)} \exp\{-p^2 l_0^2\} \times \times |1 + p^2 L^2|^{-(3+\alpha)/2}, \quad (5)$$

where

$$A(\alpha) = \frac{\Gamma(2+\alpha)}{4\pi^2} \sin(\alpha\pi/2)$$

with the parameters $\alpha = 2/3$, $C_n^2 = 1.3 \cdot 10^{-9}$ m$^{-2/3}$, $L = 25$ cm, $l_0 = 0.04$ cm (solid line) and two its estimates obtained from the set of realizations: estimate over 5000 realizations of the field is shown by dashed line, and the estimate over 500000 realizations is shown by dotted line. It is seen from this figure that the method well reconstructs the simulated random field. Maximum deviation of the estimate of the correlation function from the preset value is less than 1% of the variance at a sufficient number of tests. The estimation of the structure function is shown in Fig. 4.

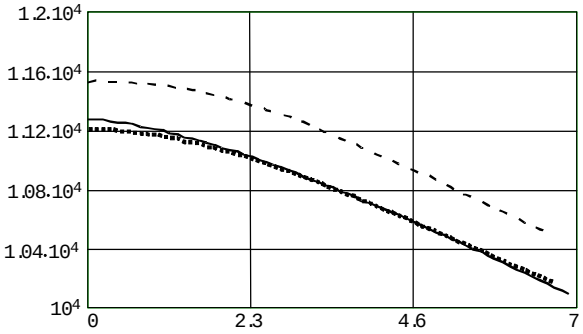


Fig. 3. Estimates of the correlation function of the random field over the set of realizations of the field with the correlation function corresponding to the spectrum (5) at $\alpha = 2/3$, $C_n^2 = 1.3 \cdot 10^{-9}$ m$^{-2/3}$, $L = 25$ cm, $l_0 = 0.24$ cm. ——— – correlation function; - - - - - – estimate over 5000 realizations of the field cross section; ····· – estimate over 500000 realizations of the field cross section.

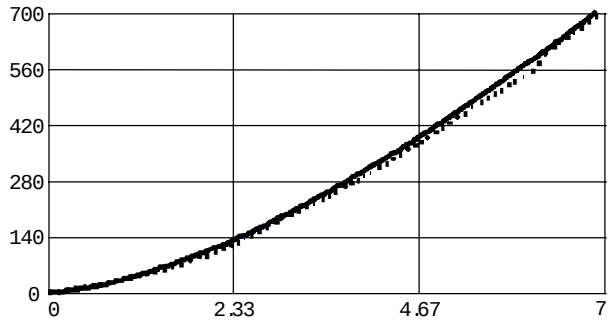


Fig. 4. Estimates of the structure function of the random field for the spectrum (5) at $\alpha = 2/3$, $C_n^2 = 1.3 \cdot 10^{-9}$ m$^{-2/3}$, $L = 25$ cm, $l_0 = 0.24$ cm. ——— – structure function; ····· – estimate over 5000 realizations of the field.

Thus, the method is proposed for numerical simulation of a uniform isotropic random field with a present correlation function that efficiently works in a wide scale range of spatial inhomogeneities. There is the possibility of introducing the field anisotropy only for certain frequency ranges. This method is convenient for analysis of the effect of different scales on the processes under study and the effect of different factors on the statistical characteristics of the laser beams propagating through the medium with random inhomogeneities. One can apply it to statistical testing in the problems of the laser beam propagation through a turbulent medium.

References

1. V.P. Kandidov, Usp. Fiz. Nauk **166**, No. 12, 1309–1338 (1996).
2. J.A. Fleck, J.R. Morris, and M.D. Feit, Appl. Phys. **10**, 129 (1976).
3. R.J. Noll, J. Opt. Soc. Am. **66**, 207 (1976).
4. B.J. Herman and L.A. Strugala, SPIE **1221**, 183–192 (1990).
5. R.G. Lane, A. Glindemann, and J.C. Dainty, Waves in random media **2**, No. 3, 209–224 (1992).
6. V.P. Kandidov, Atmos. Oceanic Opt. **11**, No. 1, 85–88 (1998).
7. T.A. Sheremet'eva and G.N. Filippov, in: *Abstracts of Reports at International Conference "Applied Optics–98"* (State Optical Institute, St. Petersburg, 1998) pp. 58–59.
8. V.P. Aksenov and Yu.N. Isaev, Atmos. Oceanic Opt. **7**, No. 7, 506–509 (1994).

0235-6880/00/05 492-05 $02.00