DESIGN OF A MEASURING DEVICE FOR ADAPTIVE ATMOSPHERIC OPTICAL SYSTEMS

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Wavefront reconstruction with the help of Zernike polynomials is considered in the situation where the wavefront is measured by phase detectors and local tilt detectors, and measurement noise is taken into account.

The design of a measuring device is a component part of the design of an adaptive optical system. It has been shown[1,2] that its optimization is closely connected with the characteristics of the whole system. For practical purposes it is sometimes expedient to consider the measuring device separately in the belief that the wavefront corrector ideally reproduces only a limited number of characteristic forms of the phase distortions of a light wave. For circular receiving apertures it is sometimes convenient to expand the phase of a light wave. The quality of such measuring systems can be characterized by the rms error of the reconstruction of the wavefront by the Zernike polynomials as measured by actual detectors.

The purpose of this stet is to investigate the quality of wavefront reconstruction by Zernike polynomials for different positions of the phase and local tilt detectors, taking their measurement noise into account.

Let us consider the measurement of the wavefront by means of the phase detectors. The result of their measurement is the value of the phase of the light at n points $\mathbf{r}_k$:

$$\varphi(\mathbf{r}_k)=\Phi(\mathbf{r}_k)+\delta\Phi(\mathbf{r}_k), \qquad (1)$$

where Φ is the precise value of the phase, $\delta\Phi(\mathbf{r}_k)$ is the detector noise which has variance σ_1^2 and not correlated with Φ or $\delta\Phi(\mathbf{r}_k)$, $j \neq k$.

The expansion coefficients β_i of the wavefront over the first m Zernike polynomials Z_1 are determined by minimizing the error

$$\Delta_1=\frac{1}{n}\sum_{k=1}^{n}\left[\varphi'(\mathbf{r}_k)-\sum_{1=2}^{m}\beta_1 Z_1(\mathbf{r}_k)\right]^2,$$

$$\varphi'(\mathbf{r})=\varphi(\mathbf{r})-\frac{1}{n}\sum_{k=1}^{n}\varphi(\mathbf{r}_k). \qquad (2)$$

We will not consider the constant polynomial $Z_1(\mathbf{r}) = 1$. Then,

$$\beta_1=\sum_{k=1}^{n}\gamma_{1k}\varphi(\mathbf{r}_k),\quad \gamma_{1k}=\sum_{j=2}^{m}a_{1j}Z_j(\mathbf{r}_k)$$

$$-\frac{1}{n}\sum_{j=2}^{m}\sum_{1=1}^{n}a_{1j}Z_j(\mathbf{r}_1), \qquad (3)$$

where a_{ij} are the elements of the inverse matrix of the matrix with the elements

$$\sum_{k=1}^{n}Z_1(\mathbf{r}_k)Z_j(\mathbf{r}_k).$$

The tilt detectors characterize the average values ξ_i and η_i of the local tilts of the wavefront in the subaperture fields. The following values[2] can be approximately taken as the results of such detector measurements:

$$\xi_k=\frac{1}{S_k}\left(\frac{\partial\Phi(\mathbf{r}_k+\mathbf{r})}{\partial x},1\right)_{\Omega_k}+\delta\xi_k,$$

$$\eta_k=\frac{1}{S_k}\left(\frac{\partial\Phi(\mathbf{r}_k+\mathbf{r})}{\partial y},1\right)_{\Omega_k}+\delta\eta_k; \qquad (4)$$

or

$$\xi_k=(\Phi(\mathbf{r}_k+\mathbf{r}),x)_{\Omega_k}/(x,x)_{\Omega_k}+\delta\xi_k;$$

$$\eta_k=(\Phi(\mathbf{r}_k+\mathbf{r}),y)_{\Omega_k}/(y,y)_{\Omega_k}+\delta\eta_k,\quad k=1,\ldots,N, \qquad (5)$$

where $\mathbf{r} = (x, y)$ is the position vector in Cartesian coordinates; N is the number of detectors; the parentheses a $(,)_{\Omega_k}$ denotes the scalar product of any two functions in the fields of sub-aperture Ω_k (integral from the product of functions); S_k is the area Ω_k; $\delta\xi_k$ and $\delta\eta_k$ are noise levels of the measurements with variance σ_2^2 which are uncorrelated with each other and with the exact values of wavefront tilts.

According to the data of the tilt detector measurements one can reconstruct the value of the phase of the wavefront at separate points[4,5]. Let us consider this problem in detail. To simplify calculations for a great number of the tilt detectors uniformly filling the area of the receiving aperture Ω we may turn from

Eqs. (4) and (5) to the continuous model and assume that the following values are measured

$$\frac{\partial\Phi}{\partial x}=X+\delta\xi,\ \frac{\partial\Phi}{\partial y}+Y+\delta\eta, \tag{6}$$

where X and Y are the measured values of the tilts. Using the apparatus of the calculus of the variations[6], we minimize of the functional of the measurement errors

$$\Delta_2=(\delta\xi^2+\delta\eta^2,1)_\Omega. \tag{7}$$

We thereby obtain the internal Neumann problem

$$\nabla^2\Phi = \frac{\partial}{\partial x}X + \frac{\partial}{\partial y}Y;$$
$$\left.\frac{\partial\Phi}{\partial n}\right|_\Gamma = X\cos(\mathbf{n},x) + Y\cos(\mathbf{n},y), \tag{8}$$

where Γ is the boundary of Ω and n is the outer normal to Γ. Using the grid approximations of Eq. (8), one can construct different algorithms for the reconstruction of the phase at the points of the specially chosen grid based on the measurement data, of the wavefront tilts at these separate points. Thus, for a square grid we have the well-known equations[4,5].

$$-4\Phi_{ij}+\Phi_{i-1,j}+\Phi_{i,j-1}+\Phi_{i,j+1}+\Phi_{i+1,j}=f_{ij}, \tag{9}$$

where

$$f_{ij} = \frac{h}{2}\left(X_{i+\frac{1}{2},j+\frac{1}{2}} + X_{i+\frac{1}{2},j-\frac{1}{2}} - X_{i-\frac{1}{2},j-\frac{1}{2}} - X_{i-\frac{1}{2},j+\frac{1}{2}} + Y_{i+\frac{1}{2},j+\frac{1}{2}} + Y_{i-\frac{1}{2},j+\frac{1}{2}} - Y_{i-\frac{1}{2},j-\frac{1}{2}} - Y_{i+\frac{1}{2},j-\frac{1}{2}}\right).$$

h is the grid step, the ordered pair $(i,\ j)$ characterizes the position of a grid point with respect to the Cartesian coordinate axes.

It is not difficult to write analogous expressions for more complicated irregular grids. One should note that for a small number of tilt detectors an approach similar to that described in Ref. 4 is more correct and solves the problem directly in a discrete formulation. As a result, when the receiving aperture is densely filled with the subapertures of the tilt detectors, the latter can with some degree of approximation be replaced by the phase values at the points of a covering grid. When the dimensions of the subapertures are small compared to the distances between them, it is necessary to use expression (4) or (5).

Let us estimate the quality of the reconstruction of the wavefront using the Zernike polynomials $Z(i)$ by the functional

$$I = \frac{1}{S}\left\langle \left(\left[\Phi(r) - \sum_{i=2}^{m}\beta_i Z_i(r)\right]^2, 1\right)_\Omega \right\rangle \tag{10}$$

where S is the area of Π and the angular brackets denote the mean value of the ensemble of realizations.

Within the framework of the Kolmogorov model of the turbulent atmosphere the phase errors are characterized by the structure function

$$D_\Phi(\rho) = 6.88\left(\frac{D}{r_0}\right)^{5/3}\rho^{5/3}. \tag{11}$$

Here D is the diameter of the circular area Ω; $\mathbf{r}_0$ is the Fried correlation radius[3]; and ρ is a dimensionless distance, where $\rho \in (0,\ 1)$. First let us consider the quality of reconstruction of the wavefront by the phase-detector measuring system. We substitute Eq. (3) into Eq. (10) and take into consideration the relation between the correlation function and the structure function[3]:

$$2\langle\Phi(r_1)\Phi(r_2)\rangle = -D_\Phi(r_1-r_2) - \frac{1}{S^2}\int_\Omega\int_\Omega D_\Phi(\rho_1-\rho_2)d^2\rho_1 d^2\rho_2 + \frac{1}{S}\int_\Omega D_\Phi(r_1-\rho_1)d^2\rho_1 + \frac{1}{S}\int_\Omega D_\Phi(r_2-\rho_2)\,d^2\rho_2 \tag{12}$$

Then, using the results of Refs. 3 and 7, we obtain

$$2\langle\beta_i\beta_j\rangle = -\sum_{k=1}^{n}\sum_{l=1}^{n}\gamma_{ik}\gamma_{jl}D_\Phi(r_k-r_l) + \sigma_1^2\sum_{k=1}^{n}\gamma_{ik}\gamma_{jk};$$
$$2\langle\beta_i(\Phi,Z_i)_\Omega\rangle = -\sum_{j=1}^{n}\gamma_{jl}(Z_i(r)D_\Phi(r-r_j))_\Omega;$$
$$\frac{1}{S}\langle(\Phi,\Phi)_\Omega\rangle = 1.0299\left(\frac{D}{r_0}\right)^{5/3},\quad \frac{1}{S}(Z_i,Z_j)_\Omega=\delta_{ij} \tag{13}$$

Expressions (13) determine the value of functional (10). It is often convenient to single out an individual noise component of the functional. Then,

$$I = J + \delta J \tag{14}$$

where J is the exact value of the functional and $\delta J = \sigma_1^2\sum_{i=2}^{m}\sum_{k=1}^{n}\gamma_{ik}^2$ is its noise.

Now it Is clear that the values J and δJ can become comparable at strong noise levels of the measurements. To decrease the influence of the noise we can introduce a system of adaptation to the noise level. To do this, we formally replace B_1 by the expressions $\beta_i' = c_1\beta_1$, in which the coefficients c_1 can be determined by minimization of functional (10). We note that even in the absence of noise the coefficients c_1 will correct, to some extent, the mistakes of algorithm (3) for any unsuccessful arrangement of the detectors.

Calculations were carried out for 10 different, physically reasonable arrangement of the phase detectors. The first six variants are given in Figs. 1 and 2. By means of them it is also possible to describe the tightly packed tilt detectors which yield a further reconstruction on the phase. In these Figures the phase and phase tilt detectors are indicated by points and circles, respectively. The values of the parameter $2R/D$ for these variants are 0.9, 0.6, 0.5 in Fig. 1a, b and c and 0.5, 0.33, 025 in Fig. 2a, b and c, respectively. Four other variants were considered with

19, 31, 55, 79 phase detectors in the center and on concentric circles 2, 3, 4 and 5 spaced an equal distance R from each other where the ratio $2R/D$ is respectively equal to 0.5, 0.33, 0.25, and 0.2 (Fig. 3). Reconstruction of the wavefront by the 6 and 10 lowest-order Zernike polynomials, respectively[3,7], is also considered. The calculated results are displayed in Table 1 ($m = 6$) and Table 2 ($m = 10$):

$$J' = J(r_0/D)^{5/3}; \quad \delta J' = \delta J/\sigma_1^2; \quad I_{1j} = I(r_0/D)^{5/3},$$

I'_{2j} corresponds to I'_{1j}, where an adaptation to the noise level has been introduced. The values of this index $j = 1$, 2, and 3 correspond to the values of this parameter $\sigma_1^2(r_0/D)^{5/3} = 0.001$, 0.01, and 0.1. If the values of this parameter exceed unity, then, according to the calculations we should restrict our calculations to reconstruction of the mean aperture tilts since the measurement noise sufficiently exceeds the contribution of the higher aberrations.

These results can also be used for an approximate analysis of the reconstruction of the wavefront made by the tightly packed local tilt detectors. The noise δJ associated with the functional J in this case is determined by the equation

$$\delta J = \sum_{l=2}^{m} \sum_{k=1}^{n} \sum_{l=1}^{n} \gamma_{lk}\gamma_{jl} \langle \delta\Phi((r_k)\delta\Phi(r_l)\rangle \qquad (15)$$

TABLE 1.

	VARIANT					
	1	2	3	4	5	6
J'	0.089	0.103	0.090	0.080	0.072	0.068
$\delta J'$	0.445	0.130	0.106	0.276	0.237	0.196
I'_{11}	0.089	0.103	0.090	0.080	0.072	0.069
I'_{12}	0.093	0.104	0.091	0.083	0.074	0.070
I'_{13}	0.133	0.116	0.101	0.108	0.095	0.088
I'_{21}	0.086	0.097	0.085	0.080	0.072	0.069
I'_{22}	0.089	0.098	0.086	0.083	0.074	0.070
I'_{23}	0.117	0.112	0.098	0.105	0.093	0.087

TABLE 2.

	VARIANT					
	5	6	7	8	9	10
J'	0.0601	0.0494	0.0529	0.0431	0.0429	0.0402
$\delta J'$	0.589	0.392	0.344	0.216	0.116	0.087
I'_{11}	0.0607	0.0498	0.0533	0.0433	0.0430	0.0402
I'_{12}	0.066	0.053	0.056	0.045	0.044	0.0410
I'_{13}	0.119	0.089	0.087	0.065	0.055	0.0489
I'_{21}	0.0517	0.0470	0.0508	0.0427	0.0416	0.0401
I'_{22}	0.055	0.050	0.054	0.045	0.043	0.0403
I'_{23}	0.081	0.073	0.082	0.063	0.054	0.0484

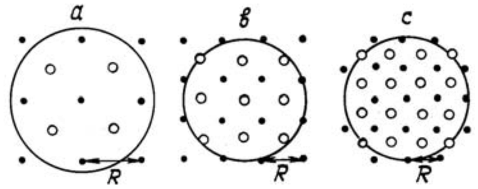

FIG. 1a, b, c. Location of the phase and tilt detectors at the, grid points of a square grid.

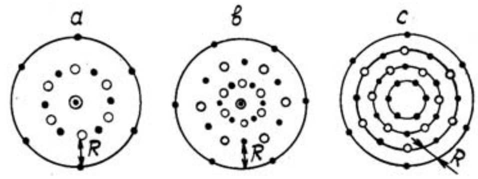

FIG. 2a, b, c. Location of the phase and tilt detectors at the grid points of a polar grid.

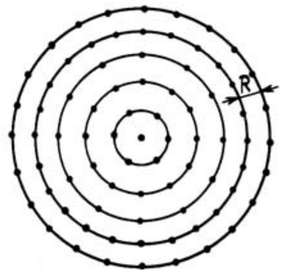

FIG. 3. Location of the phase detectors.

The values of the correlations $\langle \delta\Phi(r_k)\delta\Phi(r_l)\rangle$ can be approximately determined using Eq. (4). Since we have a linear problem, the values of Φ, x and y in Eqs. (4) can be replaced by their associated noise levels $\delta\Phi$, $\delta\xi$, and $\delta\eta$. We denote

$\alpha_1 = \langle \delta\Phi^2_{i,j}\rangle$, $\alpha_2 = \langle \delta\Phi_{i,j\pm1}\delta\Phi_{ij}\rangle = \langle \delta\Phi_{i\pm1,j}\delta\Phi_{ij}\rangle$,

$\alpha_3 = \langle \delta\Phi_{i\pm1,j\pm1}\delta\Phi_{ij}\rangle$, $\alpha_4 = \langle \delta\Phi_{ij} \times \delta\Phi_{i\pm2}\rangle = \langle \delta\Phi_{ij}\delta\Phi_{i,j\pm2}\rangle$.

Ignoring the other correlations and determining the values $\langle f^2_{ij}\rangle$, $\langle f_{ij}f_{i,j-1}\rangle$, $\langle f_{ij}f_{i+1,j-1}\rangle$, and $\langle f_{ij}f_{i,j-2}\rangle$, it is not difficult to obtain a system of linear equations which has the following solution: $\alpha_1 \approx 0.16h^2\sigma_2^2$, $\alpha_2 \approx \alpha_3 \approx 0.04h^2\sigma_2^2$, $\alpha_4 \approx 0$. One should note that the problem has not been calculated in its most general form since its solution is determined by a concrete reconstruction algorithms[8]. For square grids of the locations of the tilt detectors an analysis of the reconstruction of the phase Φ_{ij}, which allows for measurements noise, is given in Refs. 4 and 5.

Let us consider the influence of the dimensions of the subaperture fields on the quality of the reconstruction of the wavefront by the Zernike polynomials. If the diameter of the circular subapertures are smaller than the distance between them, reconstruction algorithms (8) and (9) can result in large errors. That is why in such cases the expansion coefficients β_i are determined in the following way. Let us consider a linear combination of the measured values

$$\beta_1 = \sum_{k=1}^{N} (A_{1k}\xi_k + B_{1k}\eta_k), \tag{16}$$

in which the unknown coefficients A_{ik} and B_{ik} can be found from the condition of minimization of functional (10) for structure function (11) in the absence of measurement noise ($\delta\xi_k = \delta\eta_k = 0$). Thus we obtain a system of linear algebraic equations of the following type

$$\sum_{k=1}^{N} (A_{1k} d^{\xi}_{1k} + B_{1k} d^{\xi\eta}_{1k}) = C^{\xi}_{11},$$

$$i=2,\ldots,m;$$

$$l=1,\ldots,N, \tag{17}$$

$$\sum_{k=1}^{N} (A_{1k} d^{\xi\eta}_{1k} + B_{1k} d^{\eta}_{k}) = C^{\eta}_{11},$$

where

$$d^{\xi}_{1k} = \langle \xi_1 \xi_k \rangle \quad d^{\xi\eta}_{1k} = d^{\eta\xi}_{k1} = \langle \xi_1 \eta_k \rangle \quad d^{\eta}_{1k} = \langle \eta_1 \eta_k \rangle$$

$$C^{\xi}_{11} = \frac{1}{S}\langle \xi_1 (\Phi, Z_1)_\Omega \rangle \quad C^{\eta}_{11} = \frac{1}{S}\langle \eta_1 (\Phi, Z_1)_\Omega \rangle. \tag{18}$$

The value of functional (10) of the quality will then be determined by expression (14), in which

$$J = 1.0299\left(\frac{D}{r_0}\right) - \sum_{1=2}^{m}\sum_{k=1}^{N} (A_{1k} C^{\xi}_{k1} + B_{1k} C^{\eta}_{k1}) \tag{19}$$

$$\delta J = \sigma_2^2 \sum_{1=2}^{m}\sum_{k=1}^{N} (A^2_{1k} + B^2_{1k})$$

It is not difficult to show that dispersions d^{ξ}_{kk}, and d^{η}_{kk} on circular subapertures with radius $\mathbf{r}_c$, in the absence of measurement noise, can be determined by the formula

$$d^{\xi}_{kk} = d^{\eta}_{kk} = C\left(\frac{D}{r_0}\right)^{5/3} r_c^{-1/3}, \tag{20}$$

in which the coefficient C for expressions (5) corresponds to the results of Ref. 7 and Is equal to 1.796. Then, using Eqs. (4), the calculations give $C = 1.673$. Below we will use the latter value of C. If the dimensions of the subapertures are considerably smaller than the distance between them, then to simplify the calculations the values (ξ_e and η_e in Eq. (18) can be approximately replaced by the values of the tilts at the centers of the subapertures $\partial\Phi(r_e)/\partial x$, $\partial\Phi(r_e)/\partial y$. Such calculations are made for circular subapertures with radius r_c. The first five variants of the arrangement of location of the tilt detectors were considered (Figs. 1 and 2), for which the parameter $2\ R/D$ was 0.9, 0.6, 0.5, 0.67, 0.5, respectively. The sixth variant was replaced by an equidistant distribution of 12 local tilt detectors on a circle with radius $R = 0.4\ D$. The calculated results, when reconstructing with the first six Zernike polynomials, are given in Table 3, where we have left their previous designations. It is seen from the table that the size of subapertures plays a more important role compared to the number of the local tilt detectors.

TABLE 3.

	VARIANT							
	1	1	2	3	4	4	5	6
$\frac{2r_c}{D}$	0.225	0.45	0.15	0.083	0.15	0.3	0.135	0.135
J'	0.126	0.072	0.3	0.068	0.3	0.25	0.1	0.19
$\delta J'$	0.18	0.34	0.16	0.75	0.09	0.08	0.66	0.21

The influence of measurement noise on the obtained results is considered for the first variant, which is more often used in practice. Thus, for $2r = 0.45D$, $\sigma_2^2(r_0/D)^{5/3} = 0.001$, 0.01, 0.1, 1 and 10, we have the following values of $I(r_0/D)^{5/3}$: 0.073, 0.076, 0.106, 0.414, and 3.5. After introducing a system of adaptation to the noise level these values reduce to 0.073, 0.076, 0.105, 0.314, and 0.805. Thus, the discussed technique and the model examples calculated according to it can be useful in the choice of the design of a measuring system for the reconstruction of the wavefront by lower-order Zernike polynomials.

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