On possible fractal structure of the tensor of turbulent diffusion coefficients in the atmospheric surface layer

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Components of the tensor of turbulent diffusion coefficients in the atmospheric surface layer were determined experimentally during a four-day observational period. Besides a pronounced diurnal, almost periodic, variation of these components, considerable fluctuations were found that couldn't be explained by measurement errors. An attempt is undertaken to explain such a behavior of the tensor of turbulent diffusion coefficients by the fractal structure of the process observed.

Mathematical modeling of diffusion of aerosol and gas pollutants in the atmosphere is widely used for solution of various applied problems. Within the framework of Eulerian approach to description of turbulent diffusion, use of the following semi-empirical equation[5] seems to be most fruitful method:

$$\frac{\partial \bar{C}}{\partial t} + \frac{\partial \bar{U}_i \bar{C}}{\partial x_i} - \frac{\partial}{\partial x_i} K_{ij} \frac{\partial \bar{C}}{\partial x_j} = \bar{Q}, \qquad (1)$$

where $\bar{C}$ and $\bar{U}_i$ are the mathematical expectations of the pollutant concentration and wind velocity components; K_{ij} are the components of the tensor of turbulent diffusion coefficients; $\bar{Q}$ is the term describing the pollution source; $x = x_1$ and $y = x_2$ correspond to the horizontal coordinates, and $z = x_3$ is the vertical coordinate; t is time. The over bar denotes averaging over a statistical ensemble. Repeated subscripts denote summation.

One of the main problems arising when solving this equation is correct determination of the tensor of turbulent diffusion coefficients. In Ref. 3 a recursion method of closure of the semi-empirical equation was proposed. In its first approximation, this method allows obtaining analytical equations for determination of K_{ij} depending on the series of instantaneous values of wind velocity components:

$$K_{ij} = \frac{1}{2} \int\limits_0^{\tau} [R_{ij}(\xi) + R_{ji}(\xi)] \,\mathrm{d}\xi;$$

$$R_{ij}(\xi) = \frac{1}{T - \tau} \int\limits_{\tau}^{T} U'_i(t)\, U'_j(t - \xi)\, \mathrm{d}t, \qquad (2)$$

where U'_i is the instantaneous value of fluctuations of the ith wind velocity component; τ is the characteristic time scale of fluctuations of the wind velocity components (Eulerian time scale); T is the period of averaging the instantaneous values of the wind velocity components and calculation of the mean values $\bar{U}_i$ and pulsations U'_i. Equation (2) enables one to objectively determine turbulent diffusion coefficients from experimentally obtained series of instantaneous values of the wind velocity components.

References 1 and 2 described the results of application of this method to determination of the tensor components K_{ij} in the atmospheric surface layer. To obtain the series of instantaneous values of the wind velocity components, we used a three-coordinate acoustic anemometer.[1,2] The results measured in the atmospheric surface layer allowed us to confirm, for the first time, the hypothesis on the proportionality of K_{ij} to the corresponding components of the Reynolds viscous stress tensor. Earlier this hypothesis was justified only for turbulent boundary layers modeled in wind tunnels.[7] Figure 1 exemplifies the time dependence of K_{xx} for the experiment conducted during four days on the water-land interface near Zavyalovo village situated on the bank of the River Ob. The measurements were conducted at the height of 1.5 m above the surface.

The plot is drawn based on 576 readouts obtained for $T = 10$ min. The frequency of measurement of instantaneous values of the wind velocity within each 10-minute sample was 0.5 Hz. The characteristic time scale τ was determined by estimating the time of the first zero of the autocorrelation function of fluctuations of the horizontal wind velocity component. This procedure was justified in Ref. 3. In the example considered we found that $\tau \approx 40$–50 s. The values of K_{xx} given correspond to the system of coordinates, in which the horizontal axis x is directed along the vector of the mean wind velocity. Besides the pronounced, almost periodic, diurnal variation of K_{xx}, one can see significant fluctuations of the instantaneous values of the x-component of the tensor. Data analysis showed that these fluctuations couldn't be explained by the presence of measurement errors, which make up no more than 15–20%.

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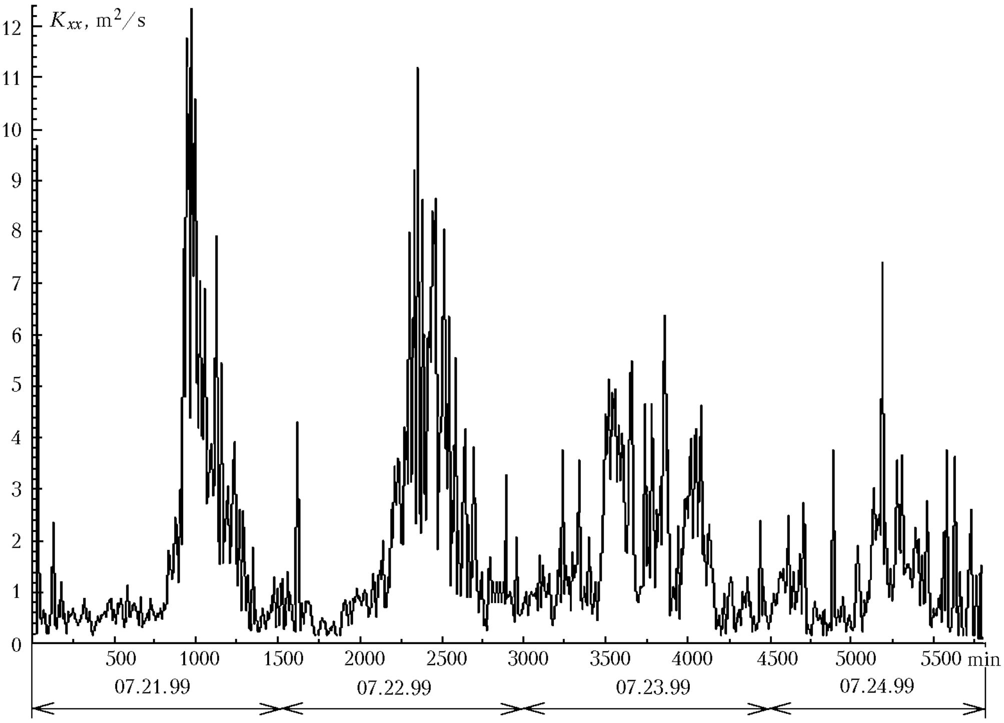

Fig. 1. Time dependence of K_{xx}.

We have undertaken an attempt to explain such a behavior of the tensor of turbulent diffusion coefficients by the fractal structure of the process observed.[8] The limits of one paper do not allow us to describe, even briefly, the main ideas of the modern theory of dynamic chaos. Therefore, let us define only those characteristics, which are considered in this paper.

In the general case, the behavior of K_{xx} should be considered as a single realization of some statistical ensemble. We do not know what number of hidden parameters (generalized coordinates) determine all terms of the statistical ensemble of K_{xx}, *that is, the dimension of the phase space is unknown.* However, the ensemble of realizations of K_{ij} can supposedly be presented by a continuous set of points distributed in the phase space. The density of the distribution of phase points is subject to random fluctuations. Statistical characteristics of this process determine the degree of dynamic chaos in the system.

In Ref. 12 a new concept of dynamic chaos was introduced for the first time, namely, a strange attractor. This name points to two its unusual properties: fractional (fractal) dimension of the phase space of the dynamic system and the property to be an attracting area for trajectories from neighboring areas of the phase space. The trajectories inside the strange attractor are dynamically unstable, and this manifests itself in the exponential divergence of the initially close trajectories. The dimension of the strange attractor is a very important characteristic of fractal objects. *In the general case, the fractal dimension determines the bulk of information needed to specify the coordinates of a point belonging to the attractor.*

Below we consider the correlation dimension of the attractor D (Ref. 10):

$$D = \lim_{E \to \infty} \left[\ln \left(\sum_{i=1}^{M(E)} p_i^2 \right) \Big/ \ln E \right], \qquad (3)$$

where $M(E)$ is the minimum number of n-dimensional cubes with the edge E needed to cover the attractor; p_i^2 is the probability that a couple of attractor points belong to the ith cube. It was shown[9] that D can be considered as a characteristic of the density of points in the statistical ensemble of phase trajectories in the phase space.

Another one important characteristic of the attractor is the entropy (this concept was introduced by Kolmogorov [4]). Below we use the correlation entropy. To construct it, we divide the phase space including the attractor into $M(E)$ non-overlapping n-dimensional cubes with the edge E. Conduct then m successive measurements, tracking the phase trajectory and marking, with the interval Δ, the cubes s_i, the trajectory has passed through. In every independent experiment, we obtain some realization in the form of a series of cubes $s_1,\ldots,s_m$. If we know the probabilities $P(s_1,\ldots,s_m)$ of all possible series of cubes, then the correlation entropy K is determined in the following way:

$$K = -\lim_{\Delta \to 0} \lim_{E \to 0} \lim_{m \to \infty} \left[-\frac{1}{m\Delta} \ln \sum_{s_1, \ldots, s_m} P^2(s_1, \ldots, s_m) \right]. \qquad (4)$$

The characteristic term, for which the system behavior can be predicted, is inversely proportional to the entropy of the process. If the entropy achieves zero, then the system becomes completely predictable. For truly random processes, the entropy is infinitely large. In the presence of the strange attractor, the entropy is positive and has a finite value. Thus, *the entropy value is a quantitative characteristic of the degree of system randomness.*

It does not follow from the above definitions that the available time series of K_{ij} are necessary and sufficient to determine the attractor characteristics. However, in Ref. 13 it was shown that a new attractor can be constructed from one realization of a dynamic variable for almost all smooth dynamic systems, and the properties of this attractor are the same as in the initial one. The numerical algorithm constructed based on the results of these references was implemented by Sychev[6] in the form of the FRACTAN 3.0 computer program. The program manipulates with time series from 512 to 16384 readouts and analyzes the phase space with the dimension from 1 to 16. We are thankful to the author of this program for the possibility to use it for analysis of experimental data.

The algorithm for calculation of the correlation dimension of the attractor for the n-dimensional space was constructed based on the following principles.[11] We consider the equation showing the relative number of couples of attractor points separated by the distance no longer than r:

$$C(r) = \frac{2}{m\,(m-1)} \sum_{i=0}^{m-2} \sum_{J=i+1}^{m-1} \Theta[r - \rho(x_i, x_j)], \quad (5)$$

where $\Theta(\xi) = \begin{cases} 1, & \xi \geq 0 \\ 0, & \xi < 0 \end{cases}$; $\rho(x_i, x_j)$ is the distance between the points x_i and x_j in the n-dimensional phase space; m is the number of points x_i on the attractor. If $C(r) \approx r^D$, then D estimates the correlation dimension of the attractor. At the unknown dimension of the phase space, calculations are made taking, successively, the values $n = 1, 2, 3, \ldots$. The correlation dimension of the attractor in this case first increases as a function of the parameter r. Then the large number of curves reach, usually, some roughly constant level, which is then taken as the sought value of D. The estimated dimension of the phase system in this case is $n \leq 2D + 1$.

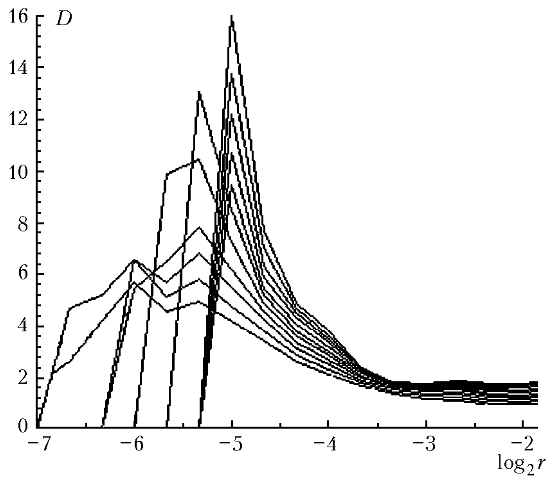

Fig. 2. Correlation dimension of the series of K_{xx} values.

Figure 2 shows an example of calculated dimension D of the series of K_{xx} values for the experiments conducted on July 21–24 of 1999. Different curves correspond to the phase space dimensions from 1 to 12. The results depicted in Fig. 2 show that roughly the same level of the curves is observed at the correlation dimension of the strange attractor $D \approx 1.5$. This corresponds to the estimated dimension of the phase space of the considered dynamic system $n \approx 4$.

The correlation entropy of the attractor K is determined in a similar way.[10] In this case, Eq. (5) is calculated depending on n and it is assumed that $C(r, n) \approx r^D \exp(-nK)$. As a result, it follows that

$$K \approx \ln \frac{C(r, n)}{C(r, n+1)} \,. \quad (6)$$

Figure 3 depicts an example of calculation of the correlation entropy K of the series of K_{xx} values for the experiment dated to July 21–24 of 1999. The curves correspond to the considered dimensions of the phase space from 1 to 12.

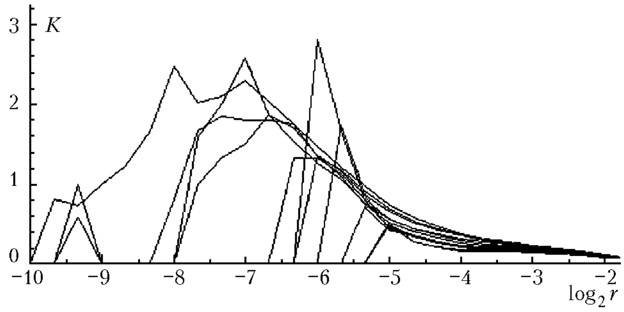

Fig. 3. Correlation entropy for the series of K_{xx} values.

It can be seen from Fig. 3 that $K \approx 0.3$. This suggests the presence of a strange attractor in the system considered, but with the low degree of chaos. The Table below generalizes the estimated dimensions of the phase space, correlation dimension of the strange attractor, and its correlation entropy for K_{xx}, K_{yy}, and K_{zz} in the experiment dated to July 21–24 of 1999.

Component	n	D	K
K_{xx}	4	1.5	0.3
K_{yy}	4.6	1.8	0.3
K_{zz}	3	1	0.2

The results obtained allow us to state that *the components of the tensor of turbulent diffusion in the atmospheric surface layer probably have the fractal structure.* In particular:

- the dimensions of phase spaces of these components vary from 3 to 4.6;
- the correlation dimensions of strange attractors vary from 1 to 1.5;
- the correlation entropy of strange attractors is about 0.2–0.3, that is, the degree of chaos found in the strange attractors is low.

Thus, it seems rather justified to continue studying the fractal structure of the components of the tensor of turbulent diffusion coefficients in the atmospheric surface layer, as well as trying to make clear its physical nature. For example, using the data tabulated, we can assume that $n = 5$ can be accepted as a very first hypothesis about the dimension of the phase space. Factor analysis of experimental data can be applied as a method for search of hidden parameters.

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