ON TAKING INTO ACCOUNT THE RANGE OF THE SOLUTION WHEN INVERTING AN EQUATION OF THE CONVOLUTION TYPE

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Regularized algorithms for solving numerically an integral equation of the convolution type that permit taking into account, together with the usually employed a priori information about the smoothness of the reconstructed function, data on the range of the function are studied. The effect of taking data of this type into account on the quality of reconstruction is studied in a numerical experiment. An iteration algorithm for reconstructing positive-definite functions is proposed and methods for adaptation under conditions of a priori uncertainty are examined.

INTRODUCTION

One of the inverse problems encountered most often in practice is the problem of solving an equation of the convolution type

$$Au \equiv \int_{-\infty}^{\infty} h(t - \tau)u(\tau)\, d\tau = f(t) \tag{1}$$

for the function $u(t)$, where A denotes the convolution operator with kernel $h(t - \tau)$, which is a function of the difference of the arguments t and τ, and $f(t)$ is the experimentally measured function.

The improper nature of this problem makes it necessary to employ *a priori* information about the solution, most often consisting of the assumption that the function sought is smooth. The smoothness of the solution is characterized quantitatively by the so-called stabilizing functional of order p[1]:

$$\Omega[u] = \int_{-\infty}^{\infty} \left\{ \sum_{n=0}^{p} q_n \left[\frac{d^n u}{dt^n} \right]^2 \right\} dt = \int_{-\infty}^{\infty} M(\omega)\ |\tilde{u}(\omega)|^2\, d\omega, \tag{2}$$

where

$$M(\omega) = \frac{1}{2\pi} \sum_{n=0}^{p} q_n \omega^{2n}, \tag{3}$$

q_n are nonnegative coefficients, $q_p > 0$, the tilde on u denotes the Fourier transform, and ω is the angular frequency. The regularized solution of Eq. (1) is found by minimizing the functional (2) with a given (in a quadratic metric) level of error on the right side of (1). This leads to the problem of finding the minimum of a smoothing functional with respect to u (Ref. 1)

$$\Phi_\alpha[u] = \int_{\infty}^{\infty} (Au - f)^2\, dt + \alpha\Omega[u] \tag{4}$$

For a given value of the regularization parameter $\alpha > 0$. The solution of this problem is found by finding the inverse Fourier transform of the function

$$\hat{\tilde{u}}(\omega) = \frac{\tilde{f}(\omega)}{\tilde{h}(\omega)} \cdot \frac{|\tilde{h}(\omega)|^2}{|\tilde{h}(\omega)|^2 + \alpha M(\omega)} . \tag{5}$$

The optimal regularized solution (with minimum mean-square error of reconstruction) can be found if the spectral power density $R_u(\omega)$ of the function sought and the spectral power density $R_\varepsilon(\omega)$ of the error $\varepsilon(t)$ in the measurement of the right side of Eq. (1) are known by setting in Eq. (5)

$$\alpha M(\omega) = R_\varepsilon(\omega)/R_u(\omega). \tag{6}$$

This solution is identical to the result of applying Wiener's optimal linear filtering.[1] The forms of *a priori* information that were examined are very general and the required quality of reconstruction cannot always be achieved. Thus is many inverse problems the region of admissible values (RAV) of the solution is known *a priori*. In particular, the solution can be positive-definite, and the appearance of negative values results in a physically meaningless solution.[2,3] In this case the minimum of the mean-square error in reconstruction is no longer a criterion for the algorithm to be optimal, and the problem of absolute minimization of the functional (4) becomes a conditional problem with constraints, in the form of inequalities, on the function $u(t)$. In its general

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form such a problem cannot be solved analytically, and this makes it necessary to use very laborious numerical methods of ainimization.[4] In this process the volume of calculations increases indefinitely as compared with the algorithm (5), realized in the frequency domain.

In this work we examine some methods for taking into account a priori assumptions about the range of the solution when inverting Eq. (1), the effect of these assumptions on the accuracy of reconstruction is investigated in a numerical experiment, and an iteration algorithm for reconstructing positive-definite functions, which makes it possible to perform the calculations more rapidly than in the case of the numerical methods of minimization, is proposed.

EFFECT OF A PRIORI CONSTRAINTS ON THE FUNCTION $U(t)$ OF THE QUALITY OF RECONSTRUCTION

The simplest method for taking into account *a priori* restrictions in the form of inequalities

$$\beta(t) < u(t) < \gamma(t) \tag{7}$$

is numerical minimization of the functional (4) as a function of N variables $u(t_1)$, ..., $u(t_N)$ taking into account Eq. (7), but for large N this problem is very time-consuming. A significant advantage is gained by reducing the problem with constraints to a problem of unconditional minimization.[4] In this work the constraints (7) were taken into account by using successively bilinear and logarithmic transformation of the range (β, γ) of the function u sought[3] (to simplify the notation we omit the arguments of the functions)

$$z(u) = \ln\left\{ \frac{au + b}{cu + d} \right\},$$

$$\frac{b}{a} = -\beta, \quad \frac{d}{c} = -\gamma, \quad ad \neq bc, \tag{8}$$

which maps this region In a single-valued fashion onto the entire number scale $z \in (-\infty, \infty)$, and in addition the ratio a/c can be arbitrary. The inverse mapping has the form

$$u(z) = \frac{b - d\exp z}{c\exp z - a}. \tag{9}$$

Substituting Eq. (9), using Eq. (8), into Eq. (1) and constructing Tikhonov's smoothing functional for the nonlinear, with respect to $z(t)$, integral equation obtained In the process, we arrive at the problem of unconditional minimization of

$$\Phi_\alpha[z] = \alpha\Omega[z] + \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} h(t-\tau)\,\frac{\beta(\tau) - \exp z(\tau)}{1 - \exp z(\tau)/\gamma(\tau)}\,d\tau - f(t)\right\}^2 dt. \tag{10}$$

Numerical minimization of Eq. (10) was performed using a package of applied programs of a dialogue system of optimization (DISO-BÉSM), developed at the Computational Center of the Academy of Sciences of the USSR,[4] with the function $z(t)$ given on a uniform grid of 16 points t_1. The search for a minimum by the method of conjugate gradients required of the order of 20 min on a BÉSM-6 computer. The results of the model calculations for different forms of the *a priori* constraints on $u(t)$ are presented in Fig. 1. The model function $u(t)$ in the form of a sum of two Gaussian curves is presented in Fig. 1a. Figure 1 also shows the function $f(t)$, obtained from Eq. (1), where $h(t)$ was assumed to be a Gaussian curve, and the results of reconstruction of $u(t)$ according to Eq. (5) (neglecting the *a priori* constraints) with $M(\omega) = 1$ and two values of the parameter α, chosen empirically. Figure 1b shows the results obtained by minimizing the functional (10) for three variants of the *a priori* constrains of the form (7):

a) $\beta(t) = 0$, $\gamma(t) - \infty$ (positive-definiteness of the solution);

b) $\beta(t) = 0$, $\gamma(t) = u_m f(t)/f_m$ (the index m denotes the maximum value and it is assumed that u_m is known); and,

c) $\beta(t) = 0$, $\gamma(t) = u_m$ at $t_0 \le t \le T$, $\gamma(t) - 0$ at $t < t_0$, $t > T$ (finite range of $u(t)$, determined by the interval $[t_0, T]$, where $f(t)$ exceeds the error level).

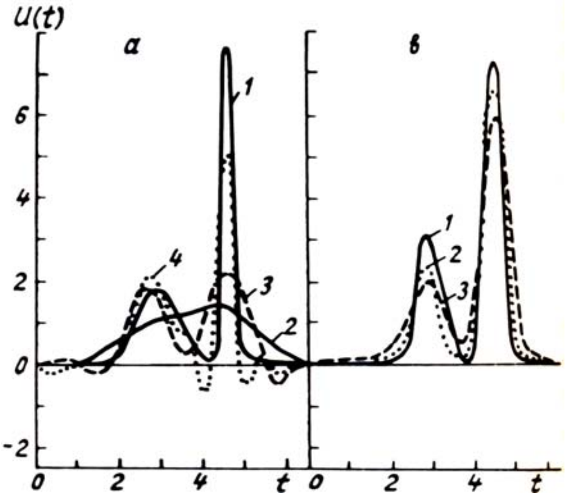

FIG. 1. The effect of a priori information about the solution on the quality of reconstruction (in the absence of noise): (a) 1) the model function u it); 2) the right side f(t) of the starting equation; 3, 4) reconstruction of u(t) by Tikhonov's method with $\alpha = 10^{-5}$ and $\alpha = 10^{-10}$; (b) reconstruction of u(t) by direct minimization of the smoothing functional with constraints of the type a, b. and c given in the text (curves 1, 2, and 3).

The regularization parameter was equal to $\alpha = 10^{-5}$ in all cases. From comparison of the results presented in Figs. 1a and b it can be concluded that the quality of the solution is significantly improved if the RAV of the function sought is taken into account. This improvement is manifested in the fact that there are no negative values of the solution and the fine structure of the solution is reconstructed in greater detail. As one can see from Fig. 1b. the form of the *a priori* constraints on $u(t)$ does not affect the reconstruction very strongly, and in the present model problem the condition that the solution be positive definite is sufficient. At the same time, the improvement in the quality of reconstruction involves large amounts of computer time. An iteration algorithm for inverting an equation of the convolution type, taking into account the positive-definiteness of the solution, is described below. This algorithm makes it possible to reduce by many factors the volume of calculations as compared with the methods of direct minimization.

ITERATION ALGORITHM FOR RECONSTRUCTING POSITIVE-DEFINITE FUNCTIONS

Let the initial approximation $\bar{u}(t)$ to the positive-definite solution of Eq. (1) be known. We shall represent $u(t)$ in the form

$$u(t) = \bar{u}(t)\cdot\Delta u(t). \tag{11}$$

We transform by means of the transformations (8) and (9), where $b = 0$ and $c = 0$, to the new function sought $z(t)$, in such a way that

$$\Delta u(t) = \exp z(t). \tag{12}$$

Substituting Eq. (12) into Eq. (1) we obtain a nonlinear equation for the function $z(t)$ whose values fall into the range from $-\infty$ to $+\infty$. Assuming that $\bar{u}(t)$ is close to the exact solution $u(t)$ and $z(t)$ is close to zero because of Eq. (12), we obtain

$$u(t) \simeq \bar{u}(t) + \bar{u}(t)z(t), \tag{13}$$

as a result of which the nonlinear equation is linearized

$$\int\limits_{-\infty}^{\infty} h(t-\tau)\bar{u}(\tau)z(\tau)\,d\tau = f(t) - A\bar{u}, \tag{14}$$

and its regularized solution can be written, analogously to Eq. (5), as

$$z(t) = \frac{1}{2\pi\bar{u}(t)} \times \int\limits_{-\infty}^{\infty} \left[\tilde{f}(\omega) - \tilde{h}(\omega)\tilde{\bar{u}}(\omega)\right] \frac{|\tilde{h}(\omega)|^2 \exp(i\omega t)}{|\tilde{h}(\omega)|^2 + \alpha M(\omega)}\,d\omega, \tag{15}$$

where

$$\alpha M(\omega) = R_\varepsilon / R_{u-\bar{u}}\ . \tag{16}$$

Iterative refinement of the solution is performed according to the scheme

$$\bar{u}^{(j)} = u^{(j-1)}, \quad R^{(j)}_{u-\bar{u}} = |R_u - R^{(j)}_{\bar{u}}|,$$
$$u^{(j)} = u^{(j-1)} \exp z^{(j)}, \quad j = 1, 2, \ldots, \tag{17}$$

starting with the zeroth-order approximation $u^{(0)} = \bar{u}$.

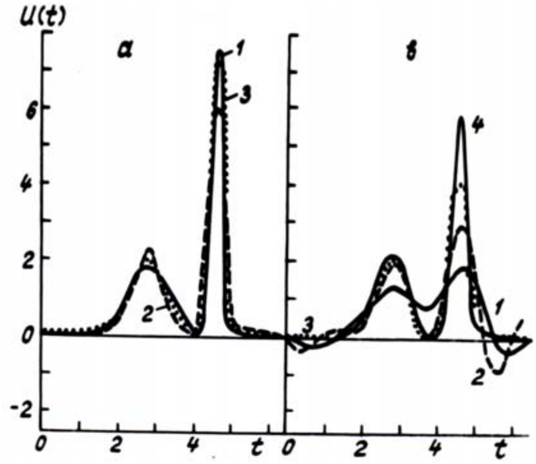

FIG. 2. Iterative reconstruction of positive-definite functions, (a) Reconstruction in absence of noise; 1) the model function u(t); 2), 3) the results of reconstruction after the first and tenth iterations; (b) reconstruction with 1% *noise on the right side of the starting equation; 1) zeroth-order regularization; 2) optimal linear filtering; 3), 4) taking into account the positive-definiteness of u(t) after* 100 *and* 400 *iterations.*

To investigate the quality of the reconstruction and the rate of convergence of the algorithm we performed a numerical experiment, whose results are presented in Fig. 2. The model function $u(t)$ was chosen to be the same sis in Fig. 1 and the modulus of the solution obtained according to (5) with $M(\omega) = 1$ and $\alpha = 10^{-15}$ (see Fig. 1a, curve 4) was chosen as the zeroth-order approximation $\bar{u}(t)$; the zero values of the initial approximation were replaced by small numbers. The results of using the iteration algorithm (15)–(17) are presented in Fig. 2a (after one and ten iterations). Figure 2b shows the results of reconstruction in the presence of an additive error $\varepsilon(t)$, distributed uniformly in the interval $[-0.01\, f_m, 0.01\, f_m]$ on the right side of Eq. (1). The curve 1 was obtained with $M(\omega) = 1$ and the parameter α determined from the mismatch; curve 2 was obtained using Eq. (6) with R_ε and R_U calculated numerically for the functions $\varepsilon(t)$ and $u(t)$. The curve 3 and 4 represent the results of the iteration algorithm after, respectively, 100 and 400 iterations (the zeroth-order approximation was assumed to be constant and equal to the average value of the model function $u(t)$), and R_ε and R_U were also assumed to be known. We shall investigate below the effect of uncertainties in the spectral power densities of the solution and the noise on the accuracy of reconstruction and the possibility of adaptation of their parameters.

RECONSTRUCTION WITH A PRIORI UNCERTAINTY

A priori information about the solution and the noise in the form of R_ε and R_U is usually not available. Nonetheless they can be estimated from the experimental data using the equations relating the statistical moments of the measured function, the solution, and the noise.[5] In application to an equation of the convolution type we can obtain from the equations for the correlation functions[6]

$$P_u(\omega)\ |\tilde{h}(\omega)|^2 + R_\varepsilon(\omega) = R_f(\omega). \tag{18}$$

We shall assume that the noise $\varepsilon(t)$ is white noise with known variance σ_ε^2 and that the solution is a stationary exponentially correlated Gaussian process, which has a spectral power density of the form

$$R_u(\omega) = \frac{1}{k_1 + k_2\omega^2}, \tag{19}$$

where the constants k_1 and k_2 can be expressed in terms of the parameters of the correlation function of this process.[7] We note that such a representation of corresponds to using a first-order Tikhonov stabilizer (3).[5] Substituting Eq. (19) into Eq. (18) and solving the equation obtained for k_1 and k_2 by the method of least squares, we obtain

$$k_1 = \frac{1}{p}\int_{-\infty}^{\infty} \Delta R(\omega)\ |\tilde{h}(\omega)|^2 d\omega - \frac{q}{p}k_2, \tag{20}$$

$$k_2 = \frac{1}{r - q^2/p}\int_{-\infty}^{\infty} \Delta R(\omega)\,|\tilde{h}(\omega)|^2\left(\omega^2 - \frac{q}{p}\right) d\omega, \tag{21}$$

where

$$p = \int_{-\infty}^{\infty} [\Delta R]^2 d\omega, \quad q = \int_{-\infty}^{\infty} [\Delta R]^2\omega^2 d\omega,$$

$$r = \int_{-\infty}^{\infty} [\Delta R]^2\omega^4 d\omega, \quad \Delta R(\omega) = R_f(\omega) - R_\varepsilon(\omega).$$

The results of numerical modeling using the adaptive estimates (20) and (21) of the parameters R_u are presented in Fig. 3. Variants with adaptation in one parameter k_1 ($k_2 = 0$), which corresponds to zeroth-order regularization (Fig. 3a), and in both parameters k_1 and k_2 (Fig. 3b) were studied. The conditions of the numerical experiment were the same as for Fig. 2b. The curves 1, 2, and 3 in Fig. 3 correspond to the results of reconstruction after 10, 100, and 200 iterations.

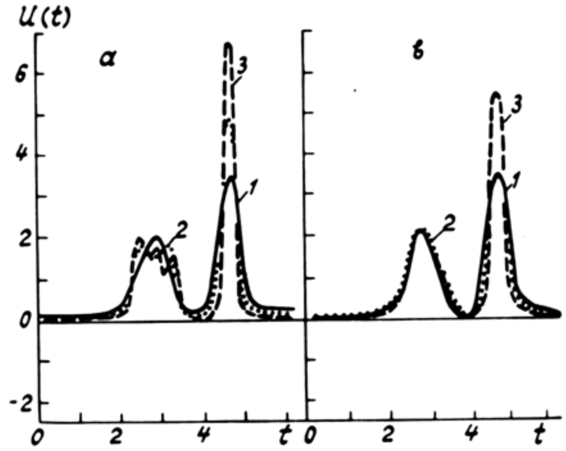

FIG. 3. Reconstruction with a priori uncertainty: 1), 2), 3) results of reconstruction after 10, 100, *and* 200 *iterations with* 1% *noise on the right side of the starting equation; a) zeroth-order regularization (adaptation in one parameter); b) first-order regularization (adaptation in two parameters).*

In conclusion we note that the implementation of the above-described algorithm using a standard FTT procedure required about three seconds of BÉSM-6 computer time for one iteration (all functions were represented on a grid with 128 points). This indicates that this algorithm is obviously more efficient than the direct-minimization methods.

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