PROBABILITY DENSITY FUNCTION OF THE "SPOTS" OF AEROSOL DEPOSITS ON THE UNDERLYING SURFACE

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This paper deals with the development of the method of determination of the mean value and standard deviation of the areas of aerosol deposition "spots" on the underlying surface. The probability density function of the aerosol deposition spots area on the underlying surface was obtained as an exact analytical solution of Kolmogorov equation. For the aerosol pollution dispersal in the air over Novosibirsk, some examples of the practical application of the probability density function are discussed.

Determination of the aerosol fallout density on the underlying surface is needed when solving some applied problems. The geometrical structure of the deposits is such that the zones with different fallout density are often separated by almost clean areas. This phenomenon was called "spottiness".[1] This phenomenon is connected with the statistical character of the process of aerosol dispersal in the atmosphere.

Methods of determining mathematical expectation and variance of the aerosol deposit spot area on the underlying surface are described in Ref. 2 using the approach developed in Ref. 3. More complete information about the distribution of spot area can be obtained using the probability density function (PDF). For instance, if this function is known, one can determine the probability that the spot area S exceeds certain preset value. The aim of the paper is to obtain PDF of spot area and to study some of its characteristics.

Within a certain domain Ω, the value S can be obtained as follows:

$$S = \iint_{\Omega} g(x, y, t)\, \mathrm{d}x\mathrm{d}y \; ;$$

$$g(x, y, t) = \begin{cases} 1, & \text{if } q(x, y, t) \ge q_0 \\ 0, & \text{if } q(x, y, t) < q_0 \end{cases} , \qquad (1)$$

where $q(x, y, t)$ is the value of the fallout density at a point of the underlying surface with the coordinates x and y at the moment t, and q_0 is a preset threshold value of the fallout density. The range of the change of the value S is $0 \le S \le S_\Omega$ where S_Ω is the area of the domain Ω and $t \ge 0$. In fact, one can always choose a domain Ω such that S is much less than S_Ω. Then one can consider that $S_\Omega = +\infty$. Let us temporarily set non-zero initial values for the area and time moment $S_0 \le S < +\infty$, $t_0 \le t$.

The change of the aerosol concentration at a given point of the space can be approximately considered as a Markovian diffusion process under certain assumptions.[3,4] The change of the fallout density can also be described as a Markovian diffusion process under the same assumptions. As a consequence, we assume that the process of the change of the aerosol deposit spot area is also Markovian diffuse.

The Fokker–Planck–Kolmogorov equation[5] for the PDF of the area transform from the initial state S_0, t_0 to the final state S, t (the PDF is denoted by $f(S, t; S_0, t_0)$) has the form

$$\frac{\partial f}{\partial t} + V(t)\,\frac{\partial f}{\partial S} - Q(t)\,\frac{\partial^2 f}{\partial S^2} = 0 \; , \qquad (2)$$

where $V(t)$ is the mean value of the local rate of the spot area change, and the coefficient $Q(t)$ multiplied by two is the local rate of the increment change of the Markovian process considered.[5]

In the general case, the "particles" of the statistical area ensemble leave the boundary $S = S_0$ at different time. So there exists a non-zero probability that a part of them is at the boundary $S = S_0$ at $t > 0$. When $S > S_0$, PDF is a continuous and smooth function. So, allowing for Ref. 6, it should have the form

$$f(S, t; S_0, t_0) = \gamma_0(t, t_0)\, \delta(S - S_0) + \theta(S - S_0)\, f_1(S, t; S_0, t_0) \; , \qquad (3)$$

where γ_0 is the probability of observing the values $S = S_0$ in a given statistical ensemble; δ is the delta function; θ is the unit step function corresponding to it; f_1 is the continuos component of the PDF.

It is obvious that the initial and boundary conditions for $S = +\infty$ are as follows:

$$f_1(S, t_0; S_0, t_0) = \delta(S - S_0) \; ; \; f_1(+\infty, t; S_0, t_0) = 0 \; . \qquad (4)$$

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To formulate the boundary condition for $S = S_0$ and derive the expression for γ_0, we use the known technique.[6] Let us substitute Eq. (3) into Eq. (2), then multiply it by an arbitrary smooth function $\varphi(S)$ and integrate this expression over S between $-\infty$ and $+\infty$. Since the function φ is arbitrary, we obtain

$$\frac{\partial f_1}{\partial t} + V(t)\frac{\partial f_1}{\partial S} - Q(t)\frac{\partial^2 f_1}{\partial S^2} = 0 ,$$

$$\frac{\partial \gamma_0}{\partial t} - Q\left.\frac{\partial f_1}{\partial S}\right|_{S = S_0} = 0 ; \quad f_1(S_0, t; S_0, t_0) = 0 , \qquad (5)$$

what, in combination with Eq. (4), formally determines the system of the initial, boundary, and additional conditions intended for determining the probability γ_0.

However, it is impossible to proceed to solving the problem (4) and (5). Note that the derivative of γ_0 with respect to t is always negative when the spot area grows monotonically. On the other hand, the derivative of f_1 with respect to S is always positive for $S = S_0$ what makes an unresolvable contradiction. In this condition, one can apply the inverse Kolmogorov equation[5] which can be taken instead of Eq. (2) and relate[1] it to the initial state S_0, t_0

$$\frac{\partial f}{\partial t_0} + V(t_0)\frac{\partial f}{\partial S_0} + Q(t_0)\frac{\partial^2 f}{\partial S_0^2} = 0 . \qquad (6)$$

By making use of the procedure used in the derivation of Eq. (5), we obtain

$$\frac{\partial f_1}{\partial t_0} + V(t_0)\frac{\partial f_1}{\partial S_0} + Q(t_0)\frac{\partial^2 f_1}{\partial S_0^2} = 0 ; \quad (S_0 \le S,\ t_0 \le t) ;$$

$$\frac{\partial \gamma_0}{\partial t_0} - Q\left.\frac{\partial f_1}{\partial S_0}\right|_{S_0 = S} = 0 ; \quad f_1(S_0, t; S_0, t_0) = 0 . \qquad (7)$$

The correctness of this procedure can be verified by solving the problem. Now, we note that the initial and the boundary conditions for $S_0 = S$ are contradictory at the initial time moment. This inconvenience can be removed by the transition from the PDF $f(S, t; S_0, t_0)$ to the distribution function of the area $F(S, t; S_0, t_0)$ which can be obtained by integration of Eq. (3) over S between $-\infty$ and S. The structure of the function F has the form $F(S, t; S_0, t_0) = \theta(S - S_0)\, F_1(S, t; S_0, t_0)$. Thus, let us formulate the problem in the following form

$$\frac{\partial F}{\partial t_0} + V(t_0)\frac{\partial F}{\partial S_0} + Q(t_0)\frac{\partial^2 F}{\partial S_0^2} = 0 ;$$

$$F(+\infty, t; S_0, t_0) = 1 ; \ F(S, t_0; S_0, t_0) = \begin{cases} 1, & S > S_0 \\ 0, & S < S_0 \end{cases} . \qquad (8)$$

One more boundary condition for F determines the PDF normalization but we do not fix it at this step. The following relation

$$F_1 = 1 + \frac{1}{2}\left[\mathrm{erf}\left(\frac{S - S_0 - \beta_1'}{\beta_2'}\right) - \mathrm{erf}\left(\frac{S + S_0 + \beta_1'}{\beta_2'}\right)\right] ;$$

$$\beta_1' = \int\limits_{t_0}^{t} V(t_1)\mathrm{d}t_1 ; \quad \beta_2' = 2\left[\int\limits_{t_0}^{t} Q(t_1)\mathrm{d}t_1\right]^{1/2} , \qquad (9)$$

where erf is the probability integral being an exact solution of the equation (8), and the function F satisfies the initial and boundary conditions for Eq. (8).

By differentiating F with respect to S we obtain the relation for the PDF of the area

$$f(S, t) = \left[1 - \mathrm{erf}\left(\frac{\beta_1}{\beta_2}\right)\right]\delta(S) + \theta(S)\, f_1(S, t) ; \qquad (10)$$

$$f_1(S, t) = \frac{1}{\pi^{1/2}\beta_2}\left\{\exp\left[-\left(\frac{S - \beta_1}{\beta_2}\right)^2\right] - \right.$$

$$\left. - \exp\left[-\left(\frac{S + \beta_1}{\beta_2}\right)^2\right]\right\} ;$$

$$\beta_1 = \int\limits_{0}^{t} V(t_1)\mathrm{d}t_1 ; \quad \beta_2 = 2\left[\int\limits_{0}^{t} Q(t_1)\mathrm{d}t_1\right]^{1/2} .$$

Let us consider some properties of this solution. It is easy to see that PDF (10) is normalized by unity. The parameter β_1 is the mathematical expectation of the spot area. It and the variance of spot area can be obtained in accordance with Ref. 2. To apply the PDF (10) in practice, let us connect the value of Q and, consequently, the parameter β_2 with the variance σ^2 of the spot area. Calculations lead to the following expression[3]

$$\frac{\sigma^2}{\beta_1^2} = \frac{1 - \gamma_0}{2\beta_0^2} - \gamma_0 + \frac{1}{\pi^{1/2}\beta_0}\exp(-\beta_0^2) ; \quad \beta_0 = \frac{\beta_1}{\beta_2} . \qquad (11)$$

The probability γ_0 takes zero value and the form of the PDF is close to the normal distribution and $2^{1/2}\sigma = \beta_2$ in the case when β_0 tends to infinity (in fact, it is already fulfilled when $\beta_0 > 2$, see Ref. 3). When $\beta_0 = 0$, the PDF degenerates into the delta function.

In the general case, the process of wind rise of particles can occur simultaneously with their deposition. For instance, this process is observed when wind blows aerosols containing radioactive nuclides off the underlying surface. These processes compete. If deposition dominates over the rise, β_1 grows, otherwise β_1 decreases. It is easy to verify that the PDF (10) describes the process considered adequately. It should be noted that the solution presented is valid only if the value S_0 is zero. The solution for S_0 different from zero is not yet obtained what does not exclude the application of approximate numerical methods to solving the above problem in the general case.

TABLE I. Calculated values of the probabilities P_i ($\beta_1 > S$) for S = 1.5·10^8, 8.3·10^7, 1.4·10^7, 4.0·10^6 m^2.

q_0, g/m^2	β_1, m^2	σ, m^2	P_1	P_2	P_3	P_4
$1.0 \cdot 10^{-6}$	$1.5 \cdot 10^{8}$	$4.0 \cdot 10^{7}$	0.50	0.95	1.00	1.00
$2.0 \cdot 10^{-6}$	$1.3 \cdot 10^{8}$	$8.5 \cdot 10^{7}$	0.41	0.70	0.86	0.86
$5.0 \cdot 10^{-6}$	$8.3 \cdot 10^{7}$	$3.9 \cdot 10^{7}$	0.04	0.50	0.95	0.96
$1.0 \cdot 10^{-5}$	$5.4 \cdot 10^{7}$	$3.6 \cdot 10^{7}$	0.00	0.22	0.83	0.85
$2.5 \cdot 10^{-5}$	$1.4 \cdot 10^{7}$	$4.6 \cdot 10^{7}$	0.04	0.08	0.11	0.11
$4.1 \cdot 10^{-5}$	$4.0 \cdot 10^{6}$	$8.5 \cdot 10^{5}$	0.00	0.00	0.00	0.50

To illustrate the application of the algorithms proposed and obtain realistic estimations of pollution characteristics under typical meteorological conditions, a series of calculations on propagation of ash released by the heat and power station HPS-2 of Novosibirsk was performed. A part of the results related to the subject of the present paper is given in the table. A hypothetical example of a snap-action discharge of 32 kg of ash at 15:00 on the 1st of June under the wind velocity of 2 m/s at the level of 2 m. The designations used in the table are as follows: P_i is the probability that the mathematical expectation of spot area β_1 with the standard deviation σ exceeds the area S at which the density of the fallout is larger than the ultimate value q_0 on the territory considered; the probabilities P_i for i = 1, 2, 3, 4 are obtained for S = $1.5 \cdot 10^8$, $8.3 \cdot 10^7$, $1.4 \cdot 10^7$, and $4.0 \cdot 10^6$ m^2, the corresponding values q_0 (g/m^2) equal $1.0 \cdot 10^{-6}$, $5.0 \cdot 10^{-6}$, $5 \cdot 10^{-5}$; $4.1 \cdot 10^{-5}$. One can see that the decrease of S results in an increase the value of P_i. The decrease of β_1 decreases P_i. The calculations also demonstrate that the decrease of the standard deviation of spot area σ at a constant value of β_1 leads to an increase of the probability P_i. When $S = \beta_1$, the probability P_i is equal to 0.5. The values of the probabilities P_i are quite realistic what makes an additional proof of the results obtained in the paper.

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