ON COMPUTATIONAL MODELS FOR DENSITIES AND FLUXES OF SOLAR RADIATION

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We formulate linear and nonlinear exact and approximate mathematical computational models for spherical and hemispherical densities and fluxes of solar radiation in scattering and absorbing inhomogeneous plane parallel layers. In particular, we consider Rayleigh and conservative layers. We place emphasis on the formulation of boundary conditions for these models.

The principal spectral energy characteristics entering the models of climate, weather forecasting schemes, radiative and spectral radiative budget calculations, simulations of the dynamics of ozone in the troposphere and stratosphere, of physicochemical kinetics of the atmosphere, and of the state of environment,[1–3] are mostly the integral radiative characteristics density and flux. One needs for accurate and fast algorithms to calculate solar radiation densities and fluxes in the atmosphere, ocean, and clouds in order to simulate mathematically radiative processes in the terrestrial environment within a wide spectral range from UV to IR. Engineering techniques of radiative correction for use in remote sensing of natural objects, of underlying surfaces and of the "atmosphere–ocean" system, which are called the two-stream, diffuse, P_1–approximations of the technique of spherical harmonics,[4–9] the Sobolev approximations,[10–13] the Eddington,[14] and δ-Eddington[15] approximations etc.,[16] are in fact based on computing angle-integrated radiative characteristics, which are different from fluxes and densities by their norms. Highly accurate algorithms used to solve numerically the equation of radiation transfer in the optically middle–dense and dense layers, which employ nonlinear procedures to speed up the convergence of iterations of the quasidifusion type[17], and the technique of average fluxes[18,19] often include calculating densities or fluxes, using nonlinear coefficients as an auxiliary part of the problem.

This paper presents linear and nonlinear exact and approximate models for calculating spherical and hemispherical densities and fluxes of solar radiation in homogeneous and heterogeneous conservative and nonconservative absorbing and scattering layers with either Lambertian or absolutely black boundaries.[20–22]

Refs. 10–13 cite the V.V. Sobolev approximate analytical solutions for spherical densities and fluxes of solar radiation in a homogeneous layer, which coincide with the Eddington approximation and the P_1–approximation of the technique of spherical harmonics and offer an algorithm to perform such calculations in heterogeneous layers, by sewing explicit solutions for homogeneous layers at their boundaries. Such an algorithm is described in Ref. 14; it has found wide use in computer versions of the radiation transfer blocks of climate, weather forecast, photochemistry models, and models for remote sensing. In particular, it is used in the international LOWTRAN 7 code (1990 version). However, such an approach does not always yield a stable solution or necessary accuracy.

We propose fast discrete computational algorithms for radiation densities and fluxes in models of the type of the "equation of diffusion" (i.e. ordinary second order differential equation of the first, second, or third types),[23–24] or of the system of two common differential equations of the first type.[25,26] To do this we construct homogeneous conservative differential schemes, which are solved by the technique of right-hand or flux through-put.[27,28] The background angular distributions of solar radiation or the transmission functions not corrected for multiple scattering, are calculated by integrating over the characteristics of the transfer equation, while its integral of collisions is computed using densities and fluxes.

INTEGRAL RADIATIVE CHARACTERISTICS

The intensity of solar radiation $\Phi(z, \vartheta, \varphi)$ repeatedly scattered into a direction described by zenith angle $\vartheta\in[0,\pi]$, $\mu=\cos\vartheta\in[-1, 1]$, and azimuth $\varphi\in[0, 2\pi]$ at the level $z\in[0, H]$ in a plane-parallel layer is determined by the boundary–value problem

$$\begin{cases} \mu \dfrac{\partial\Phi}{\partial z} + \sigma_t(z)\, F(z, \mu, \varphi) = B(z, \mu, \varphi) + F(z, \mu, \varphi), \\ \Phi\Big|_{\Gamma_0} = 0, \ \Phi\Big|_{\Gamma_H} = f_H, \end{cases} \quad (1)$$

Here

$$B(z, \mu, \varphi) \equiv \frac{\sigma_s(z)}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \Phi(z, \mu', \varphi')\, \gamma(z, \cos\chi)\, d\mu'\, d\varphi' \quad (2)$$

is the integral of collisions;

$$F(z, \mu, \varphi) = a(z)\, \gamma(z, \cos\chi_0),$$
$$a(z) \equiv \frac{S_\lambda}{4}\, \sigma_s(z) \exp\left[-\frac{\tau(z)}{\mu_0}\right], \quad (3)$$
$$\cos\chi = \mu\mu' + \sin\vartheta \sin\vartheta' \cos(\varphi - \varphi')$$

is the source;

$$f_H = \frac{q}{\pi} \int_0^{2\pi} \int_0^{1} \Phi(H, \mu, \varphi)\mu\, d\mu\, d\varphi + q\, S_\lambda\, \mu_0 \exp\left[-\frac{\tau(H)}{\mu_0}\right] \quad (4)$$

is the boundary value at the Lambertian surface with albedo q;

$$\frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \gamma(z, \cos\chi)\, d\mu\, d\varphi = \frac{1}{2} \int_{-1}^{1} \gamma(z, \cos\chi)\, d\cos\chi = 1 \quad (5)$$

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is the norm of the scattering phase function;

$$\tau(z) = \int_0^z \sigma_t(u)\, du, \quad \tau_H \equiv \tau(H) \tag{6}$$

is the optical depth of a layer, irradiated by parallel flux of solar radiation of intensity S_λ, incident on its top along the direction described by angles ϑ_0, φ_0 ($\mu_0=\cos\vartheta_0$). For convenience we use the sets

$$\Gamma_0 = \{(z, \mu): z = 0, \mu \geq 0\}, \ \Gamma_H = \{(z, \mu): z = H, \mu \leq 0\}.$$

If the spatial coordinate z is geometric, variables $\sigma_t(z)$ and $\sigma_s(z)$ are the coefficients of extinction and scattering; however, if z is the optical depth, then $\sigma_t(z) \equiv 1$, and $\sigma_s(z)$ is the albedo of a scattering act. Thus, a sufficiently general formulation of our boundary–value problem (1) makes it possible to have various presentations of the models, which would account for the δ – anisotropy of scattering, while the initial equation of radiation transfer is either transformed using similarity relations,[15] or not transformed at all.[19]

The regular approximation of the continuous solution to the problem (1), given at each point of a sphere by linear combinations[20]

$$\Phi(z, \vartheta, \varphi) = \sum_{k=0}^{\infty} Y_k(z, \vartheta, \varphi)$$

of spherical functions

$$Y_k(z, \vartheta, \varphi) = \sum_{m=0}^{k} \Phi_{ck}^m(z)\, C_k^m(\vartheta, \varphi) + \Phi_{sk}^m(z)\, S_k^m(\vartheta, \varphi),$$

results in separation of variables z, ϑ, and φ. Spherical harmonics

$$C_k^m(\vartheta, \varphi) = P_k^m(\mu) \cos m\varphi, \ k = 0, 1, 2, \ldots ; \ m = 0, 1, 2, \ldots , k;$$

$$S_k^m(\vartheta, \varphi)=(1-\delta_{m0})P_k^m(\mu) \sin m\varphi, \ k=0, 1, 2, \ldots; m=0, 1, 2, \ldots, k;$$

($m \leq k$) form an orthogonal system on a unit sphere. Here δ_{mn} is the Kronekker symbol ($\delta_{mn} = \{1$ if $m = n$, 0 if $m \neq n\}$); $P_k^m(\mu)$ are the associated Legendre functions; $P_k(\mu) = P_k^0(\mu)$ are the Legendre polynomials.

The azimuthal harmonics

$$\Phi_c^m(z, \vartheta) = \sum_{k=m}^{\infty} \Phi_{ck}^m(z)\, P_k^m(\vartheta),$$

$$\Phi_s^m(z, \vartheta) = \sum_{k=m}^{\infty} \Phi_{sk}^m(z)\, P_k^m(\vartheta),$$

$$\Phi_c^0(z, \mu) = \sum_{k=0}^{\infty} \Phi_{ck}^0(z)\, P_k(\mu) \tag{7}$$

are determined by formulas

$$\Phi_c^m(z, \vartheta) = \frac{1}{d_m \pi} \int_0^{2\pi} \Phi(z, \vartheta, \varphi) \cos m\varphi\, d\varphi,$$

$$\Phi_s^m(z, \vartheta) = \frac{1}{\delta_m \pi} \int_0^{2\pi} \Phi(z, \vartheta, \varphi) \sin m\varphi\, d\varphi,$$

$\delta_m = \{2$, if $m = 0$; 1, if $m > 0\}$.

Characteristics of radiation, integral over angles, are found in terms of the azimuthal and spherical harmonics. The radiation density (actinometric flux) is

$$n(z) = \int_0^{2\pi} \int_0^{\pi} \Phi(z, \vartheta, \varphi) \sin\vartheta\, d\vartheta\, d\varphi =$$

$$= 2\pi \int_{-1}^{1} \Phi_c^0(z, \mu)\, d\mu = 4\pi\, \Phi_{c0}^0(z). \tag{8}$$

The vertical flux of radiation

$$J(z)=\int_0^{2\pi} \int_0^{\pi} \Phi(z, \vartheta, \varphi) \cos\vartheta \sin\vartheta\, d\vartheta\, d\varphi=$$

$$= 2\pi \int_{-1}^{1} \Phi_c^0(z, \mu)\, \mu\, d\mu = \frac{4\pi}{3}\, \Phi_{c1}^0(z). \tag{9}$$

The flux of upward going radiation (hemispherical flux for $\mu<0$) is

$$J^{\uparrow}(z) =\int_0^{2\pi} \int_{\pi/2}^{\pi} \Phi(z, \vartheta, \varphi) \cos\vartheta \sin\vartheta\, d\vartheta\, d\varphi=2\pi \int_{-1}^{0} \Phi_c^0(z, \mu)\, \mu\, d\mu =$$

$$=-\pi \left\{\Phi_{c0}^0(z) - \frac{2}{3}\, \Phi_{c1}^0(z) + 2 \sum_{m=1}^{\infty} \Phi_{c,2m}^0(z)\, R_{2m}\right\}, \ J^{\uparrow}(z)<0, \tag{10}$$

$$R_{2m} \equiv \int_0^1 \mu\, P_{2m}(\mu)\, d\mu =(-1)^{m+1}\, \frac{(2m-3)!!}{2^m(m+1)!}\,.$$

The flux of downward going radiation (hemispherical flux for $\mu>0$) is

$$J^{\downarrow}(z) = \int_0^{2\pi} \int_0^{\pi/2} \Phi(z, \vartheta, \varphi)\cos\vartheta \sin\vartheta\, d\vartheta\, d\varphi= 2\pi \int_0^1 \Phi_c^0(z, \mu)\mu\, d\mu =$$

$$=\pi \left\{\Phi_{c0}^0(z) + \frac{2}{3}\, \Phi_{c1}^0(z) + 2 \sum_{m=1}^{\infty} \Phi_{c,2m}^0(z)\, R_{2m}\right\}, J^{\downarrow}(z) > 0. \tag{11}$$

The vertical flux of repeatedly scattered radiation is

$$J(z) = J^{\downarrow}(z) + J^{\uparrow}(z).$$

The horizontal flux of radiation is

$$W(z)=\int_0^{2\pi} \int_0^{\pi} \Phi(z, \vartheta, \varphi) \sin\vartheta \sin\vartheta\, d\vartheta\, d\varphi=2\pi \int_{-1}^{1} \Phi_c^0(z, \mu) \sin\vartheta\, d\mu =$$

$$= \pi \left\{\pi\Phi_{c0}^0(z) + 2 \sum_{m=1}^{\infty} \Phi_{c,2m}^0(z)\, R_{2m}^1\right\},$$

$$R_{2m}^1 \equiv \int_{-1}^{1} P_k(\mu)\, P_1^1(\mu)\, d\mu = \begin{cases} \pi/2 & \text{for } k = 0; \\ 0 & \text{for } k = 2m+1, \ m \geq 0; \\ \neq 0 & \text{for } k = 2m, \ m \geq 1. \end{cases}$$

The horizontal flux of radiation in the plane of the solar vertical[11] is

$$G(z) = \int_0^{2\pi}\int_0^{\pi} \Phi(z, \vartheta, \varphi) \sin\vartheta \cos\varphi \sin\vartheta \, d\vartheta \, d\varphi = \frac{4\pi}{3} \Phi^1_{c1}(z).$$

The hemispherical densities are

$$n^{\downarrow}(z) = \int_0^{2\pi}\int_0^{\pi/2} \Phi(z, \vartheta, \varphi) \sin\vartheta \, d\vartheta \, d\varphi = 2\pi \int_0^1 \Phi^0_c(z, \mu) \, d\mu =$$

$$= 2\pi \left\{ \Phi^0_{c0}(z) + \sum_{m=0}^{\infty} \Phi^0_{c,2m+1}(z) \, R^0_{2m+1} \right\}; \quad (12)$$

$$n^{\uparrow}(z) = \int_0^{2\pi}\int_{\pi/2}^{\pi} \Phi(z, \vartheta, \varphi) \sin\vartheta \, d\vartheta \, d\varphi = 2\pi \int_{-1}^0 \Phi^0_c(z, \mu) \, d\mu =$$

$$= 2\pi \left\{ \Phi^0_{c0}(z) - \sum_{m=0}^{\infty} \Phi^0_{c,2m+1}(z) \, R^0_{2m+1} \right\}; \quad (13)$$

$$R^0_{2m+1} \equiv \int_0^1 P_{2m+1}(\mu) \, d\mu = \frac{(-1)(-3)\ldots(-2m+1)}{2^{m+1}(m+1)!},$$

$m = 0, 1, 2, \ldots$

The average (spherical) cosine or the asymmetry coefficient for the brightness phase function is

$$\bar{\mu}(z) = \frac{J(z)}{n(z)} = \frac{1}{3} \frac{\Phi^0_{c1}(z)}{\Phi^0_{c0}(z)}. \quad (14)$$

The average (hemispherical) cosines are

$$\mu^{\downarrow}(z) = J^{\downarrow}(z) / n^{\downarrow}(z), \qquad \mu^{\uparrow}(z) = J^{\uparrow}(z) / n^{\uparrow}(z). \quad (15)$$

The K–integrals (second order moments) are

$$K(z) \equiv \int_0^{2\pi}\int_0^{\pi} \Phi(z, \vartheta, \varphi) [\cos\vartheta]^2 \sin\vartheta \, d\vartheta \, d\varphi =$$

$$= 2\pi \int_{-1}^1 \Phi^0_c(z, \mu) \, \mu^2 \, d\mu = \frac{4\pi}{3} \left[\Phi^0_{c0}(z) + \frac{2}{5} \Phi^0_{c2}(z) \right]. \quad (16)$$

The coefficient of diffusion is

$$D(z) = \frac{K(z)}{n(z)} = \frac{1}{3} + \frac{2 \, \Phi^0_{c2}(z)}{15 \, \Phi^0_{c0}(z)}. \quad (17)$$

The hemispherical K–inegrals are

$$K^{\downarrow}(z) \equiv \int_0^{2\pi}\int_0^{\pi/2} \Phi(z, \vartheta, \varphi) [\cos\vartheta]^2 \sin\vartheta \, d\vartheta \, d\varphi =$$

$$= 2\pi \int_0^1 \Phi^0_c(z, \mu) \, \mu^2 \, d\mu = 2\pi \left\{ \frac{1}{3} \Phi^0_{c0}(z) + \frac{1}{4} \Phi^0_{c1}(z) + \frac{2}{15} \Phi^0_{c2}(z) + \right.$$

$$\left. + \sum_{m=1}^{\infty} \Phi^0_{c,2m+1}(z) \, R^2_{2m+1} \right\}; \quad (18)$$

$$K^{\uparrow}(z) \equiv \int_0^{2\pi}\int_{\pi/2}^{\pi} \Phi(z, \vartheta, \varphi) [\cos\vartheta]^2 \sin\vartheta \, d\vartheta \, d\varphi =$$

$$= 2\pi \int_{-1}^0 \Phi^0_c(z, \mu) \, \mu^2 \, d\mu = 2\pi \left\{ \frac{1}{3} \Phi^0_{c0}(z) - \frac{1}{4} \Phi^0_{c1}(z) + \frac{2}{15} \Phi^0_{c2}(z) - \right.$$

$$\left. - \sum_{m=1}^{\infty} \Phi^0_{c,2m+1}(z) \, R^2_{2m+1} \right\}, \quad (19)$$

$$R^2_{2m+1} \equiv \int_0^1 \mu^2 \, P_{2m+1}(\mu) \, d\mu =$$

$$= \begin{cases} 1/24 \text{ at } m = 1, \\ (-1)^{m-1} \dfrac{(m-1)\, m\, (m+1) \ldots (2m-3)}{2^{2m-1}(m+2)!} \text{ at } m \geq 2. \end{cases}$$

The hemispherical diffusion coefficients are

$$D^{\downarrow}(z) = K^{\downarrow}(z) / n^{\downarrow}(z), \; D^{\uparrow}(z) = K^{\uparrow}(z) / n^{\uparrow}(z). \quad (20)$$

If the scattering phase function is presented by the series expansion over Legendre polynomials

$$\gamma(z, \cos\chi) = \sum_{k=0}^{\infty} \omega_k(z) \, P_k(\cos\chi), \quad (21)$$

one can separate angular variables using the theorem of summation[21], and then separate out the azimuthal harmonics which, in the general case, are calculated using the integrals[19,20,30]

$$\gamma^0(z, \mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} \gamma(\cos\chi) \, d\varphi = \sum_{k=0}^{\infty} \omega_k(z) \, P_k(\mu) \, P_k(\mu'), \quad (22)$$

$$\gamma^m(z, \mu, \mu') = \frac{1}{\pi \, \delta_m} \int_0^{2\pi} \gamma(\cos\chi) \cos m(\varphi - \varphi') \, d(\varphi - \varphi') =$$

$$= \frac{1}{\pi \, \delta_m} \int_{-1}^{1} \frac{\gamma(\cos\chi) \, T_m(y) \, dy}{\sqrt{1 - y^2}}, \quad (22)$$

where $y = \cos(\varphi - \varphi')$, $-1 \leq y \leq 1$, $T_m(y) = \cos m(\varphi - \varphi') = \cos(m \arccos y)$ are the mth order Chebyshev polynomials of the first type.

If the scattering phase function is presented in the form of a series expansion (21) over Legendre polynomials, we have

$$\gamma^+_0(\mu) \equiv \int_0^1 \gamma^0(z, \mu, \mu') d\mu' =$$

$$= 1 + \frac{1}{2} \omega_1 \mu + \sum_{m=1}^{\infty} \omega_{2m+1}(z) \, P_{2m+1}(\mu) \, R^0_{2m+1};$$

$$\gamma^-_0(\mu) \equiv \int_{-1}^0 \gamma^0(z, \mu, \mu') \, d\mu' =$$

$$= 1 - \frac{1}{2} \omega_1 \mu - \sum_{m=1}^{\infty} \omega_{2m+1}(z) \, P_{2m+1}(\mu) \, R^0_{2m+1};$$

$$\gamma_0(\mu) \equiv \int_{-1}^1 \gamma^0(z, \mu, \mu') \, d\mu' = \gamma^+_0(z, \mu) + \gamma^-_0(z, \mu) = 2.$$

The characteristics of backscattering

$$\gamma^{\downarrow}_0(z) \equiv \int_0^1 \Phi^0_c(z, \mu) \, d\mu \int_{-1}^0 \gamma^0(z, \mu, \mu') \, d\mu' \Big/ \int_0^1 \Phi^0_c(z, \mu) \, d\mu =$$

$$= \frac{2\pi \, \Gamma^{\uparrow}(z)}{n^{\uparrow}(z)} \quad (23)$$

and

$$\gamma_0^{\uparrow}(z) \equiv \int\limits_{-1}^{0} \Phi_c^0(z,\,\mu)\,d\mu \int\limits_{0}^{1} \gamma^0(z,\,\mu,\,\mu')\,d\mu' \,/\, \int\limits_{-1}^{0} \Phi_c^0(z,\,\mu)\,d\mu = \frac{2\pi\,\Gamma^{\uparrow}(z)}{n^{\uparrow}(z)} \quad (24)$$

may be expressed in terms of spherical harmonics employing presentations for Φ_c^0 (7), γ^0 (22), $n^{\downarrow}$ (12), and $n^{\uparrow}$ (13), so that

$$\Gamma^{\downarrow}(z) = \Phi_{c0}^0(z)\left[1 - \sum_{k=0}^{\infty}\omega_{2k+1}(z)\,(R_{2k+1}^0)^2\right] + \sum_{k=0}^{\infty}\Phi_{c,\,2k+1}^0(z)\,R_{2k+1}^0 - \sum_{k=0}^{\infty}\Phi_{c,\,2k+1}^0(z)\omega_{2k+1}(z) \times R_{2k+1}^0 T_{2k+1}^2 - \sum_{k=1}^{\infty}\Phi_{c,\,2k}^0(z)\sum_{n=0}^{\infty}\omega_{2n+1}(z)\,R_{2n+1}^0\,T_{2k,\,2n+1};$$

$$\Gamma^{\uparrow}(z) = \Phi_{c0}^0(z)\left[1 + \sum_{k=0}^{\infty}\omega_{2k+1}(z)\,(R_{2k+1}^0)^2\right] - \sum_{k=0}^{\infty}\Phi_{c,\,2k+1}^0(z)\,R_{2k+1}^0 - \sum_{k=0}^{\infty}\Phi_{c,\,2k+1}^0(z)\,\omega_{2k+1}(z) \times R_{2k+1}^0 T_{2k+1}^2 + \sum_{k=1}^{\infty}\Phi_{c,\,2k}^0(z)\sum_{n=0}^{\infty}\omega_{2n+1}(z)\,R_{2n+1}^0\,T_{2k,\,2n+1};$$

$$T_n^2 \equiv \int\limits_0^1 [P_n(\mu)]^2\,d\mu = \frac{1}{2\,n+1};$$

$$T_{2k,\,2n+1} \equiv \int\limits_0^1 P_{2k}(\mu)\,P_{2n+1}(\mu)\,d\mu = \frac{(2\,k)!\,(2\,n+1)!}{2^{2k+2n+1}\,(2\,k-2\,n-1)\,(k+n+1)\,k!\,n!}.$$

Using the exact presentations for $n^{\downarrow}$ (12), $n^{\uparrow}$ (13), $J^{\uparrow}$ (10), and $J^{\downarrow}$ (11) one may establish the following exact relations:

$$J^{\downarrow} + J^{\uparrow} = \frac{4\pi}{3}\,\Phi_{c1}^0(z), \quad n^{\downarrow} + n^{\uparrow} = 4\pi\,\Phi_{c0}^0(z),$$

$$J^{\downarrow} - J^{\uparrow} = 2\pi\left\{\Phi_{c0}^0(z) + 2\sum_{m=1}^{\infty}\Phi_{c,2m}^0(z)\,R_{2m}\right\},$$

$$\Phi_{c1}^0 = \frac{3}{4\pi}\,(J^{\downarrow} + J^{\uparrow}),$$

$$n^{\downarrow} - n^{\uparrow} = 4\pi\sum_{m=0}^{\infty}\Phi_{c,2m+1}^0(z)\,R_{2m+1}^0, \qquad \Phi_{c0}^0 = \frac{1}{4\pi}\,(n^{\downarrow} + n^{\uparrow}).$$

In context of the P_1–approximation of the technique of spherical harmonics, when

$$\Phi_c^0(z,\,\mu) = \Phi_{c0}^0(z) + \Phi_{c1}^0(z)\,P_1(\mu), \qquad P_1(\mu) = \mu, \quad (25)$$

radiative characteristics take the following values:

$$D(z) = \frac{1}{3}, \quad K(z) = \frac{4\pi}{3}\,\Phi_{c0}^0(z), \quad W(z) = \pi^2\,\Phi_{c0}^0(z), \quad (26)$$

$$J^{\downarrow} = \pi\,\{\Phi_{c0}^0(z) + \frac{2}{3}\,\Phi_{c1}^0(z)\}, \quad J^{\downarrow} = -\pi\,\{\Phi_{c0}^0(z) - \frac{2}{3}\,\Phi_{c1}^0(z)\}, \quad (27)$$

$$n^{\downarrow} = 2\pi\,\{\Phi_{c0}^0(z) + \frac{1}{2}\,\Phi_{c1}^0(z)\}, \quad n^{\downarrow} = 2\pi\,\{\Phi_{c0}^0(z) - \frac{1}{2}\,\Phi_{c1}^0(z)\}, \quad (28)$$

$$\mu^{\uparrow}(z) = -\frac{1}{2}\left[1 - \frac{1}{6}\,\frac{\Phi_{c1}^0(z)}{\Phi_{c0}^0(z) - \frac{1}{2}\,\Phi_{c1}^0(z)}\right],$$

$$\mu^{\downarrow}(z) = \frac{1}{2}\left[1 + \frac{1}{6}\,\frac{\Phi_{c1}^0(z)}{\Phi_{c0}^0(z) + \frac{1}{2}\,\Phi_{c1}^0(z)}\right], \quad (29)$$

$$K^{\downarrow}(z) = 2\pi\left\{\frac{1}{3}\,\Phi_{c0}^0(z) + \frac{1}{4}\,\Phi_{c1}^0(z)\right\},$$

$$K^{\uparrow}(z) = 2\pi\left\{\frac{1}{3}\,\Phi_{c0}^0(z) - \frac{1}{4}\,\Phi_{c1}^0(z)\right\}, \quad (30)$$

$$D^{\downarrow}(z) = \frac{1}{3} + \frac{1}{12}\,\frac{\Phi_{c1}^0(z)}{\Phi_{c0}^0(z) + \frac{1}{2}\,\Phi_{c1}^0(z)},$$

$$D^{\uparrow}(z) = \frac{1}{3} - \frac{1}{12}\,\frac{\Phi_{c1}^0(z)}{\Phi_{c0}^0(z) - \frac{1}{2}\,\Phi_{c1}^0(z)}, \quad (31)$$

$$\Gamma^{\downarrow}(z) = \Phi_{c0}^0(z)\left[1 - \sum_{k=0}^{\infty}\omega_{2k+1}(z)\,(R_{2k+1}^0)^2\right] + \frac{1}{2}\,\Phi_{c1}^0(z)\left[1 - \frac{\omega_1(z)}{3}\right],$$

$$\Gamma^{\uparrow}(z) = \Phi_{c0}^0(z)\left[1 + \sum_{k=0}^{\infty}\omega_{2k+1}(z)\,(R_{2k+1}^0)^2\right] - \frac{1}{2}\,\Phi_{c1}^0(z)\left[1 + \frac{\omega_1(z)}{3}\right],$$

and the following approximate relations are valid:

$$J^{\downarrow} - J^{\uparrow} = 2\pi\,F_{c0}^0(z), \; n^{\downarrow} - n^{\uparrow} = 2\pi\,\Phi_{c1}^0(z),$$

$$J = J^{\pm} + J^{\uparrow} = \frac{2}{3}\,(n^{\downarrow} - n^{\uparrow}), \; n = n^{\downarrow} + n^{\uparrow} = 2(J^{\downarrow} - J^{\uparrow}),$$

$$\Phi_{c0}^0(z) = \frac{1}{2\pi}\,(J^{\downarrow} - J^{\uparrow}),$$

$$\Phi_{c1}^0(z) = \frac{1}{2\pi}\,(n^{\downarrow} - n^{\uparrow}),$$

$$\Phi_{c1}^0(z) = \frac{3}{7\pi}\,(2\,J^{\downarrow} - n^{\uparrow}),$$

$$\Phi_{c0}^0(z) = \frac{3}{7\pi}\left(\frac{2}{3}\,n^{\uparrow} - J^{\downarrow}\right),$$

$$n^{\downarrow} = \frac{1}{4}\,(7\,J^{\downarrow} - J^{\uparrow}), \; n^{\uparrow} = \frac{1}{4}\,(J^{\downarrow} - 7\,J^{\uparrow}),$$

$$n^{\downarrow} = \frac{1}{2}\,(n + \frac{2}{3}\,J), \; n^{\uparrow} = \frac{1}{2}\,(n - \frac{3}{2}\,J),$$

$$J^{\downarrow} = \frac{1}{12}\,(7\,n^{\downarrow} - n^{\uparrow}), \; J^{\uparrow} = \frac{1}{12}\,(n^{\downarrow} - 7\,n^{\uparrow}),$$

$$J^{\downarrow} = \frac{1}{4}\,(n + 2\,J), \; J^{\uparrow} = -\frac{1}{4}\,(n - 2\,J),$$

$$\mu^{\downarrow}(z) = \frac{7\,n^{\downarrow} - n^{\uparrow}}{12\,n^{\downarrow}} = \frac{4\,J^{\downarrow}}{7\,J^{\downarrow} - J^{\uparrow}}\,,$$

$$\mu^{\uparrow}(z) = \frac{n^{\downarrow} - 7\,n^{\uparrow}}{12\,n^{\uparrow}} = \frac{4\,J^{\uparrow}}{J^{\downarrow} - 7\,J^{\uparrow}}\,,$$

$$\mathrm{D}^{\downarrow}(z) = \frac{1}{3} + \frac{1}{12}\,\frac{n^{\downarrow} - n^{\uparrow}}{n^{\downarrow}} = \frac{5}{12} - \frac{1}{12}\,\frac{n^{\uparrow}}{n^{\downarrow}} = = \frac{1}{3} + \frac{1}{2}\,\frac{J^{\downarrow} + J^{\uparrow}}{7\,J^{\downarrow} - J^{\uparrow}}\,,$$

$$\mathrm{D}^{\uparrow}(z) = \frac{1}{3} - \frac{1}{12}\,\frac{n^{\downarrow} - n^{\uparrow}}{n^{\uparrow}} = \frac{5}{12} - \frac{1}{12}\,\frac{n^{\downarrow}}{n^{\uparrow}} = = \frac{1}{3} - \frac{1}{2}\,\frac{J^{\downarrow} + J^{\uparrow}}{J^{\downarrow} - 7\,J^{\uparrow}}\,.$$

If the scattering phase function involves two components then

$$\gamma_0^{+}(z,\,\mu) = 1 + \frac{1}{2}\,\omega_1(z)\,\mu, \quad \gamma_0^{-}(z,\,\mu) = 1 - \frac{1}{2}\,\omega_1(z)\,\mu,$$

$$\gamma_0^{\downarrow}(z) = 1 - \frac{1}{2}\,\omega_1(z)\,\mu^{\downarrow}(z), \quad \gamma_0^{\uparrow}(z) = 1 + \frac{1}{2}\,\omega_1(z)\,\mu^{\uparrow}(z).$$

For the Rayleigh scattering phase function

$$\gamma_R(\cos\chi) = \frac{3}{4}\,[1 + \cos^2\chi],$$

$$\gamma_R^0(\mu,\,\mu') = 1 + \frac{1}{2}\,P_2(\mu)\,P_2(\mu'),$$

with coefficients ω_0=1, ω_1=0, ω_2=0.5 and norm given by expression (5), we have

$$\gamma_0^{+}(z,\,\mu) = 1, \quad \gamma_0^{-}(z,\,\mu) = 1, \quad \gamma_0^{\downarrow}(z) = 1, \quad \gamma_0^{\uparrow}(z) = 1. \qquad (32)$$

P_1 AND P_2–APPROXIMATIONS OF THE TECHNIQUE OF SPHERICAL HARMONICS

As is seen from the above, all the integral radiative characteristics n (8), J (9), W, K (16), D (17), $\bar{\mu}$ (14), $n^{\downarrow}$ (12), $n^{\uparrow}$ (13), $J^{\downarrow}$(11), $J^{\uparrow}$(10), $\mu^{\downarrow}$, $\mu^{\uparrow}$ (15), $\gamma_0^{\downarrow}$ (23), $\gamma_0^{\uparrow}$ (24), $K^{\downarrow}$ (18), $K^{\uparrow}$ (19), $\mathrm{D}^{\downarrow}$, $\mathrm{D}^{\uparrow}$ (20) are determined by zeroth azimuthal harmonic $\Phi_c^0\,(z,\,\mu)$ which is the solution to the boundary–value problem[19,20]

$$\begin{cases} \mu\dfrac{\partial\Phi_c^0}{\partial z} + \sigma_t(z)\,\Phi_c^0(z,\mu) = \dfrac{\sigma_s(z)}{2}\displaystyle\int_{-1}^{1}\Phi_c^0(z,\,\mu')\,\times \\ \times\,\gamma^0(z,\,\mu,\,\mu')\,d\mu' + a(z)\,\gamma^0(z,\,\mu,\,\mu_0), \\ \Phi_c^0\Big|_{\Gamma_0} = 0,\ \Phi_c^0\Big|_{\Gamma_H} = \Phi^*, \end{cases} \qquad (33)$$

$$\Phi^* \equiv 2\,q\int_0^1 \Phi_c^0(H,\,\mu)\,\mu\,\mathrm{d}\,\mu + f_H^*,$$

$$f_H^* \equiv q\,S_\lambda\,\mu_0\,\exp\left[-\frac{\tau(H)}{\mu_0}\right],$$

obtained by integrating equation (1) over azimuth $\varphi \in [0,2\pi]$ with the weight $1/2\pi$.

Spherical densities n (8) are exactly determined by spherical harmonic $\Phi_{c0}^0\,(z)$, fluxes J (9) by $\Phi_{c1}^0\,(z)$, K – integrals (16) and the coefficients of diffusion D (17) using $\Phi_{c2}^0\,(z)$ and $\Phi_{c0}^0\,(z)$, average cosines $\bar{\mu}$ (14) are determined by $\Phi_{c1}^0\,(z)$ and $\Phi_{c0}^0(z)$. Other characteristics may be calculated only approximately, using spherical harmonics, since they are presented by infinite series.

The system of equations for spherical harmonics[4–8] is infinite. In context of the P_n– approximation of the technique of spherical harmonics, expansion (7) is formally truncated, so that all the components $\Phi_{ck}^0\,(z)$ with numbers k>n are omitted and, as a consequence, the components to be accounted for in the expansion of the scattering phase function (22) with the index k varing from 0 to K and $k \le K \le n$.

We write out those equations which contain the harmonics Φ_{c0}^0, Φ_{c1}^0, Φ_{c2}^0 (Ref. 20)

$$\frac{1}{3}\,\frac{\mathrm{d}\,\Phi_{c1}^0}{\mathrm{d}\,z} + [\sigma_t(z) - \sigma_s(z)]\,\Phi_{c0}^0(z) = a(z); \qquad (34)$$

$$\frac{2}{5}\,\frac{\mathrm{d}\,\Phi_{c2}^0}{\mathrm{d}\,z} + \frac{\mathrm{d}\,\Phi_{c0}^0}{\mathrm{d}\,z} + \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_1(z)}{3}\right]\times$$

$$\times\,\Phi_{c1}^0(z) = a(z)\,\omega_1(z)\,\mu_0; \qquad (35)$$

$$\frac{3}{7}\,\frac{\mathrm{d}\,\Phi_{c3}^0}{\mathrm{d}\,z} + \frac{2}{3}\,\frac{\mathrm{d}\,\Phi_{c1}^0}{\mathrm{d}\,z} + \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_2(z)}{5}\right]\times$$

$$\times\,\Phi_{c2}^0(z) = a(z)\,\omega_2(z)\,P_2(\mu_0). \qquad (36)$$

The harmonic F_{c2}^0 is omitted within the P_1– approximation, so that a closed system of equations (34) is obtained, and

$$\frac{\mathrm{d}\Phi_{c0}^0}{\mathrm{d}z} + \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_1(z)}{3}\right]\mathrm{F}_{c1}^0(z) = a(z)\,\omega_1(z)\mu_0. \qquad (37)$$

Omitting the Φ_{c3}^0 component within the P_2– approximation, we find an explicit expression from equations (34) and (36)

$$\Phi_{c2}^0(z) = A(z) + B(z)\,\Phi_{c0}^0(z), \qquad (38)$$

$$A(z) = 2\,a(z)\left[\frac{\omega_2(z)}{2}\,P_2(\mu_0) - 1\right] \Big/ \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_2(z)}{5}\right],$$

$$B(z) = 2\,[\sigma_t(z) - \sigma_s(z)] \Big/ \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_2(z)}{5}\right],$$

using which one may approximately estimate $\Phi_{c2}^0\,(z)$ within the P_1– approximation using spherical harmonic $\Phi_{c0}^0\,(z)$ and correct the radiative characteristics containing $\Phi_{c2}^0\,(z)$. Note that, within the P_1– approximation, the radiative characteristics corresponding to isotropic and Rayleigh scattering phase functions coincide with each other. However, they differ from each other within the P_2– approximation, since the component $\Phi_{c2}^0\,(z)$ accounts for the anisotropy of Rayleigh scattering. Using the presentation (38) for $\Phi_{c2}^0\,(z)$ one may reduce equation (35) to the form

$$r(z)\,\frac{d\Phi^0_{c0}}{dz} + t(z)\,\Phi^0_{c0}(z) + s(z)\,\Phi^0_{c1}(z) = p(z), \qquad (39)$$

$$r(z) = 1 + \frac{2}{5}\,B(z)\,,\quad t(z) = 1 + \frac{2}{5}\,\frac{d\,B}{d\,z}\,,$$

$$s(z) = \sigma_t(z) - \frac{\sigma_s(z)\,\omega_1(z)}{3}\,,$$

$$p(z) = a(z)\,\omega_1(z)\,\mu_0 - \frac{2}{5}\,\frac{d\,A}{d\,z} = C(z)\,\exp\left[-\frac{\tau(z)}{\mu_0}\right],$$

so then the harmonics $\Phi^0_{c0}(z)$ and $\Phi^0_{c1}(z)$ may be calculated within the P_2–approximation using the system of equations (34) and (39).

If one introduces the coefficient of asymmetry of the scattering phase function

$$g(z) = \int\limits_{-1}^{1} \gamma(\mu)\,\mu\,d\,\mu \Big/ \int\limits_{-1}^{1} \gamma(\mu)\,d\,\mu = \frac{\omega_1(z)}{3}$$

and proceeds to the optical depth (6), then the system of equations (34) and (37) may be written in the form

$$\frac{1}{3}\,\frac{d\,\Phi^0_{c1}}{d\,\tau} + [1 - \omega_s(\tau)]\,\Phi^0_{c0}(\tau) = \frac{S_\lambda}{4}\,\omega_s(\tau)\,\exp\left[-\frac{\tau}{\mu_0}\right], \qquad (40)$$

$$\frac{d\,\Phi^0_{c0}}{d\,t} + [1 - \omega_s(\tau)\,g(\tau)]\,\Phi^0_{c1}(\tau) =$$

$$= \frac{3}{4}\,S_\lambda\,\omega_s(\tau)\,g(\tau)\,\mu_0\,\exp\left[-\frac{\tau}{\mu_0}\right]. \qquad (41)$$

Comparing the system of equations (40)–(41) with Sobolev approximation,[10–13] we have

$$\overline{\Phi} = \Phi^0_{c0}\,,\quad \overline{H} = \Phi^0_{c1} / 3,\quad n = 4\pi\,\overline{\Phi},\ J = 4\,\pi\,\overline{H}.$$

Note that, according to Refs. 14–15, the spherical harmonics $I_0 = \Phi^0_{c0}$, $I_1 = \Phi^0_{c1}$, and the system (40)–(41) coincides with the equations from the Eddington approximation. Note also that one–sided downward fluxes introduced in Ref.14 coincide with our fluxes ($F^{\downarrow} = J^{\downarrow}$), while the upward ones have opposite signs ($F^{\uparrow} = -J^{\uparrow}$).

EXACT AND APPROXIMATE EQUATIONS FOR SPHERICAL DENSITY AND FLUX

Let us integrate equation (33) over μ within the interval $[-1,1]$ with unit weight and with weight μ, using the expansion of the zeroth azimuthal harmonic of the scattering phase function (22) and definitions of n (8), J (9), and D (17), to obtain a system of exact equations

$$\frac{dJ}{dz} + [\sigma_t(z) - \sigma_s(z)]\,n(z) = 4\,\pi\,a(z), \qquad (42)$$

$$\frac{d\,[\mathrm{D}(z)\,n(z)]}{dz} + \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_1(z)}{3}\right] J(z) = \frac{4\pi}{3}\,a(z)\,\omega_1(z)\,\mu_0. \qquad (43)$$

When the right–hand side of equation (43) is zero, the system (42)–(43) is called the "equations of quasi–diffusion"[17].

By excluding the density $n(z)$ from the system (42)–(43) we obtain exact equation for vertical flux

$$\frac{d}{d\,z}\,\frac{\mathrm{D}(z)}{\sigma_t(z) - \sigma_s(z)}\,\frac{d\,J}{d\,z} - \left[\sigma_t(z) - \frac{\sigma_s(z)\,\omega_1(z)}{3}\right] J(z) =$$

$$= -\left\{\frac{4\pi}{3}\,a(z)\,\omega_1(z)\,\mu_0 - \frac{d}{d\,z}\,\frac{\mathrm{D}(z)\,4\pi\,a(z)}{\sigma_t(z) - \sigma_s(z)}\right\}. \qquad (44)$$

By excluding the flux $J(z)$ we obtain exact equation of the "diffusion equation" type that describes vertical profile of the radiation density

$$\frac{d}{dz}\,\frac{3}{3\sigma_t(z) - \sigma_s(z)\,\omega_1(z)}\,\frac{d\,[\mathrm{D}(z)\,n(z)]}{dz} - [\sigma_t(z) - \sigma_s(z)]\,n(z) =$$

$$= -\left\{4\pi\,a(z) - \frac{d}{dz}\,\frac{4\pi\,a(z)\,\omega_1(z)\,\mu_0}{3\sigma_t(z) - \sigma_s(z)\,\omega_1(z)}\right\}. \qquad (45)$$

For a conservative layer with zero absorption, when $\sigma_t(z) = \sigma_s(z)$, both density and flux are found from the exact equations

$$\frac{d\,J}{d\,z} = 4\pi\,a(z), \qquad (46)$$

$$3\,\frac{d\,[\mathrm{D}(z)\,n(z)]}{d\,z} + \sigma_t(z)\,[3 - \omega_1(z)]\,J(z) = 4\pi\,a(z)\,\omega_1(z)\,\mu_0. \qquad (47)$$

Density satisfies the exact "equation of diffusion"

$$\frac{d}{d\,z}\,\frac{3}{\sigma_t(z)\,[3 - \omega_1(z)]}\,\frac{d\,[\mathrm{D}(z)\,n(z)]}{d\,z} =$$

$$= -\left\{4\pi\,a(z) - \frac{d}{d\,z}\,\frac{4\pi\,a(z)\,\omega_1(z)\,\mu_0}{\sigma_t(z)\,[3 - \omega_1(z)]}\right\}, \qquad (48)$$

and from Eq. (46) the flux may be written explicitly using quadratures. Equation (46) is solved explicitly[10–12] for the case of a homogeneous layer with a Lambertian boundary. Systems (42)–(43) and (46)–(47), as well as "equations of diffusion" (44), (45), and (48) contain a nonlinear coefficient D(z). Within the P_1–approximation D(z) = 1/3 = const, and the above problems become linear.

EXACT AND APPROXIMATE EQUATIONS FOR HEMISPHERICAL DENSITIES AND FLUXES

Let us integrate equation (33) over μ within the interval [0,1] and [–1,0] and refer to the definitions of $n^{\downarrow}$ (12), $n^{\uparrow}$ (13), $J^{\downarrow}$ (11), $J^{\uparrow}$ (10), $\mu^{\downarrow}$, $\mu^{\uparrow}$ (15), $\gamma_0^{\downarrow}$ (23), $\gamma_0^{\uparrow}$ (24). After certain identical transformations we obtain a system of exact equations for hemispherical fluxes with nonlinear parameters $\mu^{\downarrow}$, $\mu^{\uparrow}$, $\gamma_0^{\downarrow}$, $\gamma_0^{\uparrow}$

$$\frac{d\,J^{\downarrow}}{d\,z} + \left[\frac{\sigma_t(z) - \sigma_s(z)}{\mu^{\downarrow}(z)} + \frac{\sigma_s(z)\,\gamma_0^{\downarrow}(z)}{2\,\mu^{\downarrow}(z)}\right] J^{\downarrow}(z) -$$

$$- \frac{\sigma_s(z)\,\gamma_0^{\uparrow}(z)}{2\,\mu^{\uparrow}(z)}\,J^{\uparrow}(z) = 2\pi\,a(z)\,\gamma_0^{+}(z,\,\mu_0), \qquad (49)$$

$$\frac{d\,J^{\uparrow}}{d\,z} + \left[\frac{\sigma_t(z)-\sigma_s(z)}{\mu^{\uparrow}(z)} + \frac{\sigma_s(z)\,\gamma_0^{\uparrow}(z)}{2\,\mu^{\uparrow}(z)}\right] J^{\uparrow}(z) -$$

$$-\frac{\sigma_s(z)\,\gamma_0^{\downarrow}(z)}{2\,\mu^{\downarrow}(z)}\,J^{\downarrow}(z) = 2\pi\,a(z)\,\gamma_0^{-}(z,\,\mu_0). \qquad (50)$$

A similar system of differential equations was formulated by É.P. Zege[9] for the two–stream approximation of radiation transfer through a homogeneous layer. Such an approach was initially suggested by E.S. Kuznetsov.[31]

Using definitions of $\mu^{\downarrow}$, $\mu^{\uparrow}$ (15) and Eqs. (49)–(50) we obtain a system of exact equations for hemispherical densities

$$\frac{d\,[\mu^{\downarrow}(z)\,n^{\downarrow}(z)]}{dz} + \left\{\sigma_t(z) - \frac{\sigma_s(z)}{2}\,[2 - \gamma_0^{\downarrow}(z)]\right\} n^{\downarrow}(z) -$$

$$-\frac{\sigma_s(z)}{2}\,\gamma_0^{\uparrow}(z)\,n^{\uparrow}(z) = 2\pi\,a(z)\,\gamma_0^{+}(z,\,\mu_0), \qquad (51)$$

$$\frac{d\,[\mu^{\uparrow}(z)\,n^{\uparrow}(z)]}{dz} + \left\{\sigma_t(z) - \frac{\sigma_s(z)}{2}\,[2 - \gamma_0^{\uparrow}(z)]\right\} n^{\uparrow}(z) -$$

$$-\frac{\sigma_s(z)}{2}\,\gamma_0^{\downarrow}(z)\,n^{\downarrow}(z) = 2\pi\,a(z)\,\gamma_0^{-}(z,\,\mu_0). \qquad (52)$$

Such a system is used to make iterations in the technique of average fluxes[18] in order to faster convergence.

Within the P_1–approximation the system (49)–(50) becomes linear and closed

$$4\,\frac{d\,J^{\downarrow}}{dz} + [7\,\sigma_t(z) - 4\,\sigma_s(z) - \sigma_s(z)\,\omega_1(z)]\,J^{\downarrow}(z) +$$

$$+ [4\,\sigma_s(z) - \sigma_t(z) - \sigma_s(z)\,\omega_1(z)]\,J^{\uparrow}(z) = 8\pi\,a(z)\,\gamma_0^{+}(z,\,\mu_0), \qquad (53)$$

$$4\,\frac{d\,J^{\uparrow}}{dz} - [7\,\sigma_t(z) - 4\,\sigma_s(z) - \sigma_s(z)\,\omega_1(z)]\,J^{\uparrow}(z) -$$

$$-[4\,\sigma_s(z) - \sigma_t(z) - \sigma_s(z)\,\omega_1(z)]\,J^{\downarrow}(z) = 8\pi\,a(z)\,\gamma_0^{-}(z,\,\mu_0). \qquad (54)$$

Within the P_1–approximation the system (51)–(52) contains the nonlinear parameters $\mu^{\downarrow}(z)$ and $\mu^{\uparrow}(z)$

$$\frac{d\,[\mu^{\downarrow}(z)\,n^{\downarrow}(z)]}{dz} + \left\{\sigma_t(z) - \frac{\sigma_s(z)}{2}\left[1 + \frac{\omega_1(z)}{2}\,\mu^{\downarrow}(z)\right]\right\} n^{\downarrow}(z) -$$

$$-\frac{\sigma_s(z)}{2}\left[1 + \frac{\omega_1(z)}{2}\,\mu^{\uparrow}(z)\right] n^{\uparrow}(z) = 2\pi\,a(z)\,\gamma_0^{+}(z,\,\mu_0), \qquad (55)$$

$$\frac{d\,[\mu^{\uparrow}(z)\,n^{\uparrow}(z)]}{dz} + \left\{\sigma_t(z) - \frac{\sigma_s(z)}{2}\left[1 - \frac{\omega_1(z)}{2}\,\mu^{\uparrow}(z)\right]\right\} n^{\uparrow}(z) -$$

$$-\frac{\sigma_s(z)}{2}\left[1 - \frac{\omega_1(z)}{2}\,\mu^{\downarrow}(z)\right] n^{\downarrow}(z) = 2\pi\,a(z)\,\gamma_0^{-}(z,\,\mu_0). \qquad (56)$$

In the case of the Rayleigh scattering the terms involving $\omega_1(z)$ are omitted from systems (53)–(54) and (55)–(56), instead values (32) are used. In the case of a conservative layer it is sufficient to set $\sigma_t(z) = \sigma_s(z)$.

By integrating equation (33) over μ on the intervals [0, 1], and [−1, 0] with weights 1 and μ, one may obtain a system of exact equations, which simultaneously yields hemispherical densities $n^{\downarrow}$, $n^{\uparrow}$, and fluxes $J^{\downarrow}$, $J^{\uparrow}$ (Ref. 11).

BOUNDARY CONDITIONS

As demonstrated above, radiative characteristics may be expressed in terms of spherical harmonics. Therefore, we shall formulate boundary conditions for the models of fluxes and densities, written in the form of a system of ordinary differential equations of the first order, in a way similar to that used in the method of spherical harmonics,[5,22,32] i.e. based on the boundary–value problem for a zeroth order azimuthal harmonic (33). Demanding that the physical condition of balance of radiation fluxes at the boundary of the layer with the "vacuum" be satisfied

$$\text{at } \Gamma_0\text{: } \int_0^1 \Phi_c^0(0,\,\mu)\,\mu\,d\,\mu = 0, \qquad (57)$$

and

$$\text{at } \Gamma_H\text{: } \int_{-1}^0 [\Phi_c^0(H,\,\mu) - \Phi^*]\,\mu\,d\,\mu = 0, \qquad (58)$$

when

$$\mu\,\Phi_c^0(0,\,\mu)\,\Big|_{\Gamma_0} \equiv 0, \quad \mu\,[\Phi_c^0(H,\,\mu) - \Phi^*]\,\Big|_{\Gamma_H} \equiv 0,$$

and assuming that the expansion (7) exists, with the account for orthogonality of the Legendre polynomials within the interval [0, 1] we obtain the following conditions for the Fourier coefficients[7] (m = 0, 1, 2, ...):

$$\int_0^1 \mu\,\Phi_c^0(0,\,\mu)\,P_{2m}(\mu)\,d\,\mu = 0,$$

$$\int_{-1}^0 \mu\,[\Phi_c^0(H,\,\mu) - \Phi^*]\,P_{2m}(\mu)\,d\,\mu = 0. \qquad (59)$$

Using recursive relations for the Legendre polynomials, expressions (59) may be written in the equivalent form (m=0,1,2,...)

$$\int_0^1 \Phi_c^0(0,\,\mu)\,P_{2m+1}(\mu)\,d\,\mu = 0,$$

$$\int_{-1}^0 [\Phi_c^0(H,\,\mu) - \Phi^*]\,P_{2m+1}(\mu)\,d\,\mu = 0. \qquad (60)$$

Expressions (60) are so–called Marshak conditions.[5,32,33] As shown in Ref. 5 such approximate boundary conditions bring the lowest error of the technique of spherical harmonics, i.e. are "the best" in the sense of variational principle, which minimizes the values of a quadratic functional within the P_{2m+1}–approximation.

Let us find Marshak conditions for the P_1–approximation of the technique of spherical harmonics for systems of equations (34)–(35), (34) and (37), (34) and (39), and (40)–(41). By substituting expansion (7) into (57) and (58) we obtain

$$\text{at } \Gamma_0: \frac{1}{2}\Phi^0_{c0}(0) + \frac{1}{3}\Phi^0_{c1}(0) = 0, \tag{61}$$

$$\text{at } \Gamma_H: (1-q)\frac{1}{2}\Phi^0_{c0}(H) = (1+q)\frac{1}{3}\Phi^0_{c1}(H) + \frac{1}{2}f^*_H . \tag{62}$$

Let us now substitute the exact presentation Φ^0_{c1} and the P_1–approximation for Φ^0_{c0} into the Marshak conditions (61), (62) in terms of the hemispherical fluxes

$$J^{\downarrow}(0) = 0, \qquad -J^{\uparrow}(H) = q\,J^{\downarrow}(H) + \pi f^*_H . \tag{63}$$

As is seen, Marshak conditions within the P_1–approximation are exact for the boundary values of hemispherical fluxes. Within the same approximation the boundary conditions from Sobolev model[10–13] coincide with Marshak conditions. The same conditions are used in Eddington approximation.[14]

Below we present exact presentations of the boundary value in terms of

– spherical harmonics

$$\Phi^* = f^*_H + q\,\Phi^0_{c0}(H) + \frac{2}{3}q\,\Phi^0_{c1}(H) + 2q\sum_{m=1}^{\infty}\Phi^0_{c,2m}(H)\,R_{2m}; \tag{64}$$

– spherical densities and fluxes

$$\Phi^* = f^*_H + q\frac{n(H)}{4\pi} + 2q\frac{J(H)}{4\pi} + 2q\sum_{m=1}^{\infty}\Phi^0_{c,2m}(H)\,R_{2m}; \tag{65}$$

– hemispherical densities and fluxes

$$\Phi^* = f^*_H + \frac{q}{\mathrm{p}}J^{\downarrow}(H); \qquad \Phi^* = f^*_H + \frac{q}{\mathrm{p}}\mu^{\downarrow}(H)\,n^{\downarrow}(H). \tag{66}$$

Within the P_1–approximation we have

$$\Phi^* = f^*_H + q\,\Phi^0_{c0}(H) + \frac{2}{3}q\,\Phi^0_{c1}(H);$$

$$\Phi^* = f^*_H + q\frac{n(H)}{4\pi} + q\frac{J(H)}{4\pi} . \tag{67}$$

The exact and the approximate computational models for spherical fluxes and densities of solar radiation are reduced to two basic forms, i.e. to

– a system of ordinary differential equations of the first order

$$\frac{d\,w}{d\,z} + \alpha(z)\,\nu(z) = \varphi(z)\exp\left[-\frac{\tau(z)}{\mu_0}\right], \tag{68}$$

$$\frac{d\,\nu}{d\,z} + \beta(z)\,w(z) = p(z)\exp\left[-\frac{\tau(z)}{\mu_0}\right], \tag{69}$$

with the boundary conditions

$$w(0) = \alpha_0\,\nu(0) + \varphi_0, \qquad w(H) = \alpha_H\,\nu(H) + \varphi_H, \tag{70}$$

–or to a single equation of the second order of the type of the "equation of diffusion"

$$\frac{d}{d\,z}m(z)\frac{d\,u}{d\,z} - k^2(z)\,u = -f(z) \tag{71}$$

with the source

$$f(z) = F(z)\exp\left[-\frac{\tau(z)}{\mu_0}\right]$$

and the boundary conditions

$$m(z)\frac{d\,u}{d\,z}\Big|_{z=0} = \kappa_0\,u(0) + \beta_0,$$

$$m(z)\frac{d\,u}{d\,z}\Big|_{z=H} = \kappa_H\,u(H) + \beta_H. \tag{72}$$

Usually the "diffusion models" (71)–(72) are derived from the system of two equations (68)–(69) with the boundary conditions (70) by excluding one of the components, $w(z)$ or $\nu(z)$[21].

To make formulation of the problem on calculating spherical fluxes and densities complete, one needs to add two boundary conditions to the system of equations (68)–(69), which would, in a certain sense, account for boundary conditions (33). One may choose such conditions arbitrarily, since the boundary conditions (33) may be approximately satisfied in different ways. Equation (68) is obtained by integrating over μ along the interval [–1, 1] with unit weight, and equation (69) – by the same operation with weight μ. Similar transformations appear inappropriate for formulation of the boundary conditions (70), since conditions (33) are single–sided and they cannot be integrated over $\mu \in [-1, 1]$.

Let us integrate conditions (33) over μ for Γ_0 along the interval [0, 1], and for Γ_H along the interval [–1, 0], to obtain the exact relations

$$\text{for } \Gamma_0: \Phi^0_{c0}(0) + \frac{1}{2}\Phi^0_{c1}(0) + \sum_{m=1}^{\infty}\Phi^0_{c,2m+1}(0)\,R^0_{2m+1} = 0,$$

or

$$2\,n(0) + 3\,J(0) + 8\pi\sum_{m=1}^{\infty}\Phi^0_{c,2m+1}(0)\,R^0_{2m+1} = 0;$$

$$\text{for } \Gamma_H: \Phi^0_{c0}(H) - \frac{1}{2}\Phi^0_{c1}(H) - \sum_{m=1}^{\infty}\Phi^0_{c,2m+1}(H)\,R^0_{2m+1} = \Phi^*, \tag{73}$$

or

$$2\,n(H) - 3\,J(H) - 8\pi\sum_{m=1}^{\infty}\Phi^0_{c,2m+1}(H)\,R^0_{2m+1} = 8\pi\,\Phi^*. \tag{74}$$

With the account for Eq. (64) condition (73) can be reduced to the form

$$(1-q)\,\Phi^0_{c0}(H) = f^*_H + \left(\frac{1}{2} + \frac{2}{3}q\right)\Phi^0_{c1}(H) +$$

$$+ \sum_{m=1}^{\infty}\Phi^0_{c,2m+1}(H)\,R^0_{2m+1} + 2q\sum_{m=1}^{\infty}\Phi^0_{c,2m}(H)\,R_{2m},$$

and using (65) condition (74) may be written as follows:

$$(1-q)\frac{n(H)}{4\pi} = f^*_H + \left(\frac{3}{2} + 2q\right)\frac{J(H)}{4\pi} +$$

$$+ \sum_{m=1}^{\infty}\Phi^0_{c,2m+1}(H)\,R^0_{2m+1} + 2q\sum_{m=1}^{\infty}\Phi^0_{c,2m}(H)\,R_{2m}.$$

Within the P_1–approximation

$$2\,\Phi^0_{c0}(0) + \Phi^0_{c1}(0) = 0,$$

$$(1-q)\,\Phi^0_{c0}(H) = f^*_H + \left(\frac{1}{2} + \frac{2}{3}q\right)\Phi^0_{c1}(H); \tag{75}$$

$$2\,n(0) + 3\,J(0) = 0,$$

$$2(1-q)\,n(H) = 8\pi f_H^* + (3+4q)\,J(H). \quad (76)$$

We integrate condition (33) over μ with the weight μ for Γ_0 along the interval [0, 1], and for Γ_H along the interval [–1, 0]

$$\text{for } \Gamma_0\text{: } \Phi_{c0}^0(0) + \frac{2}{3}\Phi_{c1}^0(0) + 2\sum_{m=1}^{\infty}\Phi_{c,2m}^0(0)\,R_{2m} = 0$$

or

$$n(0) + 2\,J(0) + 8\pi\sum_{m=1}^{\infty}\Phi_{c,2m}^0(0)\,R_{2m} = 0;$$

$$\text{for } \Gamma_H\text{: } \Phi_{c0}^0(H) - \frac{2}{3}\Phi_{c1}^0(H) + 2\sum_{m=1}^{\infty}\Phi_{c,2m}^0(H)\,R_{2m} = 0 \quad (77)$$

or

$$n(H) - 2\,J(H) + 8\pi\sum_{m=1}^{\infty}\Phi_{c,2m}^0(H)\,R_{2m} = 0. \quad (78)$$

Using condition (64) we find from (77) that

$$(1-q)\,\Phi_{c0}^0(H) =$$

$$= f_H^* + (1+q)\frac{2}{3}\Phi_{c1}^0(H) - 2(1-q)\sum_{m=1}^{\infty}\Phi_{c,2m}^0(H)\,R_{2m},$$

and using condition (66) we find from (78) that

$$(1-q)\,n(H) =$$

$$= 4\pi f_H^* + 2(1+q)\,J(H) - (1-q)\,8\pi\sum_{m=1}^{\infty}\Phi_{c,2m}^0(H)\,R_{2m}.$$

Within the P_1–approximation Marshak conditions are

$$\Phi_{c0}^0(0) + \frac{2}{3}\Phi_{c1}^0(0) = 0,$$

$$(1-q)\,\Phi_{c0}^0(H) = f_H^* + (1+q)\frac{2}{3}\Phi_{c1}^0(H); \quad (79)$$

$$n(0) + 2\,J(0) = 0,$$

$$(1-q)\,n(H) = 4\pi f_H^* + 2(1+q)\,J(H). \quad (80)$$

As is seen, boundary conditions (75)–(76) and (79)–(80), constructed by different means as it were, only differ in values of their coefficients within the P_1–approximation. Note that the second approach is the one used to formulate practically all the Marshak conditions.

The exact and approximate models for calculating hemispherical densities ($w = \mu^{\downarrow}n^{\downarrow}$, $v = \mu^{\uparrow}n^{\uparrow}$) or fluxes ($w = J^{\downarrow}$, $v = J^{\uparrow}$) are described by systems of differential equations

$$\frac{d\,w}{d\,z} + a(z)\,w + b(z)\,v = \varphi(z)\exp\left[-\frac{\tau(z)}{\mu_0}\right], \quad (81)$$

$$\frac{d\,v}{d\,z} + c(z)\,v + d(z)\,w = p(z)\exp\left[-\frac{\tau(z)}{\mu_0}\right] \quad (82)$$

with the boundary conditions

$$\xi_0\,v(0) = \alpha_0\,w(0) + \varphi_0,\ \xi_H\,v(H) = \alpha_H\,w(H) + \varphi_H. \quad (83)$$

When setting the boundary conditions to calculate hemispherical densities and fluxes with the first technique (that is using the definitions of $n^{\downarrow}$ (12), $n^{\uparrow}$ (13), $\mu^{\downarrow}$, $\mu^{\uparrow}$ (15), $J^{\downarrow}$ (11), $J^{\uparrow}$ (10)), we obtain the exact relations from conditions (33)

$$\text{for } \Gamma_0\text{: } n^{\downarrow}(0) = 0, \quad (84)$$

$$\text{for } \Gamma_H\text{: } n^{\uparrow}(H) = 2\pi\,\Phi^*, \quad (85)$$

which may be written in different ways using expression (66)

$$n^{\uparrow}(H) = 2\pi f_H^* + 2q\,J^{\downarrow}(H);$$

$$n^{\uparrow}(H) = 2\pi f_H^* + 2q\,\mu^{\downarrow}(H)\,n^{\downarrow}(H); \quad (86)$$

$$\frac{1}{\mu^{\uparrow}(H)}J^{\uparrow}(H) = 2\pi f_H^* + 2q\,\mu^{\downarrow}(H)\,n^{\downarrow}(H);$$

$$\frac{1}{\mu^{\uparrow}(H)}J^{\uparrow}(H) = 2\pi f_H^* + 2q\,J^{\downarrow}(H). \quad (87)$$

Let us now use presentations of $\mu^{\downarrow}$ and $\mu^{\uparrow}$ in the P_1–approximation to write conditions (86), (87) in the following form:

$$(6+q)\,n^{\uparrow}(H) = 12\pi f_H^* + 7q\,n^{\downarrow}(H), \quad (88)$$

$$-7\,J^{\uparrow}(H) = 8\pi f_H^* + (8q-1)\,J^{\downarrow}(H). \quad (89)$$

In accordance with the second approach, we use (66) to transform the exact boundary conditions

$$\text{for } \Gamma_0\text{: } J^{\downarrow}(0) = 0, \quad (90)$$

$$\text{for } \Gamma_H\text{: } J^{\uparrow}(H) = \pi\,\Phi^* \quad (91)$$

into the form

$$-J^{\uparrow}(H) = \pi f_H^* + q\,J^{\downarrow}(H),\quad -J^{\uparrow}(H) = \pi f_H^* + q\,\mu^{\downarrow}(H)\,n^{\downarrow}(H),$$

$$-\mu^{\uparrow}(H)\,n^{\uparrow}(H) = \pi f_H^* + q\,\mu^{\downarrow}(H)\,n^{\downarrow}(H),$$

$$-\mu^{\uparrow}(H)\,n^{\uparrow}(H) = \pi f_H^* + q\,J^{\downarrow}(H).$$

Within the P_1–approximation we have

$$(7+q)\,n^{\uparrow}(H) = 12\pi f_H^* + (7q+1)\,n^{\downarrow}(H). \quad (92)$$

Expressions (90)–(91), obtained according to the second technique are the exact boundary conditions for the system (49)–(50). Meanwhile, to get rid of the nonlinear parameters, conditions (84) and (88), obtained following the first technique, may be used for system (51)–(52). Condition (92), obtained by the second technique within P_1–approximation is also applicable.

Most algorithms of radiative correction, developed for problems of remote sensing, are formulated on the basis of approximate models of radiation transfer and involve approximate solutions to the mathematical problems generated, so that the data to be processed may be parametrized. However, representative algorithms are also necessary, which would permit standard computations of high accuracy to verify engineering, express–analysis, and routine techniques, so that their application ranges may be assessed to a needed accuracy. Based on unified methods, we proposed accurate and approximate computational models for characteristics of solar radiation widely applicable to fast algorithms of radiative correction. It is recommended to use fast throughput techniques to obtain stable accurate numerical results.

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