COMPENSATION FOR THE MOIRE EFFECT OF DIGITIZED IMAGES

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Distortions of an image caused by their digitizing with a frequency lower than Nyquist's frequency are analyzed. The expediency of the low–frequency prefiltration of the images are shown. A method of restoration with the help of several copies of image being digitized with a displacement, which makes it possible to reduce the moire effect, is proposed.

Analog–to–digital conversion of images, which incorporates spatial discretization and quantization of their intensity distributions, is an obligatory operation of the state–of–the–art opto–computing systems. The accuracy of its performance largely predetermines the possible ways of subsequent processing of the images. Thus, if the optical system, which forms an image, has the angular resolution λ/D, where λ is the wavelength of light and D is the diameter of the receiving aperture then, in order to store the information with this resolution, the discretization step Δ must not exceed $\Delta_0 = \lambda/2D$. This condition is well known as Nyquist's criterion. When it is violated the overlap of the image spectra occurs, which distorts the fine details of the image and is called the moire–effect.[1] In this paper, we shall examine some possible ways of compensating for this effect. In so doing, for simplicity of mathematical operations, we will restrict ourselves to the one–dimensional analysis.

In the case of the one–dimensional digitizer, the output intensity distribution $I_{\rm D}(x)$ is related to the initial intensity distribution $I_0(x)$ of the image as

$$I_{\rm D}(x) = I_0(x) \sum_{j=-\infty}^{\infty} \delta(x - j\Delta) . \quad (1)$$

For the corresponding Fourier spectra, which are defined according to the rule

$$\tilde{I}(f) = \int_{-\infty}^{\infty} {\rm d}x\, I(x) \exp\{2\pi i f x\} , \quad (2)$$

the relation follows from Eq. (1)

$$\tilde{I}_{\rm D}(f) = \Delta^{-1} \sum_{j=-\infty}^{\infty} I_0\left(f - j\Delta^{-1}\right) , \quad (3)$$

i.e., the spectrum of the digitized image is obtained by an infinite repetition of the initial image spectrum of the image displaced by the distances multiple of Δ^{-1}. Since the spectrum of the optical image is nonzero only when $|f| \le f_{\rm D}$, where $f_{\rm D} = D/\lambda$ is the diffraction frequency of the cut–off, then when Nyquist's condition $\Delta \le \Delta_0$ holds, where $\Delta_0 = (2f_{\rm D})^{-1}$, the zeroth order term in Eq. (3) can be easily separated out, e.g., by transmitting $I_{\rm D}(f)$ through the low–frequency filter $\tilde{P}(f)$ which is equal to unity in the interval $|f| < (2\Delta)^{-1}$ and to zero outside it. As a result, the initial image can be restored. When $\Delta > \Delta_0$, the overlap of the terms of the zeroth order and of the higher order occurs. This results in distortion of information at the frequencies, which satisfy the inequality $(\Delta^{-1} - f_0) < |f| \le f_0$, and is manifested in false details of the image. The effect of overlap can be eliminated, if prior to digitizing the image is transmitted through the above–mentioned low–frequency filter.[1] In so doing, the information at the frequencies $\Delta^{-1}/2 < |f|^{-1} \le f_{\rm D}$ will be lost. Since $\Delta^{-1}/2 > (\Delta^{-1} - f_{\rm D})$, the image thus obtained proves to be a more accurate copy of the initial image.

It is assumed that in real systems this low–frequency filtration is realized in a natural way due to averaging of their intensities over the photosensitive surface of the elementary sensor of the digitizer (recorder). Mathematically, this is equivalent to digitizing of the smoothed distribution

$$I(x) = \int_{-\infty}^{\infty} {\rm d}y\, I_0(y)\, P({\rm x} - y) , \quad (4)$$

rather than $I_0(x)$, where $P(x)$ is the effective function of the sensor response. For the spectrum $\tilde{I}(f)$, it follows from Eq. (4) that

$$\tilde{I}(f) = \tilde{I}_0(f) \cdot \tilde{P}(f) . \quad (5)$$

$P(x)$ is usually assumed to be equal to Δ^{-1} in the interval $|x| \le \Delta/2$ and to zero outside it, so that the following expression turns out to be valid for $\tilde{P}(f)$:

$$\tilde{P}(f) = \frac{\sin \pi f \Delta}{\pi f \Delta} . \quad (6)$$

Such a filter does decrease the distortions caused by overlapping of the spectra, but it simultaneously attenuates the undistorted image. As a result, the image quality may be as low as previously. Results of the mathematical experiment are shown in Fig. 1. The initial image *a*, which is specified by the 64·64 array of digits for $\Delta_0 = 1$ was digitized with the discretization step $\Delta = 2$ in two different ways: *b* – by simply digitizing in a step and *c* – by pretransmitting through the filter $P(f_x)\cdot P(f_y)$. The normalized error ε of restoration of the images *b* and *c*, in comparison with *a*, was defined by the formula

$$\varepsilon = \frac{\int |I_0(f) - I_{\rm D}(f)|^2 {\rm d}f}{\int |I_0(f)|^2 {\rm d}f} . \quad (7)$$

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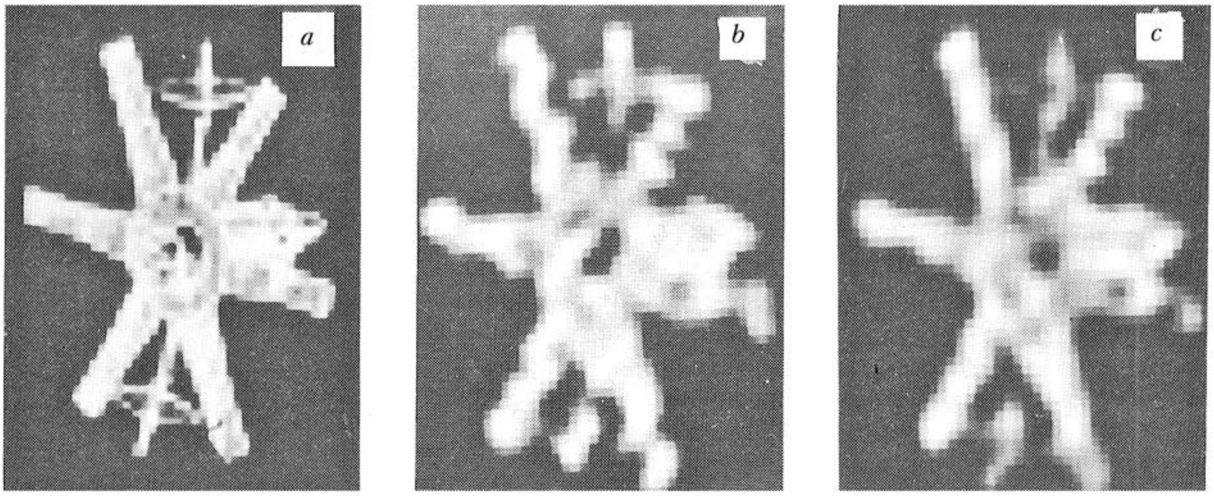

FIG. 1. The results of digitizing of the images: the initial model image (a), simple digitizing in a step (b), and digitizing with smoothing over the 2·2 pixels (c).

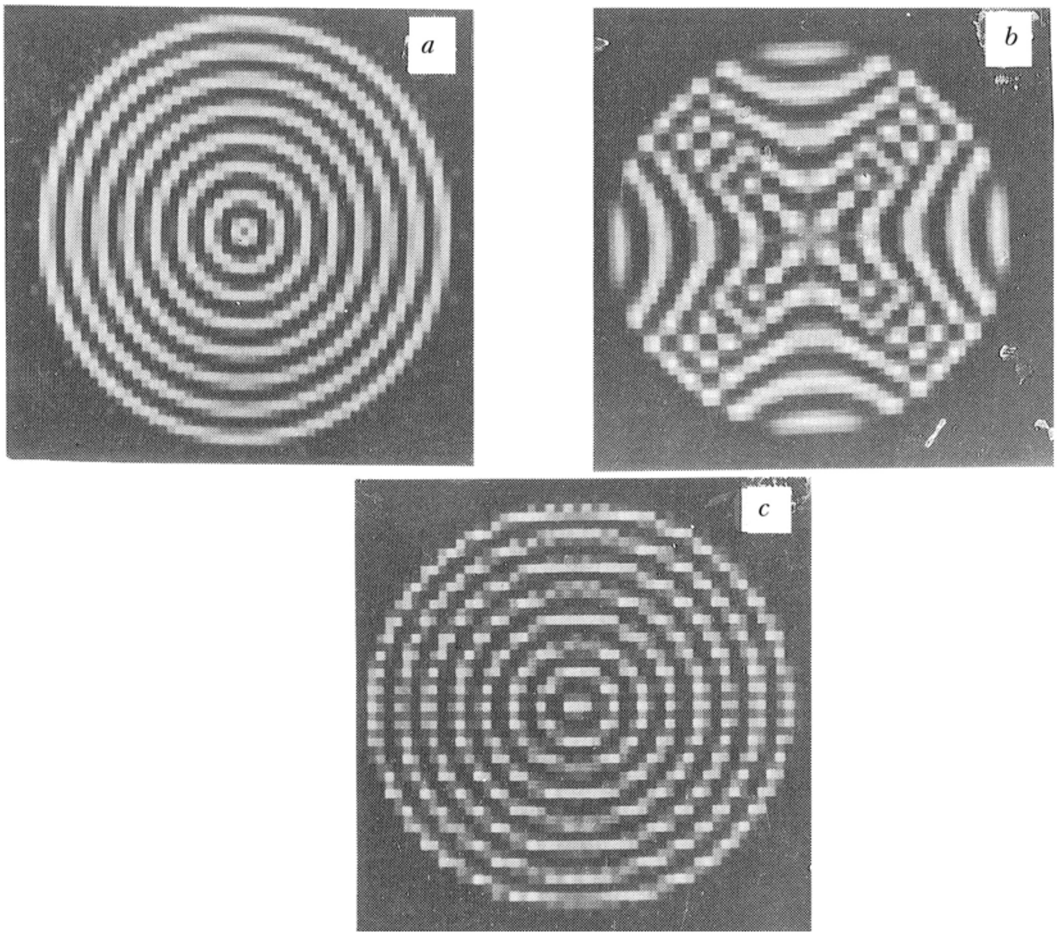

FIG. 2. The results of restoration of the images with the help of the displaced digitized images: the initial model image (a), its characteristic digitizing in a step (b), and the image restored with the help of 50 displaced copies of the form b (c).

This error was equal to 0.163 and 0.167 for the images *b* and *c*, respectively. It should be noted that in both cases the effective resolution of the image was not better than 2Δ.

It is possible to propose another way for decreasing the error due to the overlap of the spectra. It is based on the use of several copies of the initial image. Thus, if the digitizer is displaced by the distance θ, the following relation is valid for the spectrum of the new image:

$$\tilde{I}^{\theta}_{D}(f) = \Delta^{-1} \sum_{j=-\infty}^{\infty} \tilde{I}_0(f - j\Delta^{-1}) \exp\left\{-\frac{2\pi i}{\Delta} j\theta\right\} . \qquad (8)$$

In the case in which θ is known, $2\Delta_0 \geq \Delta > \Delta_0$, and not more than two terms enter in Eqs. (3) and (8) at each frequency *f*, these equations can be regarded as a system of two linear equations in two unknown values. The solution of this system permits us to find exactly the initial spectrum $\tilde{I}_0(f)$. In particular, at $\theta = \Delta/2$, the following equality is valid:

$$\tilde{I}_0(f) = \frac{1}{2}\left\{\tilde{I}^{\,0}_{D}(f) + \tilde{I}^{\,\Delta/2}_{D}(f)\right\} . \qquad (9)$$

If the quantity θ is random, but uniformly distributed over the interval $(-\Delta/2;\ \Delta/2)$, then in order to approximately restore $\tilde{I}_0(f)$, it is sufficient to average Eq. (8) over the series of realizations. Since

$$\left\langle \exp\left(-\frac{2\pi i\Delta}{\Delta} j\theta\right)\right\rangle = \begin{cases} 1, & \text{at } j = 0, \\ 0, & \text{at } j \neq 0, \end{cases} \qquad (10)$$

then

$$\langle \tilde{I}^{\,\theta}_{D}(f)\rangle = I(f) . \qquad (11)$$

When the image itself, rather than the digitizer, is displaced by θ, the spectrum is equal to

$$\tilde{I}^{\,\theta}_{D}(f) = \Delta^{-1}\{2\pi i f\theta\} \times \sum_{j=-\infty}^{\infty} \tilde{I}_0(f - j\Delta^{-1}) \exp\left\{-\frac{2\pi i}{\Delta} j\theta\right\} , \qquad (12)$$

When θ is known, in order to find $\tilde{I}_0(f)$ we must solve again the system of equations (3) and (12), while for random and uniformly distributed quantity θ one should average the spectra. However, in contrast to Eq. (11), the product of the form $\tilde{I}^{\theta}_{D}(f) \exp\{-2\pi i f\theta\}$, rather than the spectra themselves must be averaged; otherwise, after averaging we will arrive at the spectrum of the form (3) of the digitized image. In order to estimate the quantity θ in this case, we can use the method of determination of the displacement of the intensity distribution centroid. This method is based on measuring the displacement of the centroid of the image with respect to the initial one. The centroid of the image is found according to the formulas

$$x_c = \left(\int\!\!\int x \cdot \tilde{I}^{\theta}_{\Delta}(x, y)\,dx\,dy\right) \Big/ \left(\int\!\!\int \tilde{I}^{\theta}_{\Delta}(x, y)\,dx\,dy\right), \qquad (13)$$

$$y_c = \left(\int\!\!\int y \cdot \tilde{I}^{\theta}_{\Delta}(x, y)\,dx\,dy\right) \Big/ \left(\int\!\!\int \tilde{I}^{\theta}_{\Delta}(x, y)\,dx\,dy\right), \qquad (14)$$

The error in the determination of the centroid using this method is small given that the discretization step $\Delta \leq 2\Delta_0$. Indeed, the integrals, which enter in the right side of equalities (13) and (14), can be regarded as the Fourier transform at the point $f_x = 0$ and $f_y = 0$, while the amplitude of the spurious spectra at this point equals 0 when $\Delta \leq 2\Delta_0$. If $\Delta > 2\Delta_0$, the error in determining the centroid of the image with Δ. Figure 2 illustrates this method. Figure 2*a* shows the initial image, Fig. 2*b* – the image at $\Delta = 2\Delta_0$, and Fig. 2*c* – the image restored with the help of 50 identical copies. The error ε equals 0.413 and 0.147, respectively. Thus, by way of processing of several copies of the image digitized with the discretization step Δ, we can restore the image, which has the quality corresponding to the discretization step at least $\Delta/2$.

REFERENCES

1. U. Prett, *Digital Processing of Images* [Russian translation] (Mir, Moscow, 1983).
2. Zh. Maks, *Methods and Procedures for Signal Processing in Physical Measurements* [Russian translation] (Mir, Moscow, 1983).