OPTIMAL MODE EXPANSION OF PHASE RECONSTRUCTED FROM MEASUREMENTS OF WAVE FRONT TILTS IN A TURBULENT ATMOSPHERE. PART II. NUMERICAL EXPERIMENT AND ERRORS OF ALGORITHMS

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We describe in this paper algorithms and some results of constructing optimal mode expansion of random phase of an optical wave in turbulent atmosphere. We also present the obtained analytical form of the fundamental Karhunen–Loeve–Obukhov (K–L–O) modes represented in Zernike basis.

In the first part of the paper we have formulated a semi–analytical approach to the problem of obtaining the optimal mode expansion of a random phase which is registered by wave front sensors in the systems of atmospheric adaptive optics. The use of such an expansion makes it possible to minimize the ensemble–averaged error of approximation of random phase. The approach is based on the theory of statistically orthogonal Karhunen–Loeve–Obukhov expansions and allows us to use the statistical information inherent in the spatial correlation function of the field being expanded in order to construct the expansions of random fields within the confines of the local area.

In contrast to numerical method described in Ref. 2 this approach allows us to optimize the mode expansion over any given set of functions such as Zernike polynomials where aberrations of wave front are presented through tilts, defocusing, distortion, coma and others higher order aberrations or Uolsh functions[3] which are basic ones for zonal step compensation.

The proposed approach is not completely analytical. At different stages of constructing optimal expansion the numerical methods are used, and as a consequence the errors occur due to truncation of an infinite series of functions expansion.

The problems of accuracy and results of numerical simulations at constructing optimal basis for representation of optical wave phase in a turbulent atmosphere with Kolmogorov–Obukhov power spectrum of refractive index are discussed in the paper.

We dwell at first on the problem of the representation error of adaptive integral equation kernel (formula (8) in Ref. 1). In most interesting cases the phase structure function $D_s(\rho)$ is the kernel of this equation. Let us use the expansion of phase structure function in terms of Bessel functions[1]

$$D_s(\rho) = \sum_{p=0}^{P} a_p\, J_0\left(\mu_p \frac{\rho}{2R}\right); \tag{1}$$

$$a_p = \frac{2}{R^2\,[J_0'(\mu_p)]^2} \int\limits_0^{2R} \rho\, D_s(\rho)\, J_0\left(\mu_p \frac{\rho}{2R}\right) d\rho. \tag{2}$$

(All designations in this paper are the same as in Ref. 1.) For the Kolmogorov spectrum of atmospheric turbulence[3] we can write

$$D_s(\rho) = 6.88 \left(\frac{\rho}{r_0}\right)^{5/3}, \tag{3}$$

where $r_0 = 0.185\,137\,1\,(C_n^2\, L / \lambda^2)^{-3/5}$ is the Fried radius defined by the structure constant of refractive index C_n^2, pathlength of wave propagation L, and wavelength λ. To calculate the expansion coefficients a_p we use the standard Gaussian method. The error of calculation of integral (2) was not more then 10^{-8}.

The error of calculation of Δ with the use of approximation (1) was estimated by such a way

$$\Delta \le \max_{0 \le \rho \le 2R} \left| D_s(\rho) - \sum_{p=0}^{P} a_p\, J_0\left(\mu_p \frac{\rho}{2R}\right)\right| . \tag{4}$$

For example, at $\Delta = 10^{-6}$ the number of terms P in expansion (1) is equal to 30. In accordance with Eq. 13 in Ref. 1 the same number defines the dimension of Gramme square matrix. To diagonalize the matrices the Jacobi method was used. It allowed us to find the spectrum of eigenvalues Λ_k and calculate the coefficients of expansion $K_j^l(\rho)$ in terms of Bessel functions. According to the increase order of Λ_k the sequence of functions was chosen. This sequence of functions Ψ_k is a K–L–O adaptive modes (polynomials).

The forms of Ψ_k for $k = 2, \ldots, 13$ are presented in Fig. 1. The radial parts of the first polynomials are plotted in Fig. 2 in comparison with those of Zernike polynomials.

From Figs. 1 and 2 it follows that Zernike polynomials are close to adaptive polynomials only in the case of lower–order modes. If Zernike basis is taken as an expansion basis, the given accuracy can obviously be achieved even with several first terms of a new series.

In the first part of the paper we have described the transformation of the function representation in a conventional Zernike basis into adaptive Zernike basis by use of the following formula:

$$K_j^l(\rho) = \sum_{n=1}^{N} w_{j\,n}^{\,l}\, R_n^l(\rho). \tag{5}$$

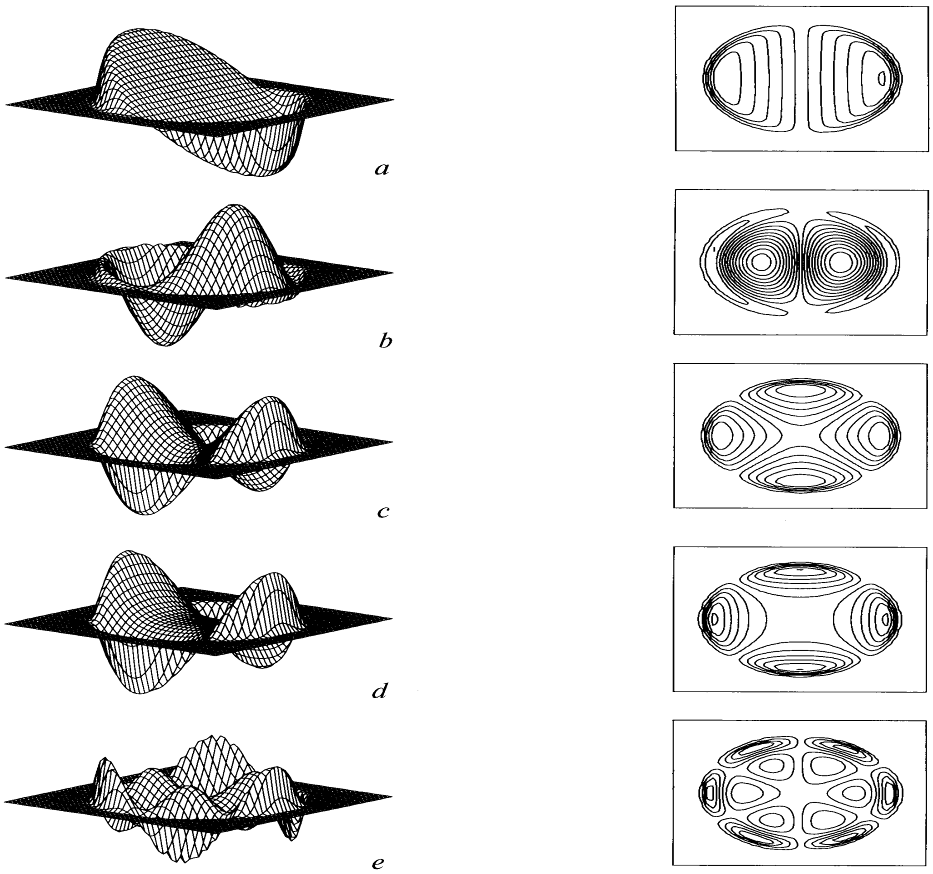

FIG. 1. The K–L–O modes Ψ_k: for k = 2 (a), 8 (b), 4 (c), 13 (d), and 7 (e).

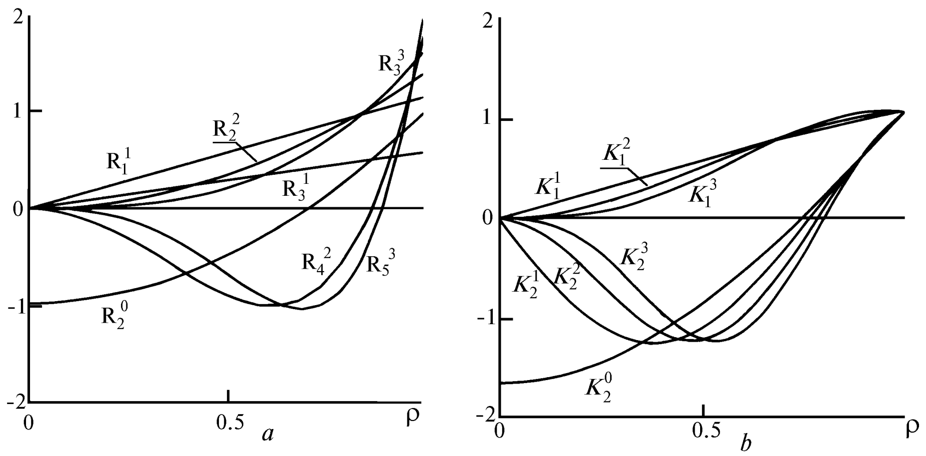

FIG. 2. Radial parts of polynomials: Zernike polynomials (a) and K–L–O ones (b).

Let us present the explicit form of the matrices of such a transformation $w^{l}_{j\,n}$ for five successive values of the azimuth index l and the index of radial disintegration j. The choice of the given quantity of indices (and consequently the choice of dimension of vector–lines) is quite sufficient to the mean–square error of random phase approximation $\langle\varepsilon^2\rangle$ be not more then 10^{-6}. Below the values of radial component of K–L–O functions and Zernike polynomials for different indexes j form matrix–columns and, in turn, matrix–lines $\mathbf{w}^{l}_{j}$ form square matrices so that

$$\begin{pmatrix} K_1^0 \\ K_2^0 \\ K_3^0 \\ K_4^0 \\ K_5^0 \end{pmatrix} = \begin{pmatrix} 0.9754 & 0.2205 & -0.0014 & -0.0001 & 0.0000 \\ -0.2203 & 0.9739 & -0.0554 & 0.0043 & 0.0000 \\ -0.0104 & 0.0518 & 0.9218 & -0.3761 & 0.0777 \\ -0.0030 & 0.0157 & 0.3451 & 0.7250 & -0.5958 \\ -0.0012 & 0.0067 & 0.1677 & 0.5770 & 0.7993 \end{pmatrix} \times \begin{pmatrix} R_0^0 \\ R_2^0 \\ R_4^0 \\ R_6^0 \\ R_8^0 \end{pmatrix} \quad (6)$$

$$\begin{pmatrix} K_1^{\pm1} \\ K_2^{\pm1} \\ K_3^{\pm1} \\ K_4^{\pm1} \\ K_5^{\pm1} \end{pmatrix} = \begin{pmatrix} 0.9995 & -0.0319 & 0.0018 & 0.0000 & 0.0000 \\ 0.0310 & 0.9540 & -0.2945 & 0.0472 & -0.0034 \\ 0.0072 & 0.2677 & 0.7790 & -0.5428 & 0.1637 \\ 0.0029 & 0.1187 & 0.4794 & 0.5413 & -0.6805 \\ 0.0013 & 0.0563 & 0.2768 & 0.6404 & 0.7142 \end{pmatrix} \times \begin{pmatrix} R_1^{\pm1} \\ R_3^{\pm1} \\ R_5^{\pm1} \\ R_7^{\pm1} \\ R_9^{\pm1} \end{pmatrix}; \quad (7)$$

$$\begin{pmatrix} K_1^{\pm2} \\ K_2^{\pm2} \\ K_3^{\pm2} \\ K_4^{\pm2} \\ K_5^{\pm2} \end{pmatrix} = \begin{pmatrix} 0.9838 & -0.1784 & 0.0192 & -0.0007 & 0.0000 \\ 0.1675 & 0.8758 & -0.4406 & 0.1030 & -0.0125 \\ 0.0563 & 0.3794 & 0.6201 & -0.6331 & 0.2535 \\ -0.0256 & -0.1908 & -0.4715 & -0.2655 & 0.6808 \\ -0.0140 & -0.1101 & -0.3200 & -0.4065 & 0.0827 \end{pmatrix} \times \begin{pmatrix} R_2^{\pm2} \\ R_4^{\pm2} \\ R_6^{\pm2} \\ R_8^{\pm2} \\ R_{10}^{\pm2} \end{pmatrix}; \quad (8)$$

$$\begin{pmatrix} K_1^{\pm3} \\ K_2^{\pm3} \\ K_3^{\pm3} \\ K_4^{\pm3} \\ K_5^{\pm3} \end{pmatrix} = \begin{pmatrix} 0.9599 & -0.2770 & 0.0429 & -0.0029 & 0.0002 \\ 0.2523 & 0.7898 & -0.5348 & 0.1609 & -0.0265 \\ 0.1028 & 0.4371 & 0.4741 & -0.6743 & 0.3322 \\ -0.05212 & -0.2503 & -0.4601 & -0.1023 & 0.6468 \\ -0.0308 & -0.1583 & -0.3361 & -0.3328 & 0.2137 \end{pmatrix} \times \begin{pmatrix} R_3^{\pm3} \\ R_5^{\pm3} \\ R_7^{\pm3} \\ R_9^{\pm3} \\ R_{11}^{\pm3} \end{pmatrix}; \quad (9)$$

Note, that the sum of the squares of elements in each line and each column are equal to one and the same value. This value should coincide with the norm of function $K_j^l(\rho)$. In our case the norm equals unity. Obviously, that closeness to unity may serve as a criterion of accuracy of calculation of the coefficients of K–L–O basis expansion in terms of Zernike polynomials.

The analytical form of the five first K–L–O polynomials derived by calculation of matrix–lines $\mathbf{w}_j^l$ for structure function (3) is presented by the formulas

$$\Psi_{1,2}(\rho,\theta) = K_1^1(\rho)\begin{Bmatrix}\cos\theta \\ \sin\theta\end{Bmatrix} =$$
$$= \left[2\rho\, 0.9995 - 0.03195\,(3\rho^3 - 2\rho)\sqrt{8}\,\right]\begin{Bmatrix}\cos\theta \\ \sin\theta\end{Bmatrix},$$

$$\Psi_{3,4}(\rho,\theta) = K_1^2(\rho)\begin{Bmatrix}\cos 2\theta \\ \sin 2\theta\end{Bmatrix} =$$
$$= \left[0.9838\rho^2\sqrt{6} - 0.1784(4\rho^4 - 3\rho^2)\sqrt{10} + 0.0192\sqrt{14}\,(15\rho^6 - \right.$$
$$\left. - 20\rho^4 + 6\rho^2)\right]\begin{Bmatrix}\cos 2\theta \\ \sin 2\theta\end{Bmatrix},$$

$$\Psi_5(\rho,\theta) = K_1^0(\rho) =$$
$$= \left[0.9754\, 2\rho - 0.2205\,(2\rho^2 - 1)\sqrt{3} + 0.0014\,(6\rho^4 - 6\rho^2 + 1)\sqrt{5}\,\right].$$

These polynomials were chosen from the elements $K_j^l(\rho)$ of the matrix–columns (6)–(9) after arrangement of the eigenvalues $\lambda_l^{(j)}$ in descending order.

Thus, we have investigated in this paper the efficiency of a semi–analytical approach to the problem of constructing the basis of random phase representation in the systems of coherent adaptive optics optimal for atmospheric turbulence. We described the algorithms of numerical realization of this approach and estimated the errors caused by truncation of infinite series representing the K–L–O functions. We also presented some results of numerical simulations and analytical expressions of the fundamental K–L–O modes expanded in Zernike basis. The developed approach and algorithms can serve as a base for creation software for the wave–front correctors control based on principles of mode or zonal step correction.

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