SVD ALGORITHM FOR PROBLEM OF DETECTION AND RANGING OF GROUND-BASED OPTICAL RADIATION SOURCES WITH A SATELLITE SYSTEM

A.V. Fabrikov

Scientific Center of Optical-Physical Measurements, Moscow
Received October 30, 1995

In relation to a general problem of detection and ranging of ground-based sources from satellites, modern approaches to a solution of overdetermined set of linear equations are discussed for the case of random errors in determination of their left and right sides. The singular expansion algorithm is widely used in these approaches and the total least-squares method (TLSM) that implements this algorithm and linearizes the Frobenius norm of an extended matrix used to compensate for the errors. Computational aspects of the TLSM inadequately covered in Russian literature and field of its applications are outlined.

1. The procedure for the expansion of a linear operator in singular numbers, called the SVD algorithm,[1] is widely used in signal processing.[1–4] It ensures higher accuracy and robustness as compared with all the other approaches to a solution of least-squares method (LSM) problems.[4] The SVD algorithm is especially efficient in the cases in which errors are present not only in the right side of equations but also in their left side and the problem is being solved not by the classical method but by the total or extended least-squares method (TLSM).[5] It is the case which is considered in the present paper in connection with detection and ranging of ground-based or circumterrestrial radiation sources with the NAVSTAR or GLONASS satellite system.[6]

Here, we consider a point-size omnidirectional source of light pulses, but our consideration also can be extended to a wider class of problems of detection and ranging of sufficiently high-power radiation sources (not only optical ones) by the differential– ranging scheme.

In the differential-ranging scheme, the source coordinates (x, y, z) are determined from the differences f_{ik} of path lengths from the source to the space vehicles (SVs) that recorded a signal:

$$f_{ik} \equiv f_{ik}(x, y, z) = r_i - r_k, \quad k, i = 1, 2, \ldots, N,$$

$$r_l = \sqrt{(x_l - x)^2 + (y_l - y)^2 + (z_l - z)^2},$$

$$l = 1, 2, \ldots, N, \qquad (1)$$

where (x_l, y_l, z_l) are the coordinates of the lth SV. In the experiment, one measured not f_{ik}, but the time delays Δt_{ik} connected with them by the relation

$$c\Delta t_{ik} \approx f_{ik}. \qquad (2)$$

The delays can be measured by one of the two ways: *a*) by fixing the moments of signal recording by the ith and kth SVs separately followed by calculation of $\Delta t_{ik} = t_i - t_k$ or *b*) by joint processing of recorded signal copies and finding the temporal shift at which the copies superimpose in the best way. The latter can be realized with the use of a correlator, adaptive filter, or another device for comparing two weakly disturbed, temporally shifted, and scaled copies of the same signal (see, for instance, Ref. 7)

$$s_1(t) = s(t) + n_1(t), \; s_2(t) = as(t - \tau) + n_2(t),$$

where τ is the time delay, a is the scaling factor, $n_1(t)$ and $n_2(t)$ are noisy processes. The first approach yields $m = N - 1$ linearly and statistically independent random variables Δt_{ik} (estimation based on noisy data is always a random variable), and in the second approach, all variables Δt_{ik} are statistically independent, i.e., $m = N(N - 1)/2$.

Let us formalize the above-described scheme for convenient further consideration. Introducing a single-parameter enumeration of all the pairs of indices (i, k) to which the statistically independent Δt_{ik} correspond, we write down initial equations (2) of the differential-ranging scheme in the form

$$\mathbf{f} \simeq \Delta, \qquad (3)$$

where $\mathbf{f} = [f_1,\ldots,f_m]^{\mathrm{T}}$ and $\Delta = [\Delta_1,\ldots,\Delta_m]^{\mathrm{T}}$ are the $m\times 1$ column vectors with elements $f_l = f_{ik}$ and $\Delta_l = c\Delta_{ik}$, $l = 1,\ldots, m$ respectively; l is the index of the pair (i, k); T denotes matrix transposition.

The vector $\mathbf{f}$ is a nonlinear function of the parameter vector

$$\boldsymbol{\theta} = [x, y, z]^{\mathrm{T}}, \qquad (4)$$

0235-6880/96/07 651-05 $02.00

i.e., $\mathbf{f} \equiv \mathbf{f}(\theta)$. Using the Taylor expansion
$\mathbf{f} = \mathbf{f}(\theta_0) + A(\theta - \theta_0) + \ldots$

and considering two terms, we obtain the linearized version of equation (3)

$$A^{(0)} \mathbf{x}^{(0)} \simeq \mathbf{b}^{(0)}, \quad (5)$$

with the help of which θ is estimated. Here,

$$\mathbf{x}^{(0)} = \theta - \theta_0, \quad \mathbf{b}^{(0)} = \Delta - \mathbf{f}(\theta_0), \quad (6)$$

the $m\times 3$ matrix A is the Jacobi matrix $A = \mathrm{d}\mathbf{f}/\mathrm{d}\theta\,|_{\theta = \theta_0}$ whose elements are functionals

$$A_{ik}^{(0)} = \left.\frac{\partial f_i}{\partial x_k}\right|_{\theta = \theta_0}, \quad i = 1, \ldots, m, \quad k = 1, 2, 3, \quad (7)$$

$x_1 = x, \quad x_2 = y, \quad x_3 = z,$

θ_0 are the nominal values of θ.

The nominal value of θ_0 initially chosen for *a priori* reason is then adjusted by calculations. The estimation $\hat{\theta}^{(j-1)}$ of the vector θ obtained at the $(j - 1)$th iteration is used as θ_0 at the following jth iteration. Here,

$$A^{(j)} \mathbf{x}^{(j)} \simeq \mathbf{b}^{(j)}, \quad (8)$$

$$\mathbf{x}^{(j)} = \theta - \hat{\theta}^{(j-1)}, \quad \mathbf{b}^{(j)} = \Delta - \mathbf{f}(\theta^{(j-1)}),$$

$$A_{ik}^{(j)} = \left.\frac{\partial f_i}{\partial x_k}\right|_{\theta = \theta^{(j-1)}}, \quad (9)$$

$i = 1, \ldots, m, \quad k = 1, 2, 3.$

2. In the typical case $m > 3$, the set of equations (3) and, consequently, (5) and (8) are overdetermined and, generally speaking, incompatible. The latter circumstance is expressed in the equations by the sign of approximate equality. They have no solution in the common sense but admit estimation of $\mathbf{x}$ optimal against one or another criterion. As is well-known,[3] the LSM estimation $\hat{\mathbf{x}}$ of the parameter vector $\mathbf{x}$ such that

$$A\mathbf{x} \simeq \mathbf{b} \quad (10a)$$

or

$$[A\,|\,\mathbf{b}]\begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix} \simeq 0 \quad (10b)$$

is the solution of the normal equations

$$A^{\mathrm{T}} A\, \mathbf{x} = A^{\mathrm{T}} \mathbf{b}. \quad (11)$$

The sign "$\simeq$" is replaced here by "=" because multiplication by A^{T} made the components of the data vector $\mathbf{b}$ orthogonal to the space of columns A to vanish. This transformed the system of equations (10) into a compatible one. For the $m\times n$ matrix A of total rank $r = \mathrm{rank}(A) = n$, the matrix $A^{\mathrm{T}}A$ is invertible; moreover, there exists a unique solution of Eq. (11)

$$\hat{\mathbf{x}} = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}} \mathbf{b}, \quad (12)$$

which is just the LSM estimation of $\mathbf{x}$.

If $r < n$, unambiguous solution of Eq. (11) can be obtained by imposing additional requirement of norm minimization on $\mathbf{x}$. The estimation satisfying the requirement of LSM and having the minimum Euclidean norm as compared with all the other estimations satisfying this requirement (MNLSM) is expressed by the formula

$$\hat{\mathbf{x}} = A^{+} \mathbf{b} = (A^{\mathrm{T}} A)^{+} A^{\mathrm{T}} \mathbf{b}, \quad (13)$$

where A^{+} is the generalized inverse or pseudoinverse[8] matrix corresponding to A.

The explicit expression for A^{+} is easy to find by SVD calculation of the matrix A

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{n} \lambda_i\, \mathbf{u}_i\, \mathbf{v}_i^{\mathrm{T}}, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \quad (14)$$

where U and V are orthogonal matrices, Σ is a diagonal matrix, λ_i are singular numbers, and $\mathbf{u}_i$ and $\mathbf{v}_i$ are left and right singular vectors of A. The SVD exists for an arbitrary matrix and is unique. There are well-developed algorithms for its calculation (see, for instance, Ref. 8). The formula for SVD of a generalized inverse matrix

$$A^{+} = \sum_{i=1}^{r} \lambda_i^{-1}\, \mathbf{v}_i\, \mathbf{u}_i^{\mathrm{T}}. \quad (15)$$

follows immediately from Eq. (14). Substitution of Eq. (15) into Eq. (13) yields the solution of problem (10) in the form

$$\hat{\mathbf{x}} = \sum_{i=1}^{r} \lambda_i^{-1} (\mathbf{u}_i^{\mathrm{T}} \mathbf{b}_i) \mathbf{v}_i\,. \quad (16)$$

Here, $\mathbf{u}_i^{\mathrm{T}}\mathbf{b}_i$ is the scalar product of the vectors $\mathbf{u}_i$ and $\mathbf{b}_i$ and $r \leq n$ is the rank of the matrix A corresponding to the number of nonzero λ_i in Eq. (14). For $r = n$, Eqs. (15) and (12) coincide and

$$A^{+} = (A^{\mathrm{T}} A)^{-1} A^{\mathrm{T}}. \quad (17)$$

This is easy to verify by substituting singular expansions of A and A^{T} into Eq. (16) and taking into account the equalities $A^{\mathrm{T}}A = V\Sigma^{-2}V^{\mathrm{T}}$, $(A^{\mathrm{T}}A)^{-1} = V\Sigma^{-2}V^{\mathrm{T}}$, $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = V\Sigma^{-1}U^{\mathrm{T}}$.

3. The equations (10) and (11) are not equivalent. The passage from Eq. (10) to Eq. (11) is unambiguous, but the converse is not true. The sense of this passage is to correct the data vector $\mathbf{b}$ with a certain correction vector $\mathbf{r}$ added to it. The vector $\mathbf{r}$ is such that the equations

$$A\mathbf{x} = \mathbf{b} + \mathbf{r}, \quad \mathbf{b} + \mathbf{r} \in \text{range}(A) \tag{18}$$

become compatible because the vector $\mathbf{b} + \mathbf{r}$ lies in the space of columns of the matrix A. The requirement of Euclidean norm minimization is imposed on $\mathbf{r}$, i.e., the solution of the system of equations (18) is an LSM solution of the system of equations (10) only if the condition

$$\|\mathbf{r}\|_2 = \sqrt{\sum_{i=1}^{m} r_i^2} = \min \tag{19}$$

is satisfied simultaneously with Eq. (18).

It is easy to see that

$$\mathbf{r} = A\,\hat{\mathbf{x}} - \mathbf{b} = (AA^{+} - I)\mathbf{b} = (UU^{\mathrm{T}} - I)\mathbf{b}. \tag{19a}$$

The incompatibility of the system of equations (10) is often caused by “noisy” right and left sides of equation (10). Errors in modeling manifesting themselves through inaccurate assignment of elements of the matrix A play an important part together with measurement errors being the source of distortions of the elements of $\mathbf{b}$. Allowance for this circumstance leads to generalization of the classical least-squares method, namely, to the total or extended LSM (TLSM).[2,4]

According to TLSM, the initial equation (10) is replaced by the approximation

$$(A + E)\mathbf{x} = \mathbf{b} + \mathbf{r} \tag{20a}$$

or equivalently

$$(B + D)\mathbf{z} = 0, \tag{20b}$$

where

$$B = [A \quad \mathbf{b}],\ D = [E \quad \mathbf{r}] = [d_{ij}],\ \ \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix}.$$

Here, normalization that makes the elements of the vectors $\mathbf{b}$ and $\mathbf{x}$ dimensionless is assumed. The correction terms E and $\mathbf{r}$ are subject to the conditions that the vector $\mathbf{b} + \mathbf{r}$ belongs to the space of columns of the matrix $A + E$:

$$(\mathbf{b} + \mathbf{r}) \in \text{range}(A + E), \tag{21a}$$

and that the Frobenius norm of the matrix D is minimum:[4]

$$\|D\|_F^2 = \text{trace}DD^{\mathrm{T}} = \left(\sum_i \sum_j d_{ij}^2\right) = \min. \tag{21b}$$

The problem of the optimal approximation of initial incompatible system (10) by compatible system (20) with the minimum Frobenius norm (the sum of all elements squared) of the correction matrix D is solved with the help of equations (21).

The equations (20b) admit a solution only if the rank of the matrix $B + D$ is less than the number of its columns $n + 1$ at least by unity. According to the well-known Eckart–Young theorem (see, for instance, Ref. 9), the matrix B_1 obtained from the singular expansion of B

$$B = \sum_{i=0}^{n+1} \sigma_i\, \xi_i\, \eta_i^{\mathrm{T}}, \quad \sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n \ge \sigma_{n+1}, \tag{22}$$

neglecting the last term $\sigma_{n+1}\, \xi_{n+1}\, \eta_{n+1}^{\mathrm{T}}$

$$B_1 = \sum_{i=0}^{n} \sigma_i\, \xi_i\, \eta_i^{\mathrm{T}}, \quad \sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n \tag{23a}$$

is the closest to B from the viewpoint of the Frobenius norm among all the matrices of rank n. Moreover,

$$\|D\|_F^2 = \|B - B_1\|_F^2 = \\ = \text{trace}\{\sigma_{n+1}^2 \xi_{n+1}\, \eta_{n+1}^{\mathrm{T}} \eta_{n+1}\, \xi_{n+1}^{\mathrm{T}}\} = \sigma_{n+1}^2. \tag{23b}$$

If the rank of B_1 is less than n (e.g., equals $r < n$), then

$$B_1 = \sum_{i=0}^{r} \sigma_i\, \xi_i\, \eta_i^{\mathrm{T}}, \quad \sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_r, \tag{24a}$$

and

$$\|D\|_F^2 = \sum_{i=1}^{n-r} \sigma_i^2. \tag{24b}$$

A solution of equation (20b) should be the $(n + 1)\times 1$ vector orthogonal to the space of columns of B_1 and hence parallel to the vector η_{n+1}. Let us represent the latter as

$$\eta_{n+1} = [\eta'^{\mathrm{T}}_{n+1} \,|\, \eta_{n+1,\,n+1}]^{\mathrm{T}}. \tag{25}$$

Then

$$\begin{bmatrix} \mathbf{x} \\ -1 \end{bmatrix} = -\frac{1}{\eta_{n+1,\,n+1}} \begin{bmatrix} \eta' \\ \hline \eta_{n+1,\,n+1} \end{bmatrix}$$

and hence

$$\hat{\mathbf{x}}_{\mathrm{TLSM}} = -\frac{1}{\eta_{n+1,\,n+1}}\, \eta'_{n+1} \equiv$$

$$= -\frac{1}{\eta_{n+1,\,n+1}} \begin{bmatrix} \eta_{n+1,\,1} \\ \eta_{n+1,\,2} \\ \cdots \\ \eta_{n+1,\,n} \end{bmatrix}. \tag{26a}$$

This is the TLSM solution obtained by the straightforward application of the SVD algorithm. In more general case of multiple singular numbers $\sigma_{r+1} = \ldots = \sigma_{n+1}$ it can be written in the form

$$\hat{\mathbf{x}}_{\text{TLSM}} = -\frac{1}{\alpha}\sum_{k=r+1}^{n+1}\eta_{k,n+1}\,\boldsymbol{\eta}'_k \quad \alpha = \sum_{k=r+1}^{n+1}|\eta_{k,n+1}|^2. \quad (26b)$$

Here, $\boldsymbol{\eta}'_k$ is the vector obtained from $\boldsymbol{\eta}_k$ by neglecting the last element and retaining the first n elements.

4. The TLSM solution (26) can be written in another form similar to Eq. (15) or (12) in order to compare it with the LSM solution. A straightforward substitution of $B[A \quad B]$ and $\boldsymbol{\eta}_i = [\boldsymbol{\eta}_i'^{\text{T}} \quad \eta_{i,n+1}]^{\text{T}}$ into the equation for the eigenvalues

$$B^{\text{T}}B\boldsymbol{\eta}_i = \sigma_i^2\boldsymbol{\eta}_i, \quad i = r+1, \ldots, n+1,$$

easily demonstrates that

$$\boldsymbol{\eta}'_i = -\eta_{i,n+1}(A^{\text{T}}A - \sigma_i^2 I)^+ A^{\text{T}}\mathbf{b}.$$

With allowance for Eq. (26*b*), it follows that

$$\hat{\mathbf{x}}_{\text{TLSM}} = -\frac{1}{\alpha}\sum_{k=r+1}^{n+1}|\eta_{k,n+1}|^2(A^{\text{T}}A - \sigma_i^2 I)^+ A^{\text{T}}\mathbf{b}.$$

If all σ_i^2, $i = r+1, \ldots, n+1$, can be considered equal to a certain $\sigma \in (\sigma_{n+1}, \sigma_{n+1} + \varepsilon)$ where ε is the error in estimation of singular numbers, then

$$\hat{\mathbf{x}}_{\text{TLSM}} = (A^{\text{T}}A - \sigma^2 I)^+ A^{\text{T}}\mathbf{b}, \quad (27)$$

which is close to Eq. (13). Let us suppose that none of the λ_i coincides with σ. Then $(A^{\text{T}}A - \sigma^2 I)^+ = (A^{\text{T}}A - \sigma^2 I)^{-1}$ and, taking into account the equality $A^{\text{T}} = \sum^{r} \lambda_i \mathbf{v}_i \mathbf{u}_i^{\text{T}}$, we have

$$(A^{\text{T}}A - \sigma^2 I)^+ A^{\text{T}} = \sum_{i=1}^{r}\frac{\lambda_i}{\lambda_i^2 - \sigma^2}\mathbf{v}_i\,\mathbf{u}_i^{\text{T}}.$$

In this case, Eq. (27) takes the form

$$\hat{\mathbf{x}}_{\text{TLSM}} = \sum_{i=1}^{r}\frac{1}{\lambda_i}\frac{1}{1 - \sigma^2/\lambda_i^2}(\mathbf{u}_i^{\text{T}}\mathbf{b})\mathbf{v}_i, \quad (28)$$

which is close to Eq. (16).

For $r = n$, the criterion of estimation optimization (21*b*) used in the TLSM can be written in the form

$$\frac{||A\mathbf{x} - \mathbf{b}||_2}{\sqrt{||\mathbf{x}||_2^2 + 1}} = \min, \quad (29a)$$

which is more convenient to compare the LSM and TLSM estimations. The conditions (29*a*) and (21*a*) are equivalent if $r = n$. To verify this, let us write down Eq. (29*a*) in the following way:

$$\left(\max_{\mathbf{z}}\frac{||D\mathbf{z}||_2^2}{||\mathbf{x}||_2^2}\right) = \min. \quad (29b)$$

However, the left side of Eq. (29*b*) is nothing but the operator norm of the $m\times(n+1)$ matrix D coinciding with its Frobenius norm for $r = n$. This demonstrates the equivalence of conditions (21*b*) and (29*a*). The requirement that the vector $\mathbf{z}$ should belong to the space of matrix D columns following from Eq. (20) and equivalent to the maximization of the parameter $||D\mathbf{z}||_2^2/||\mathbf{z}||^2$ is taken into account when going from Eq. (29*a*) to Eq. (29*b*). The TLSM criterion (29*a*) differs from LSM criterion (19) by an additional factor $1/\sqrt{||\mathbf{x}||_2^2 + 1}$ having a simple geometric interpretation: it corresponds to the cosine of the angle between the discperancy vector $\mathbf{r}$ and the direction of the normal to the nearest subspace

$$\left\{\begin{bmatrix}\mathbf{a}\\ b\end{bmatrix} : \mathbf{a} \in R^n, \ b \in R, \ b = \mathbf{x}^{\text{T}}\mathbf{a}\right\}.$$

5. Detection and ranging of ground-based radiation sources with a satellite informational measuring system (IMS) leads to many problems of statistical estimation most efficiently solved by the SVD algorithm. They are smoothing and extrapolation of signals by a finite number of noisy sample data,[10] estimation of the system pulse response by discrete deconvolution,[5] estimation of the position of a few simultaneously radiating sources whose signals are superposed at the inputs of a sensor system of the satellite IMS,[3] estimation of the source coordinates from the data of observations through a cloud layer distorting the propagation path of signals,[11] etc. Paraphrasing the title of the book,[4] one can say that TLSM problems formulated usually as an overdetermined system of equations with a large number of rows or/and columns with noisy left (coefficients of equations) and right (measurement data) sides are the main fields of SVD algorithm application.

Accuracy and robustness of the TLSM solution of such problems is better or at least no worse than those of the LSM solution (with the exception of rare "pathology"). Their advantage as compared with the LSM solution increases with the decrease of the matrix conditional number and compatibility of the system.

A separate paper will be devoted to concrete results of SVD-algorithm and TLS-method application to solving some problems of statistical estimation for source detection and ranging. Preliminary data of an experiment performed for this purpose were published in Ref. 12.

REFERENCES

1. S.L. Marple, *Digital Spectral Analysis* (Englewood Cliffs, New York).
2. V.C. Klema and A.J. Laub, IEEE Trans. Automat. Contr. **AC–25**, No. 2, 164–176 (1980).
3. F. Deprettere, ed., *SVD and Signal Processing Algorithms, Applications and Architectures*, (North-Holland, 1988).

4. C.L. Lawson and R.J. Hanson, *Solving Least Squares Problems* (Englewood Cliffs, New York).
5. G. Golub and C.F. van Loan, SIAM J. Numer. Anal. **17**, No. 6, 883–893 (1980).
6. A.I. Savin, Issled. Zemli iz Kosmosa, No. 1, 40–46 (1993).
7. N.L. Stal', A.V. Fabrikov, and V.A. Fabrikov, in: *Algorithms and Structures of Information Display Systems* (Tula State Technical University, Tula, 1994), pp. 37–46.
8. G. Strang, *Linear Algebra and Its Applications* (New York, 1976).
9. G.E. Forsythe, M.A. Malcolm, and C." . Moler, *Computer Methods for Mathematical Computations* (Prentice Hall Inc., New York, 1977).
10. " .D. Rao and K.S. Arin, Proc. IEEE **80**, No. 2, 283–309 (1992).
11. V.V. " acherikov, A.V. Fabrikov, and O.I. Aldoshina, Issled Zemli iz Kosmosa (to be published).
12. A.V. Fabrikov, N.L. Stal', and V.A. Fabrikov, in: *Abstracts of Reports at the 17-th Scientific-Technical Conference on High-Speed Photography and Photonics*, Moscow (1995), p. 5.