# SOLUTION OF THE SOUNDING PROBLEM BASED ON THE QUANTITATIVE DESCRIPTION OF AN LR LIDAR

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*Received May 18, 1989*

*The solution of the problem of laser coherent sounding of the atmosphere based on a quantitative model for an LR lidar is studied. A mathematical model, that gives an adequate quantitative description of an LR lidar is constructed.*

*A methodology for developing the mathematical software for the measuring system, enabling the processing of the experimental results in the dialog mode, is constructed based on the model.*

## INTRODUCTION

The study of the properties of the atmosphere in full-scale experiments requires constant improvement apparatus as well as the improvement and development of advanced lasers and laser radiation detectors. The so-called LR lidar,[1,2] in which an integrated scheme where the same laser is both the source and detector of radiation, is implemented, is very promising for these purposes. This scheme was improved by the use of a $Q$-switched laser, coupled with a mirror (natural reflector) located at a significant distance away from the laser system.[3,4] In Refs. 3 and 4 $Q$-switching was achieved by moving periodically one of the mirrors of the laser cavity. It is well known that in this case the light is phase-and amplitude-modulated.[5,6] Mixing of the light reflected from the external mirror and returning into the cavity with the light inside the cavity (amplitude- and frequency-modulated) results in the appearance of beats. This phenomenon can be regarded as coherent reception of laser radiation.

The proposed reception system is very attractive for atmospheric sounding because, in contradiction to conventional heterodyne detection, it does not impose any serious restrictions on signal stability.[7] Since it has a narrow gain band it is also much more noise resistant than the classical sounding method using a photodetector. This makes it possible to work at any time during day and throughout the year in both the visible and infrared regions of spectrum.

The obvious advantages of coherent reception of radiation are, however, accompanied by a definite difficulty in signal processing. The complicated interaction of different parameters of the scheme, the fact that both frequency and amplitude modulation of the mixed signals are always present at the same time, and the lack of reliable information about some parameters of the system present obstacles to the development of areal measurement system for coherent sounding of the atmosphere with the help of an LR lidar.

To overcome these difficulties we present in this paper a mathematical model of an LR lidar that takes into account the most important features of the( parameters of the atmosphere is ‘proposed. Such a quantitative description of an LR lidar essentially points the way toward the solution of the problems in atmospheric optics with efficient use of automation. The fact is that the proposed model enables convenient implementation on a computer and is thus suitable for development of computer technology for data acquisition and processing, in which "...all information about the object region is stored, calculated, circulated, transformed, and processed on automated carriers ...".[8]

In Ref. 9 the problem of describing an LR lidar theoretically was studied qualitatively. Here, in order to describe in detail the coupled amplitude-phase modulation, the system of equations for slowly-varying amplitudes and phases is derived from Maxwell's equations. Taking into account the boundary conditions and making some physically justified assumptions gives a closed system of nonlinear ordinary differential equations for the slowly-varying- amplitudes, measured at characteristic points (on mirrors). The system of equations is solved numerically. A methodology and technology for constructing on this basis a measuring system with an adaptive measuring scheme are formulated.

## DERIVATION OF THE EQUATIONS FOR THE SLOW-VARYING VARIABLES

The starting point is the equation for a plane wave in a medium in the form

$$\partial_t^2 E + \frac{\omega}{Q}\partial_t E - c^2\partial_x^2 E = -\frac{1}{\varepsilon_0}\partial_t^2 P. \quad (1)$$

The slowly-varying amplitudes $E_i$ and $P_i$ and the phases $\varphi_i$ ($i = 1$ for the direct wave and $i = 2$ for the return wave) are separated according to formulas

$$G=G_1\exp\left[i\left(\omega t-kx+\varphi_1\right)\right]+G_2\exp\left[i\left(\omega t+kx+\varphi_2\right)\right]+\text{c.c.} \quad (2)$$

where $G$ is one of the quantities $E$ or $P$. Dropping terms which are quadratically small in the quantity $(1/\omega)d/dt$ and taking into account the facts that 1) the intracavity medium is nondispersive ($\omega = kc$) and 2) the phase difference between the field and the polarization is equal to $\pi/2$, i.e., $\exp(i((\varphi_1 - \varphi_2)) = i$ (this assumption will be analyzed below) we obtain the equations for the amplitudes and phases in the form

$$\dot{E}_i \pm c\partial_x E_i + \frac{\omega}{Q}E_i = \frac{\omega}{2\varepsilon_0}P_i; \quad (3)$$

$$\dot{\varphi}_i \pm c\partial_x\varphi_i - \frac{1}{2Q}(\dot{E}_i/E_i) = -\frac{1}{\varepsilon_0}(\dot{P}_i/E_i). \quad (4)$$

Here the plus sign corresponds to $i = 1$. The system of equations (3) and (4) must be closed by relations which couple the polarization of the medium with the field. To calculate the polarization of the active medium we assumed that the medium consists of $N$ two-level atoms, having the same complex dipole moment $d$ interacting resonantly with the mode of the field. For macroscopic polarization we have the expression

$$P(x,t) = N\left(d^*\rho_{12} + \text{c.c.}\right). \quad (5)$$

For the four elements $\rho_{ij}$ of the atomic density matrix we have the following system of equations:

$$\left(\partial_t + i\omega_0 + \gamma_\perp\right)\rho_{12} + idD\,E = 0; \quad (6)$$

$$\left(d_t + \gamma_\parallel\right)D - \gamma_\parallel\Delta_0 + 2i\left(d^*\rho_{12} - \text{c.c.}\right) = 0, \quad (7)$$

where $D = \rho_{11} - \rho_{22}$; $\omega_0$ is the frequency of an atomic transition $\gamma_\perp$ and $\gamma_\parallel$ are the traditional atomic relaxation constants; and, $\Delta_0$ is the equilibrium population inversion. We assumed the medium relaxes much more rapidly than the field $(1/\gamma_{\perp,\parallel})d_tE_i \ll E_i$ and the steady-state solution of the system (6) and (7) can be employed.

This system of equations can be, written in terms of harmonics by separating the spatial and temporal of the quantities $E$ and $P$. Truncating the expansion in the field amplitudes at the quadratic terms we obtain the following expressions for the forward and return waves of polarization:

$$P_1 = \varepsilon_1\left[1 - \varepsilon_2\left(E_1^2 + 2E_2^2\right)\right]E_1; \quad (8)$$

$$P_2 = \varepsilon_1\left[1 - \varepsilon_2\left(E_2^2 + 2E_1^2\right)\right]. \quad (9)$$

Here we introduce the linear susceptibility $\varepsilon_1 = N|d|^2\Delta_0/\gamma_\perp$ and the saturation parameter $\varepsilon_2 = 4|d|^2/(\gamma_\perp\gamma_\parallel)$. The relations (8) and (9) close the system of equations (3) and (4).

## REDUCTION OF THE SYSTEM OF EQUATIONS; BOUNDARY CONDITIONS

In this section we shall transfer from partial differential equations to a system of ordinary differential equations, and we shall also formulate the boundary conditions. Figure 1 illustrates the scheme of coherent reception of laser radiation. In a coordinate system at rest the stationary, moving, and external mirrors have the coordinates $l$, $a = a(t)$, and $L$, respectively. The reflection coefficients of the mirrors are equal to $R_0$, $R_1$, and $R_2$. The external loss factor is $\sigma$ (this means that the amplitude of the signal which has traversed the path and returned has decreased by a factor of exp $(2L/\sigma)$. Since the spatial dependence of the slowly-varying variables is weak we shall approximate the spatial derivatives by finite differences:

$$\partial_x E_1 \approx \frac{E_1(a)-E_1(-l)}{a+l}, \quad \partial_x\varphi = \frac{\varphi_1(a) - \varphi_1(-l)}{l + a}. \quad (10)$$

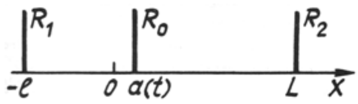


*Fig. 1*

The relations (10) make it possible to transform from partial differential equations to ordinary differential equations. The boundary conditions at the cavity mirrors make it possible to reduce the number of independent unknown functions and make the system determinate. We start with the boundary conditions for the phases:

$$-ka + \varphi_1(a) = ka + \varphi_2(a) + 2\pi n; \quad (11)$$

$$kl + \varphi_1(-l) = -kl + \varphi_2(-l) + 2\pi m; \quad (12)$$

$$\varphi_1(a) - \varphi_1(-l) = \varphi_2(-l) - \varphi_2(a), \quad (13)$$

where $m$ and $n$ are integers. The relations (11) and (12) are the conventional reflection relations at the cavity mirrors. The relation (13) expresses the symmetry condition, corresponding physically to the fact that nodes occur at the mirrors. In other words the phase accumulates in the same way during motion in both the direct and return directions along the cavity. Subtracting (12) from (11) and using (13) and the facts that $a \ll \lambda/2$ and that transitions into other modes do not occur as a result of the oscillations of the mirror (the relation $kl = \pi(n - m)$ holds) we finally obtain the relation

$$\varphi_1(a) - \varphi_1(-l) = \varphi_2(-l) - \varphi_2(a) = ka. \tag{14}$$

The relation (14) makes it possible to study immediately the question of the evolution of the phase, since it is now possible to evaluate the terms in Eqs. (4) which "couple" with the field. Thus, for example, we have

$$c\partial_x\varphi \left[\frac{1}{2Q}\frac{\dot{E}_1}{E_1}\right]^{-1} \sim \frac{cka}{l}\frac{Q\Delta_t E_1}{\Delta E_1} = \left[\frac{\omega a\Delta t}{l}\right] Q\frac{E_1}{\Delta E_1} \gg 1. \tag{15}$$

Here $\Delta t$ is the time scale of the variation of the amplitude $E_1$. The inequality (15) follows from the relations $\omega a\Delta t \sim l$, $Q \gg 1$, and $E_1 \gg \Delta E_1$, which hold with good accuracy in an experiment. Thus the contribution of the amplitudes in the equations for the phases is insignificant and they can be neglected. As a result we obtain the following relations which determine the evolution of the phases:

$$\varphi_1(x_1,\ 0) = \varphi_1(x_1,\ 0) \pm \frac{\omega}{l}\int_0^t a(t_1)dt_1,$$

$$(i=1,2), \tag{16}$$

where $x_1 = a$, $x_2 = -l$, and the plus sign is used for $i = 1$. Next, we write the boundary conditions for the amplitudes on the moving and stationary mirrors:

$$E_1(-l,\ t) = R_1E_2(-l,\ t); \tag{17}$$

$$E_2(a,\ t) = R_0E_1(a,\ t) + \exp\left[-\frac{2L}{\sigma}\right]\left[1 - R_0\right]^2$$

$$\times R_2E_1\left[t - \tau\right]\cos\left[\varphi_1(a,\ t) - \varphi_1(a,\ t-\tau) + \alpha\right], \tag{18}$$

where $\tau \equiv 2L/c$ and $\alpha = \text{const}$. The relation (17) is the condition of reflection. The relation (18) is the conventional law for adding slowly-varying amplitudes of signals with close frequencies and different amplitudes. Since in an experiment the delay time on the path $\tau$ usually satisfies the condition $\tau \ll l/(\omega a)$ we can set $E_1(t - \tau) \simeq E_1(t)$ on the right side of (18). In addition, because $\tau a \ll a$ we have

$$\varphi_1(a,\ t) - \varphi_1(a, t-\tau) = \frac{\omega}{l}\int_{t-\tau}^t a(t_1)dt_1 \simeq \frac{\omega a(t)}{l}\tau. \tag{19}$$

Thus instead of (18) we obtain boundary conditions in the form:

$$E_2(a,\ t) \simeq \gamma(t)E_1(a,\ \tau);$$

$$\gamma(t) \equiv R_0 + \exp(-2L/\sigma)(1-R_0)^2R_2\cos\left[\frac{\omega\tau}{l}a(\tau)+\alpha\right]. \tag{20}$$

Using Eq. (17) and the relation (20) we obtain the final system of ordinary differential equations for the two independent unknown functions $x(t) \equiv E_1(a,\ t)$, $y(t) \equiv E_2(-l,\ t)$:

$$\dot{x} + \nu(1-a/l)(x-R_1y) = \alpha x - \beta(1+2\gamma^2)x^3; \tag{21}$$

$$\dot{y} - \nu(1-a/l)(\gamma x-y) = \alpha y - \beta(1+2R_1^2)y^3, \tag{22}$$

where $\gamma = \gamma^{(t)}$ and $a = a(t)$. The positive constants $\alpha$ and $\beta$ (the gain and saturation constants, respectively) are given by the formulas

$$\alpha_1 = \frac{\omega}{Q}\left[\xi-1\right];\quad \beta = \frac{\omega}{Q}\frac{\gamma_\perp\gamma_\parallel}{4|d|^2}\xi$$

$$\xi = \Delta_0/\tilde{\Delta};\quad \tilde{\Delta} = (\omega/Q)\gamma_\perp/(N|d|^2), \tag{23}$$

where $\nu = c/l$ is the inverse transit time in the cavity and c is the velocity of light.

## ANALYSIS OF THE FIELD DYNAMICS

The system of equations (21) and (22) is the basic model, whose parameters can be adjusted so as to achieve adequate agreement with experiment. It describes well the dynamics of the field both at the starting stage of lasing and after beats have been established.

At the initial stage of lasing (far from saturation) the amplitudes $x$ and $y$ satisfy a system of linear system of equations, analysis of which gives the threshold condition (amplification condition)

$$\alpha - \nu\left[1 - \sqrt{R_0R_1}\right] > G, \tag{24}$$

where the time $\tau_1 = [\alpha - \nu(1 - \sqrt{R_0R_1})]^{-1}$ determined by the left side of (24), is the characteristic time over which beats are established. The system (21) and (22) was studied numerically for times $t \gg \tau_1$. The dependence of the solution on the amplitude $a_0$ and frequency $\omega_1$ of the oscillations of the mobile mirror are of greatest interest from the viewpoint of practical applications. The oscillations were assumed to be sinusoidal $a(t) = a_0\sin\omega_1 t$. The time step $\Delta t$ employed in solving the system of equations numerically must be chosen so that $\Delta t \ll \nu^{-1}$ (the "fastest" time for the problem).

The steady-state solution qualitatively represents beats with two characteristic frequencies. The lower frequency corresponds to the frequency of oscillations of the mirror and the higher frequency corresponds to the difference frequency (the difference of the cavity frequency at the time $t$ and $t + \tau$). The difference frequency can be determined by expanding $a(t)$ near $t = 0$, i.e., $a(t) \sim a_0\omega_1 t$. Thus, the beat frequency is of the order of

$$\Delta\omega_R \sim \frac{\omega\tau a_0\omega_1}{l}.$$

It follows from here that $\Delta\omega_R$ can be varied by varying the product $a_0\omega_1$. The degree of modulation (losses), which affects the amplitude modulation of the signal, can be easily determined from the linear problem. This gives the relation $k \sim a_0 c\tau_1 / l^2$. Thus the degree of amplitude modulation of the signal can be varied by varying the amplitude of the oscillations of the mobile mirror.

## SOLUTION OF THE SOUNDING PROBLEM AND METHODOLOGY FOR BUILDING THE MEASURING SYSTEM

One of the key aspects of the solution of problems in lidar sounding of the atmosphere is data processing. The lack of the accurate quantitative relations between characteristics of the atmosphere and the parameters of the lidar complex as well as the impossibility of calibrating the system are serious obstacles (unfortunately, we cannot vary the absorption coefficient in a fixed interval so as to determine how the amplitude of the beats or some other quantity changes in the process). To overcome these obstacles we propose a complex approach consisting essentially of the fact that the solution of the sounding problem (in the present case the determination of the absorption coefficient and path length) depends substantially on the model solution of the direct problem.

The mathematical model of an LR lidar described in the preceding sections is actually the solution of the direct problem of coherent sounding. Indeed, two characteristic quantities $\alpha_a$ and $\alpha_f$, associated with the degree of amplitude and frequency modulation, respectively, can be determined from the graph of the time dependence of the amplitude of the field on the stationary mirror, shown in Fig. 2 (compare with the oscillogram of heterodyne beats of Fig. 3 of Ref. 4). If necessary, they can be averaged over several periods of oscillations of the mirror. The values of this quantities depend on the absorption coefficient $\sigma$ and path length $L$ and this dependence, as computer experiments show, is unique and quite smooth. The latter facts form the basis for the method used to calibrate the measuring system. The essence of the method is as follows. The range $X$ of the variation of $L$ and $\sigma$ is fixed, so that to each pair of values $(L, \sigma)$ there corresponds a point in $X$. Then for each point from $X$ the equations of the model (21) and (22) are solved numerically, and the corresponding pair of values $(\alpha_a, \alpha_f)$, representing a point in some region $Y$, is calculated from the solutions of these equations. As a result we obtain a one-to-one mapping $X \Leftrightarrow Y$, which is the solution of the direct sounding problem. This actually solves the problem of calibrating the measuring system, and there no fundamental difficulties in solving the inverse problem, so that it reduces to simple inversion of the one-to-one mapping.

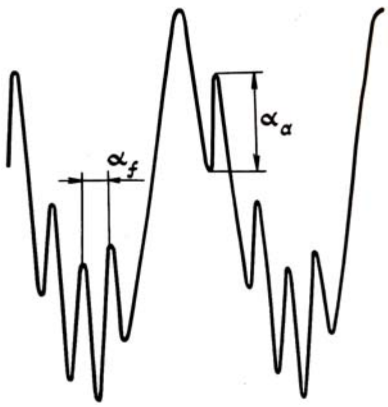


*Fig. 2*

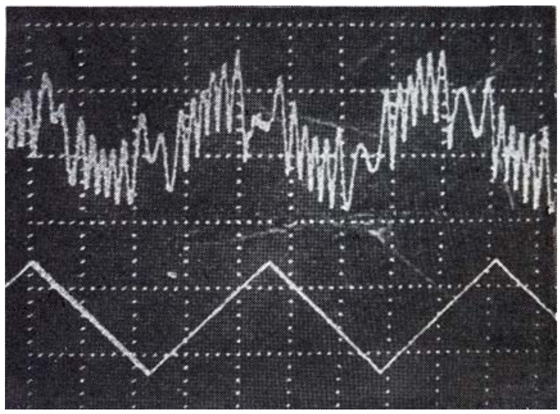

*Fig. 3*

Next, after $\alpha_a$, $\alpha_f$ from the experimental curve and using the inverse map $Y \Leftrightarrow X$, it is not difficult to construct the values of the absorption coefficient $\sigma$ and path length $L$.

The methodology described above for measuring the optical parameters of the atmosphere also gives a method for determining the nominal (in the sense of achieving a fixed accuracy) parameters of the measuring system and the nominal measuring ranges. The choice of measuring range is made based on a simple limiting criterion. Within the range the number of small beats per period of oscillation of the mirror should be constant (a value of about 7–12). The meaning of this criterion is that the curve employed for interpretation should not change qualitatively within one range (no new differential features should appear). This parameters $a_0$ and $\omega_1$ are chosen based on this criterion.

The mapping $X \Leftrightarrow Y$ corresponding to one range, is precalculated on a powerful computer and then fed in a compact matrix form into a computer-based data processing system. The number of ranges is limited only by the size of the working memory of the measuring system.

Software enabling data processing in the dialog mode as well as making it possible to create a data base allowing for different ranges of measurements has been developed based on the proposed concept.

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