OPTIMAL ESTIMATE OF THE CONCENTRATION OF THE ATMOSPHERIC CONSTITUENTS FROM LIDAR DATA OBTAINED IN THE PHOTON–COUNTING MODE

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Two suboptimal estimates of the concentrations of the atmospheric constituents applicable for processing the data obtained using the elastic and Raman scattering lidars as well as the resonance fluorescence lidar are synthesized based on the measurement of laser pulse energies during every sensing step. Two formulas are derived for the errors in these estimates. The advantage of one estimate over another and over the conventionally used intuitive estimate is demonstrated by way of example.

The lidars based on the effects of elastic scattering (ES) and Raman scattering (RS) as well as on the resonance fluorescence (RF) are capable of measuring the concentrations of the aerosol component ES and different gaseous components (RF and RS) in the atmosphere.[1] The accuracy of measuring the concentration depends strongly on the estimate employed. The intuitive estimates, whose errors may significantly exceed the errors of optimal estimates, are most often used. At the same time, transfer from the intuitive estimates to the optimal or suboptimal estimates often requires only an insignificant change in the algorithm for processing the lidar signals or simple modification of the lidars.

The error in measuring the concentration is caused by a stochastic nature of the lidar—atmosphere system and, correspondingly, by different kinds of fluctuations.[2] When synthesizing the estimates of concentration we will take into account only the shot electric flux fluctuations of the photodetector, and when analyzing these estimates, in addition, the fluctuations of the atmospheric transmission and the laser pulse energies. Let us consider the lidars, in which the laser pulse energies are measured during every sensing step by taking off a small portion of the emitted flux. Each of these lidars has one informational frequency channel operating in the photon counting mode. Since optimal estimate against the criterion of minimum variance is too complicated for such lidars, let us synthesize two simple suboptimal estimates, which would account for the measured energies, and compare their errors with the error of the conventionally used intuitive estimate disregarding the energies measured during every sensing step.

Let n_μ be the number of single—electron pulses (SEP) recorded by the detector during the strobing time t, which corresponds to the spatial strobe $L = ct/2$, during the μth sensing step of the series of N ($\mu = \overline{1, N}$) observations. The quantity n_μ, which obeys a Poisson distribution[2] comprises lidar signal and noise (background noise, dark current, etc.). The average quantity n_μ is given by the lidar equation[1] and, for all lidars being analyzed (the ES, RS, and RF lidars), may be represented in the form $\bar{n}_\mu = KTMI_\mu + m$, where K is the instrumental parameter, T is the atmospheric transmission along the propagation path to the strobe L being sensed, and backwards, M is the estimated concentration averaged over L and over N observations, I_μ is the laser pulse energy, and m is the average number of the SEP of noise. In this case the suboptimal estimates, based on different approaches to the estimate of maximum likelihood, have the forms

$$\hat{M} = \frac{1}{KT}\,\frac{\frac{1}{N}n_\mu - m}{\frac{1}{N}I_\mu}\,, \tag{1}$$

$$\hat{M} = \frac{1}{KT}\,\frac{1}{N}\sum\frac{n_\mu - m}{I_\mu}\,, \tag{2}$$

where the symbol Σ indicates summing over μ from 1 to N. Note that estimate (1) is based on normalization of the sampling averaged over the series of observations to the energy averaged over the series of observations while estimate (2), which is more widespread than estimate (1), is based on normalization for one sensing step. If the energy is not measured during each sensing step but is taken identical for all sensing step and equal to I, then the optimal estimate has the form

$$\hat{M} = \frac{1}{KT}\,\frac{\frac{1}{N}n_\mu - m}{I} \tag{3}$$

and coincides with the conventionally used intuitive estimate,[3] which is based on the sampling mean.

In calculating $\hat{M}$, instead of the unknown quantities K, T, and m, which are implemented during sensing, we may use their estimates $\hat{K}$, $\hat{T}$, and $\hat{m}$, which can be determined for K by means of preliminary calibration of the lidar, for m based on additional sampling of the SEP numbers during time t_m between the sensing steps or during time t in the auxiliary "noise" channel,[3] and for T by additional sampling the SEP numbers from the "reference" altitude (ES and RF lidars) or in the auxiliary "nitrogen" channel (RS and ES lidars).[1] In relations (1) and (2) I_μ is replaced by its estimate $\hat{I}_\mu$ obtained by direct measurement,

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and in Eq. (3) I is replaced by its estimate , taken from the nominal data of the laser or obtained by preliminarily testing the laser.

When analyzing $\hat{M}$ we assume that $\hat{K} = K$ and $\hat{m} = m$, because they can be determined with small errors with the help of a high—precision calibrating device and during time $t_m \gg t$, while $\hat{T}$ is unbiased with mean value $\bar{\hat{T}} = T$ and relative variance d_T^2. We will obtain the error of $\hat{M}$ comprising the fluctuation error δ and the statistical bias ξ (Ref. 2) for the models of $\hat{I}$ and $\hat{I}_\mu$ most important in practice.

For estimate (1) when $\hat{I}_\mu$ is well known and possesses the deterministic bias $\xi_{\mu I} = (I_\mu - \hat{I}_\mu)/\hat{I}_\mu$, we have

$$\xi \simeq \sum \xi_{\mu I} \frac{\hat{I}_\mu}{\sum \hat{I}_\mu}, \tag{4}$$

and

$$\delta \simeq \frac{1}{N^{1/2}} \left[\frac{1}{\frac{1}{N}\sum s(\hat{I}_\mu)} \left(1 + \xi + \frac{m}{\frac{1}{N}\sum s(\hat{I}_\mu)} \right) + N\delta_T^2 (1+\xi)^2 \right]^{1/2}, \tag{5}$$

and when $\hat{I}_\mu$ is unknown, random and has the statistical bias $\xi_{\mu I} = (I_\mu - \bar{\hat{I}}_\mu)/\bar{\hat{I}}_\mu$ and the variance $\delta_{\mu I}^2 = D(\hat{I}_\mu)/\bar{\hat{I}}_{\mu I}^2$ we have

$$\xi \simeq \sum \xi_{\mu I} \frac{\bar{\hat{I}}_\mu}{\sum \bar{\hat{I}}_\mu}, \tag{6}$$

$$\delta \simeq \frac{1}{N^{1/2}} \left[\frac{1}{\frac{1}{N}\sum s(\bar{\hat{I}}_\mu)} \left(1 + \xi + \frac{m}{\frac{1}{N}\sum s(\bar{\hat{I}}_\mu)} \right) + N \left(\delta_T^2 + \sum \delta_{\mu I}^2 \frac{\bar{\hat{I}}_\mu^2}{(\sum \bar{\hat{I}}_\mu)^2} \right) (1+\xi)^2 \right]^{1/2}. \tag{7}$$

Here and below, $s(x) = KTMx$. In particular cases we have the following: when the energy is measured exactly and $\hat{I}_\mu = I_\mu$ and $\xi = 0$, and δ can be found from Eq. (5); when $\hat{I}_\mu$ is stationary and $\bar{\hat{I}}_\mu \equiv \bar{\hat{I}}$ and $\delta_{\mu I}^2 \equiv \delta_I$, $\xi = \sum \xi_{\mu I}/N$ and δ can be found from Eq. (7); when $\hat{I}_\mu$ is unbiased and $\hat{I}_\mu = I_\mu$ and $\xi_{\mu I} = 0$, $\xi = 0$ and δ can be found from Eq. (7).

For estimate (2) when $\hat{I}_\mu$ is well known ($\xi_{\mu I} = (I_\mu - \hat{I})/\hat{I}$), we have

$$\xi \simeq \frac{1}{N} \sum \xi_{\mu I}, \tag{8}$$

$$\delta \simeq \frac{1}{N^{1/2}} \left[\frac{1}{N}\sum \frac{1}{s(\hat{I}_\mu)} \left(1 + \xi_{\mu I} + \frac{m}{s(\hat{I}_\mu)} \right) + N\delta_T^2 (1+\xi)^2 \right]^{1/2}, \tag{9}$$

and when $\hat{I}_\mu$ is unknown and random ($\xi_{\mu I} = (I_\mu - \bar{\hat{I}}_\mu)/\bar{\hat{I}}_\mu$ and $\delta_{\mu I}^2 = D(\hat{I}_\mu)/\bar{\hat{I}}_\mu^2$) we have

$$\xi \simeq \frac{1}{N} \sum \xi_{\mu I}, \tag{10}$$

$$\delta \simeq \frac{1}{N^{1/2}} \left[\frac{1}{N}\sum \frac{1}{s(\bar{\hat{I}}_\mu)} \left(1 + \xi_{\mu I} + \frac{m}{s(\bar{\hat{I}}_{\mu\mu})} \right) + N\delta_T^2 (1+\xi)^2 + \frac{1}{N}\sum \delta_{\mu I}^2 (1+\xi_{\mu I})^2 \right]^{1/2}. \tag{11}$$

Using Eqs. (8)—(11) it is easy to derive relations for ξ and δ of estimate (2) in the same particular cases as for estimate (1).

For estimate (3) when $\hat{I}_\mu$ is taken from the nominal data of the laser and possesses the deterministic bias $\xi_{\mu I} = (I_\mu - \hat{I})/\hat{I}$, we have

$$\xi \simeq \frac{1}{N} \sum \xi_{\mu I}, \tag{12}$$

$$\delta \simeq \frac{1}{N^{1/2}} \left[\frac{1}{s(\hat{I})} \left(1 + \xi + \frac{m}{s(\hat{I})} \right) + N\delta_T^2 (1+\xi)^2 \right]^{1/2}, \tag{13}$$

and when $\hat{I}$ has been obtained during the preliminary laser tests, similar to a sensing step, and has the statistical bias $\xi_{\mu I} = (I_\mu - \bar{\hat{I}})/\bar{\hat{I}}$ and the variance $\delta_I^2 = D(\hat{I})/\bar{\hat{I}}^2$ we have

$$\xi \simeq \frac{1}{N} \sum \xi_{\mu I}, \tag{14}$$

$$\delta \simeq \frac{1}{N^{1/2}} \times$$

$$\times \left[\frac{1}{s(\bar{\hat{I}})} \left(1 + \xi + \frac{m}{s(\bar{\hat{I}})} \right) + N(\delta_T^2 + \delta_I^2)(1+\xi)^2 \right]^{1/2}. \tag{15}$$

When there are N_p stationary steps of preliminary testing and $\hat{I}$ is determined as an arithmetical mean over these steps, the variance of $\hat{I}$ equals to δ_I^2/N_p. For $N_p = N$ estimate (3) and its error coincide with estimate (1) and its error, and for $N_p > N$ the error in estimate (3) is smaller than that in estimate (1).

For each individual lidar system and specific conditions of sensing, based on formulas (4)–(15), one can choose the most efficient estimate among estimates (1)–(3). Thus, it may turn out that for small errors in measuring the laser pulse energies during the period of observations (N) and for certain atmospheric conditions (δ_T), one of estimates (1) or (2) is most efficient. The other estimate may be efficient during another period of observations and for other atmospheric conditions. If the errors are large, it is more reasonable to avoid the energy measurements during every sensing step and employ estimate (3).

As an example we have calculated the values of δ for two models of I_μ, i.e., for a linear model

$$\frac{I_\mu}{I_m} = \frac{1}{N-1}\,\mu + \frac{1}{2}\frac{N-3}{N-1}, \tag{16}$$

and for a sinusoidal one

$$\frac{I_\mu}{I_m} = 1 + \frac{1}{2}\sin\left(\frac{2\pi}{N-1}(\mu-1)\right), \tag{17}$$

where $N \ge 2$ and $I_m = \sum I_\mu / N$. We set $s(I_m) = m = 1$, δ_T varied from 0 to 0.5, and N varied from 2 to 20. Some of the calculated results are given in Figs. 1—3.

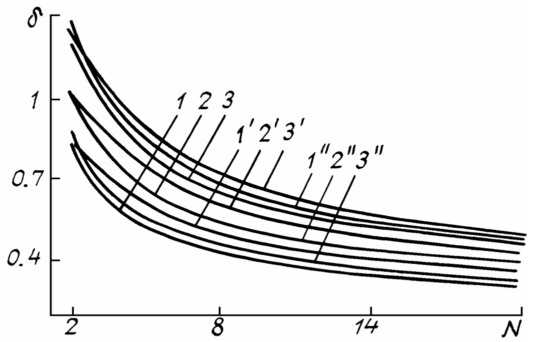

FIG. 1. Error in estimates (1) (curves 1, 2, and 3), (2) (1′, 2′, and 3′), and (3) (1″, 2″, and 3″) for the model, in which the values of $\hat{I}_\mu$ *are well known:* $\Delta I = \xi_{\mu I} = -0.2$ *(1, 1′, and 1″),* 0 *(2, 2′, and 2″), and* 0.2 *(3, 3′, and 3″).* $\delta_T = 0.2$.

Figure 1 illustrates the errors in all three estimates (1)—(3) for model (17) with δ_T being equal to 0.2, for the case of well–known $\hat{I}_\mu$ for estimates (1) and (2), and when $\hat{I}$ is prescribed for estimate (3). Note that for this case the errors in estimates (1) and (3) are independent of the model I_μ while the error in estimate (2), even though it depends on the model, does so weakly, especially for $N \ge 5$. For the exact energy measurements during the sensing steps ($\xi_{\mu I} = 0$) and when the energy is averaged over a series of observations, ($\Delta I = (\hat{I} - I_m)/I_m = 0$), estimates (1) and (3) coincide and have an identical accuracy while estimate (2) is worse than they. The same can be seen for other values of δ_T. When measuring the energies with the errors, estimate (1) is virtually always higher than estimate (2), except when $\xi_{\mu I} = 0.2$ and the values of δ_T and N are large (for $\delta_T = 0.5$, when $N > 9$), and sometimes it is worse than estimate (3) depending on the quantities $\xi_{\mu I}$ and ΔI (for example, when $\xi_{\mu I} = 0.2$ and $\Delta I = 0.2$). As δ_T increases, the magnitudes of the errors of all estimates monotonically increase.

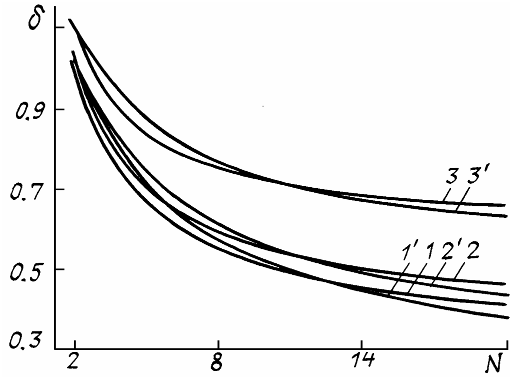

FIG. 2. Error in estimates (1) (curves 1, 2, and 3) and (2) (curves 1′, 2′, and 3′) for the model, in which the quantities $\hat{I}_\mu$ *are random and unbiased:* $\delta_T = 0$ *(1 and 1′),* 0.2 *(2 and 2′), and* 0.5 *(3 and 3′) for* $\xi_{\mu I} = 0$ and $\delta_I = 0.2$.

Figures 2 and 3 show the variant of random $\hat{I}_\mu$ described by model (17), and correspond to the unbiased energy measurements ($\xi_{\mu I} = 0$) with $\delta_I = D^{1/2}(\hat{I}_\mu)/I_m = 0.2$ and to biased energy measurements with $\delta_{\mu I} = 0.2$. Note that for this case the error in estimate (3) is independent of the model I_μ, the error in estimate (1) depends weakly on it for all N, and the error in estimate (2) depends strongly on the model only for $N \le 4$. It turns out that, when the unbiased energy has been measured (see Fig. 2), estimate (1) is higher than estimate (2) for any δ_T when $N \le 11$ and, when we have the biased readings of the energy (see Fig. 3), this is the case for any δ_T, N, and $\xi_{\mu I}$ with exception of the cases when $\xi_{\mu I} = 0.2$ and δ_T and N are large (for $\delta_T = 0.5$, when $N > 9$). In an important particular case in which $\hat{I}_\mu$ is stationary, estimate (1) is always higher in accuracy then estimates (2) and (3) while its error, just like the error in estimate (3), is independent of the model of I_μ.

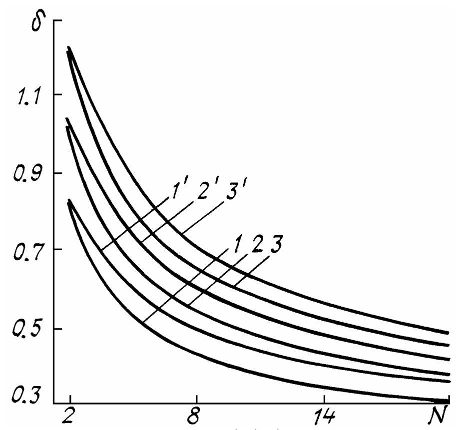

FIG. 3. Error of estimates (1) (curves 1, 2, and 3) and (2) (curves 1′, 2′, and 3′) for the model, in which the quantities $\hat{I}_\mu$ *are random and biased,* $\xi_{\mu I} = -0.2$ *(1 and 1′),* 0 *(2 and 2′), and* 0.2 *(3 and 3′), and* $\delta_{\mu I} = \delta_T = 0.2$.

Thus, we can see that estimate (1), which has been synthesized for processing of the signals of the ES, RS, and RF lidars, possesses the best combination of the accuracy characteristics in comparison with conventionally employed estimates (3) and (2). An employment of estimate (1) instead of estimate (2) will require only a simple change in the algorithm for signal processing, while its employment instead of estimate (3) will require an additional measurement of the laser pulse energies during every sensing step with an acceptable accuracy.

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