Ellipsoidal particle and its projections onto the coordinate planes

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In this paper we consider the problem of determination of the principal geometrical dimensions of a particle that has a shape close to ellipsoid of revolution from the parameters of particle's projections onto two orthogonal planes. The errors caused by unknown orientation of the particle are estimated. It is shown that these errors may be rather large for an oblate ellipsoid and rather small for the prolate one.

The method proposed earlier for analyzing suspended particles[1] consists in simultaneous recording, in one plane, of two or three particle images corresponding to projections of the particle onto the orthogonal planes. In Ref. 2 it was shown that for an arbitrarily oriented ellipsoidal particle the principal geometrical parameters can be determined from the parameters of projections (lengths and areas) onto two (ellipsoid of revolution) or three (triaxial ellipsoid) mutually perpendicular planes.

In this paper we consider some possibilities of determining the principal dimensions of a particle as applied to a relatively simple case of dealing with two images.

Let the images of a particle in the planes XOY and XOZ be known. These images (ellipses) obviously correspond to the projections of an ellipsoid onto the above-mentioned coordinate planes. The measured parameters are the lengths of projections, l_x, l_y, and l_z, of the ellipses onto the corresponding coordinate axes and the areas S_{xy} and S_{xz} of these ellipses in the corresponding planes.

Assume that the shape of a particle is described by a triaxial ellipsoid with the principal geometrical dimensions A, b, and C; then the measured parameters can be expressed through the ellipsoid dimensions as:

$$\begin{aligned}
l_x^2 &= A^2\, t_{11}^2 + \text{”}^2\, t_{12}^2 + C^2\, t_{13}^2\,,\\
l_y^2 &= A^2\, t_{21}^2 + \text{”}^2\, t_{22}^2 + C^2\, t_{23}^2\,,\\
l_z^2 &= A^2\, t_{31}^2 + \text{”}^2\, t_{32}^2 + C^2\, t_{33}^2\,, \qquad (1)\\
S_{xy}^2 &= \text{”}^2\, C^2\, t_{31}^2 + A^2\, C^2\, t_{32}^2 + A^2\, \text{”}^2\, t_{33}^2\,,\\
S_{xz}^2 &= \text{”}^2\, C^2\, t_{21}^2 + A^2\, C^2\, t_{22}^2 + A^2\, \text{”}^2\, t_{23}^2\,.
\end{aligned}$$

In Eqs. (1) parameters t_{ik} (i, k = 1, 2, 3) determine the orientation of the ellipsoid relative to the chosen coordinate system and can be expressed through the Euler angles[3]; besides, $t_{11}^2 + t_{12}^2 + t_{13}^2 = 1$; $t_{11}^2 + t_{21}^2 + t_{31}^2 = 1$, etc.

In the case of an arbitrary relation between the dimensions A, b, and C, these relations are insufficient for unambiguous determination of the above-mentioned dimensions. So, let us consider a practically important case of a triaxial ellipsoid, which only slightly differs from an ellipsoid of revolution. This is can be an ellipsoid, in which two dimensions are close to each other, and the difference between them is much less than that between others. Assume for certainty that such dimensions are b and C ($b > C$), i.e., $(b^2 - C^2)/(b^2 + C^2) = p^2 \ll 1$, and $A^2 - b^2 > b^2 - C^2$, or $b^2 - C^2 < C^2 - A^2$. Such an ellipsoid can be approximated by the ellipsoid of revolution with the equivalent radius R_{e0} and the height $H_0 = A$, and $R_{e0}^2 = (b^2 + C^2)/2$.

As seen, the principal dimensions of an ellipse can be expressed through the measured parameters as:

$$\begin{aligned}
2a_{xy}^2 &= l_x^2 + l_y^2 + [(l_x^2 + l_y^2)^2 - 4\, S_{xy}^2/\pi^2]^{1/2}\,;\\
2b_{xy}^2 &= l_x^2 + l_y^2 - [(l_x^2 + l_y^2)^2 - 4\, S_{xy}^2/\pi^2]^{1/2}\,;\\
2a_{xz}^2 &= l_x^2 + l_z^2 + [(l_x^2 + l_z^2)^2 - 4\, S_{xz}^2/\pi^2]^{1/2}\,;\\
2b_{xz}^2 &= l_x^2 + l_z^2 - [(l_x^2 + l_z^2)^2 - 4\, S_{xz}^2/\pi^2]^{1/2}\,. \qquad (2)
\end{aligned}$$

Obviously, the radius for the ellipsoid of revolution is determined by the coinciding dimensions in both planes: $a_{xy} = a_{xz}$ or $b_{xy} = b_{xz}$ (the case that both of these equalities hold true should be considered separately). For the triaxial ellipsoid under consideration we take the following quantity as the radius ("measured"):

$$\begin{aligned}
R_e^2 &= (a_{xy}^2 + a_{xz}^2)/2 \text{ at } a_{xy}^2 - a_{xz}^2 < b_{xy}^2 - b_{xz}^2\,;\\
R_e^2 &= (b_{xy}^2 + b_{xz}^2)/2 \text{ at } b_{xy}^2 - b_{xz}^2 < a_{xy}^2 - a_{xz}^2\,.
\end{aligned}$$

The first relation for R_e obviously corresponds to an oblate ellipsoid, while the second to a prolate one. The height ("measured") of the ellipsoid, as it follows from Ref. 2, is determined by the following equation:

$$H^2 = l_x^2 + l_y^2 + l_z^2 - 2\, R_e^2\,. \qquad (3)$$

Estimate now possible errors in determination of R_e, H, and the volume V. We consider only the errors caused by unknown orientation of a particle (that is, we neglect the measurement errors). The starting point is determination (selection) of R_e. It follows from Eqs. (1)–(2) that R_e^2 can vary from b^2 to C^2 depending

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on the orientation of a particle. So, the relative error $(R_e - R_{e0})/R_{e0} = \delta R_e/R_{e0}$ varies within

$$1 - (1 + p^2)^{1/2} \le \delta R_e/R_{e0} \le 1 - (1 - p^2)^{1/2} . \quad (4)$$

However, because $p^2 \ll 1$, we have that $|\delta R_e/R_{e0}| \le p^2/2$. So, R_e can be either smaller or larger than R_{e0} depending on the particle orientation.

With $\delta R_e/R_{e0}$ known from Eq. (3) it is easy to find the relative error of determination of the ellipsoid height $\delta H/H_0 = (H - H_0)/H_0$:

$$1 - [1 + p^2 (2R_{e0}/H_0)^2]^{1/2} \le \delta H/H_0 \le 1 - [1 - p^2 (2R_{e0}/H_0)^2]^{1/2} . \quad (5)$$

Here one can see a clear difference between the prolate ($R_{e0} < H_0$) and oblate ($R_{e0} > H_0$) ellipsoids. In the former case $|\delta H/H_0| \le p^2(R_{e0}/H_0)^2$. In particular, if $(R_{e0}/H_0)^2 \le 1/2$, then $|\delta H/H_0| \le |\delta R_e/R_{e0}|$. Physically, this means that as the ellipsoid in Eq. (3) becomes more prolate, the role of $2R_e^2$ decreases, and we actually determine a relatively large value at the above-indicated error of determination of its small component. It should be noted that the considered errors are not independent. Moreover, δR_e and δH have opposite signs, that is, if we "underestimate" R_e, then we "overestimate" H, and vice versa.

The situation is quite different for an oblate ellipsoid. In this case we determine a relatively small value as a difference of two large values. It is clear that the errors in this case can be much larger. In particular, if $p^2(2R_{e0}/H_0)^2 > 1$ (that is $A^2 < b^2 - C^2$), then at a certain orientation of the ellipsoid it may prove that the measured value $H^2 < 0$, and $\delta H/H_0$ is a complex value, what makes no physical sense. Therefore, from the practical point of view, the considered measurements are mainly applicable to prolate ellipsoids.

Consider now the relative error of determination of the volume $\delta V/V_0$, where $\delta V = V - V_0$; $V_0 = 4\pi AbC/3$; $V = 4\pi HR_e^2/3$.

It is easily seen that for a prolate ellipsoid

$$1 - [1 + 2p^2(1 - R_{e0}^2/H_0^2)]^{1/2} \le \delta V/V_0 \le 1 - [1 - 2p^2(1 - R_{e0}^2/H_0^2)]^{1/2} . \quad (6)$$

It formally follows from Eq. (6) that $\delta V/V_0 = 0$ at $R_{e0}^2 = H_0^2$. However, on the above assumptions about the relation between the ellipsoid dimensions ($A^2 - b^2 > b^2 - C^2$) one can see that $R_{e0}^2/H_0^2 \le 1 - p^2$. Then, taking into account that $p^2 \ll 1$, we obtain $|\delta V/V_0| \le p^2 (1 - R_{e0}^2/H_0^2)$. The minimum value $|\delta V/V_0| \approx p^4$ corresponds to a minimum prolation of the ellipsoid (the maximum value of R_{e0}^2/H_0^2). In this case the errors $\delta R_e/R_{e0}$ and $\delta H/H_0$ compensate for each other to a maximum degree, so that $\delta V/V_0 \ll \delta R_e/R_{e0}$. It is clear that at $p = 0$ (ellipsoid of revolution) $\delta V/V_0 = 0$, because we consider the errors caused only by unknown particle orientation. For an ellipsoid of revolution the orientation is insignificant in the problem under consideration.

For a strongly prolate ellipsoid ($R_{e0}^2/H_0^2 \ll 1$), as follows from Eqs. (4) and (5), $\delta H/H_0 \ll \delta R_e/R_{e0}$, and $\delta V/V_0 = 2\delta R_e/R_{e0} = p^2$ because $V \sim R_e^2$.

Now let us present some numerical estimates. Assume that $p^2 \approx 0.2$ (that is $C/b \approx 0.8$), and $R_{e0}^2/H_0^2 \approx 0.8$. Then $\delta R_e/R_{e0} \approx 0.1$; $\delta H/H_0 \approx 0.16$; $\delta V/V_0 \approx 0.04$. If at the same value $p^2 \approx 0.2$ we assume that $R_{e0}^2/H_0^2 \approx 0.5$, then $\delta R_e/R_{e0} \approx \delta H/H_0 \approx \delta V/V_0 \approx 0.1$. Finally, we have that if $R_{e0}^2/H_0^2 << 1$, then $\delta R_e/R_{e0} \approx 0.1$, $\delta H/H_0 \approx 0$, $\delta V/V_0 \approx 0.2$.

Thus, for a particle having the shape of a triaxial ellipsoid, in which the closest principal dimensions differ insignificantly (for a particle differing only slightly from an ellipsoid of revolution), it proves to be possible to determine its principal dimensions, namely, the height and equivalent radius, from the parameters of only two images corresponding to the projections onto two mutually orthogonal planes. The errors of determination of these dimensions, as well as of the particle volume, due to the unknown orientation of the particle are determined by the parameter characterizing how greatly the particle shape differs from an ellipsoid of revolution. The errors of determination of the height and volume depend also on the ratio between the equivalent radius and the height. For an oblate ellipsoid these errors can be rather large. From the practical point of view, it is preferable to use the technique considered for analysis of prolate particles (threads, fibers) because the errors for them are not very large.

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