Application of exponential series to frequency integration of the radiative transfer equation

S.D. Tvorogov

Institute of Atmospheric Optics, Siberian Branch of the Russian Academy of Sciences, Tomsk

Received July 8, 1999

The approach known in the atmospheric optics as exponential series is used to derive the equation for spectrally integral characteristics of the light upon propagation through an inhomogeneous emitting aerosol–molecular medium.

1. Statement of the problem

The equation of radiative transfer in an inhomogeneous aerosol–molecular medium

$$\mathbf{n}\ \mathrm{grad}\, I(\mathbf{r},\mathbf{n};\omega) = -\big(\varkappa\eta(\mathbf{r},\omega) + \sigma(\mathbf{r},\omega)\big)\, I(\mathbf{r},\mathbf{n};\omega) + \int \mathrm{d}\mathbf{n}'\ \varphi(\mathbf{n},\mathbf{n}';\omega) I(\mathbf{n}';\omega) + \eta(\mathbf{r},\omega) \quad (1)$$

is written for the spectral (at the frequency ω) intensity I of a beam coming at the point $\mathbf{r}$ along the unit vector $\mathbf{n}$; $\varkappa$, σ, η, and φ are the coefficients of molecular absorption, aerosol extinction, emission, and the properly normalized scattering phase function. Assume that we need a spectrally integrated parameter

$$A(\mathbf{r},\mathbf{n}) = \frac{1}{\Delta\omega}\int_{\omega'}^{\omega''} \mathrm{d}\omega\ I(\mathbf{r},\mathbf{n};\omega),\ \Delta\omega = \omega'' - \omega', \quad (2)$$

that usually appears when discussing the radiative properties of the atmosphere.

The practical problem expressed by Eqs. (1) and (2) is well known. The integral term in Eq. (1) presents certain computational difficulties. This term describes scattering whose characteristics rather slightly depend on ω. Molecular absorption, in its turn, is essentially trivial in the problem of wave propagation. However, it significantly increases the bulk computations by Eq. (2) because of a huge number of spectral lines involved. Of course, we would like to have an equation for direct calculation of Eq. (2). Here it proves helpful to use the approach called the exponential series.

Let us present this term in the context of Refs. 1–3. Assume that we consider the absorption function P for a ray of length l in a homogeneous purely molecular medium, then

$$P(l) \equiv \frac{1}{\Delta\omega}\int_{\omega'}^{\omega''} \mathrm{e}^{-\varkappa(\omega)l}\mathrm{d}\omega = \int_0^1 \mathrm{e}^{-S(g)l}\mathrm{d}g = \sum_{\nu=1} b_\nu\, \mathrm{e}^{-S(g_\nu)l}. \quad (3)$$

The newly introduced function $S(g)$ is monotonic (in spite of “peaked” $\varkappa(\omega)$) and inverse to

$$g(S) = \frac{1}{\Delta\omega}\int \mathrm{d}\omega\ ; \quad (4)$$

$$\varkappa(\omega) \le S,\ \omega \in [\omega', \omega''] .$$

We use b_ν and g_ν to denote the ordinates and abscissas of the quadrature formula selected to numerically calculate the second integral in Eq. (3).

The idea of using Eq. (3) to solve the problem posed in Eqs. (1) and (2) has been put forward in Ref. 4 as applied to a homogeneous medium without emission ($\eta = 0$ in Eq. (1)). The corresponding scenario looks like the exact transformation, while the series in Ref. 3 is a sort of approximation. Let us now present its basic elements.

The function $J(\mathbf{r}, \mathbf{n}, l; \omega)$ is introduced, which is the solution of the equation

$$\mathbf{n}\ \mathrm{grad}\, J + \frac{\partial J}{\partial l} = -(\varkappa + \sigma)\, J + \int \mathrm{d}\mathbf{n}'\ \varphi\ J(\mathbf{r},\mathbf{n}',l;\omega) \quad (5)$$

with the boundary conditions

$$J(\mathbf{r},\mathbf{n},0;\omega) = J(\mathbf{r},\mathbf{n},\infty;\omega) = 0. \quad (6)$$

Upon integration $\int_0^\infty \mathrm{d}l(\ldots)$ of Eq. (5) under the condition (6), Eq. (1) arises, i.e., ($\eta = 0$)

$$I(\mathbf{r},\mathbf{n};\omega) = \int_0^\infty J(\mathbf{r},\mathbf{n},l;\omega)\ \mathrm{d}l\ . \quad (7)$$

The substitution

$$J = E(\mathbf{r},\mathbf{n},l)\ \mathrm{e}^{-\varkappa(\omega)l} \quad (8)$$

in Eq. (5) eliminates the selective $\varkappa(\omega)$, and the equation for E

0235-6880/99/09 730-05 $02.00

$$\mathbf{n}\ \mathrm{grad}E + \frac{\partial E}{\partial l} = -\sigma\ E + \int \mathrm{d}\mathbf{n}'\ \varphi\ E(\mathbf{r},\mathbf{n}',l) \qquad (9)$$

now includes only ω–independent (within the selected range Δω) characteristics of scattering. Therefore, after multiplication of Eq. (9) by $\exp(-\varkappa l)$ and integration over ω with the account of the definition (3), we have the following equation:

$$\mathbf{n}\ \mathrm{grad}\ D(\mathbf{r},\mathbf{n},l) + P\frac{\partial E}{\partial l} = -\sigma D + \int \mathrm{d}\mathbf{n}'\ \varphi\ D(\mathbf{r},\mathbf{n}',l), \quad (10)$$

which involves the function

$$D(\mathbf{r},\mathbf{n},l) \equiv \frac{1}{\Delta\omega}\int\limits_{\omega'}^{\omega''} J(\mathbf{r},\mathbf{n},l;\omega)\mathrm{d}\omega = E(\mathbf{r},\mathbf{n},l)P(l)\,. \qquad (11)$$

Equations (2), (7), (8), and (11) give rise to the chain

$$A(\mathbf{r},\mathbf{n}) = \frac{1}{\Delta\omega}\int\limits_{\omega'}^{\omega''}\mathrm{d}\omega\int\limits_{0}^{\infty}\mathrm{d}l\ J(\mathbf{r},\mathbf{n},l;\omega) =$$
$$= \int\limits_{0}^{\infty}\mathrm{d}l\ D(\mathbf{r},\mathbf{n},l) = \int\limits_{0}^{\infty}\mathrm{d}l\ E(\mathbf{r},\mathbf{n},l)\ P(l),$$

which demonstrates the need for integration over l. It is just at this stage when we make use of Eq. (3)

$$\int\limits_{0}^{\infty} P(l)\ \frac{\partial E}{\partial l}\ \mathrm{d}l = -\int\limits_{0}^{\infty} E\ \frac{\partial P}{\partial l}\ \mathrm{d}l =$$
$$= \sum_{\nu} b_\nu\ S(g_\nu)\int\limits_{0}^{\infty}\mathrm{e}^{-S(g_\nu)l}\ E\ \mathrm{d}l \equiv \sum_{\nu} b_\nu\ S(g_\nu)\ A_\nu$$

with the obvious reference to Eq. (6). It also follows from this chain that

$$A = \int\limits_{0}^{\infty}\mathrm{d}l\ E(\mathbf{r},\mathbf{n},l)P = \sum_{\nu} b_\nu\int\limits_{0}^{\infty}\mathrm{e}^{-S(g_\nu)l}\ E\mathrm{d}l = \sum_{\nu} b_\nu A_\nu.$$

The doubtless linear independence of b_ν allows us to take A_ν as the solution of the equation

$$\mathbf{n}\ \mathrm{grad}\,A_\nu(\mathbf{r},\mathbf{n}) = -\left(\sigma + S(g_\nu)\right)\ A_\nu + \int \mathrm{d}\mathbf{n}'\ \varphi\ A(\mathbf{r},\mathbf{n}') \quad (12)$$

with the following

$$A(\mathbf{r},\mathbf{n}) = \sum_{\nu} b_\nu\ A_\nu(\mathbf{r},\mathbf{n})\,. \qquad (13)$$

Comparing Eqs. (12) and (1), we see that we have returned to the same transfer equation. However, in the new equation the function $\varkappa(\omega)$, which varies widely with ω, is replaced with the frequency-independent function $S(g_\nu)$. If the quadrature formula in Eq. (3) is chosen properly, Eq. (13) involves only several terms in spite of tens of thousands as usually in the case of direct numerical integration of Eq. (2).

The above trick is then generalized to the case of an inhomogeneous medium (all aerosol and molecular characteristics are functions of **r**), which requires the free radiation of the medium to be taken into account. Such a situation is characteristic of most "difficult" spectral region (3–8 μm) in the problem of estimation of the atmospheric radiative fluxes.

2. Solution

" y the well known rules $I = I^{(0)} + I'$, where $I^{(0)}$ is the general solution of the homogeneous (at $\eta = 0$) equation with the boundary conditions of the problem, and I' is a particular solution of the inhomogeneous equation. The latter is written in terms of the corresponding Green's function. It is clear that Eq. (2) acquires similar structure: $A = A^{(0)} + A'$.

The function $I^{(0)}$ also follows the above-considered scenario. However, because $\varkappa$ depends on **r**, now the substitution (8) adds to the right-hand side of Eq. (9) the term $(-El)\ \mathbf{n}\ \mathrm{grad}\ \varkappa(\omega,\mathbf{r})$, which is selective with respect to ω as before. This circumstance requires some explanation. The case in point now is elimination of $\varkappa$ from Eq. (1). Therefore, the exponent in Eq. (8) cannot include the term like

$$\tau = \int\limits_{z(\mathbf{n})}\varkappa(\omega,\mathbf{r}(z))\mathrm{d}z$$

with integration over the ray because then the **n**-dependent function τ finds itself under the sign $\int \mathrm{d}\mathbf{n}'(\ldots)$. By the same argument, prior transition from Eq. (1) to the integral equation is unreasonable: it gives rise to the integral of $\varkappa$ with the argument $\mathbf{r} - z\mathbf{n}$. And it is by no means reasonable to refer in Eq. (9) to the Laplace transform with respect to l.

So it becomes necessary to invoke some approximation. We introduce it based on the condition

$$\left|\,\mathrm{grad}\,\varkappa(\mathbf{r},\omega)\right| / \sigma^2 << 1\,, \qquad (14)$$

which is almost unquestionably true in the overwhelming majority of applications of atmospheric optics. The condition (14) means comparison of the "excessB term with the first term in the right-hand side of Eq. (9). Equation (9) can be formally treated as a nonstationary equation of transfer (assuming $l \sim t$, where t is time). So for it $l = 0(1/\sigma)$ in any problem with scattering.[5]

The condition (14) turns us back to Eq. (9) and the following scenario. The only difference is that $S(g_\nu) \to S(g_\nu;\ \mathbf{r})$ in the definition (4). Of course, recourse to Eq. (14) is again necessary for $S(g_\nu;\ \mathbf{r})$, but now this merely follows from the above-said.

The final result is just clear: for $A^{(0)}$ we have Eq. (13) with A_ν replaced with $A_\nu^{(0)}$, that is, solution to Eq. (12) with $S(g_\nu;\,\mathbf{r})$ in spite of $S(g_\nu)$ and the above notes on the comparison of Eqs. (12) and (1).

It is just a peculiar point arising here and associated with the exponential series. Assume that there is no scattering and the intensity of external radiation is constant within $\Delta\omega$ (it is assumed to be unity). Certainly, then we have the equation

$$A^{(0)} = \frac{1}{\Delta\omega}\int\limits_{\omega'}^{\omega''} e^{-\tau}d\omega \equiv \frac{1}{\Delta\omega}\int\limits_{\omega'}^{\omega''} d\omega\left\{\exp - \int\limits_{0}^{\infty} \varkappa(\mathbf{r}-R\mathbf{n};\omega)\,dR\right\}$$

with integration over the ray coming at the point $\mathbf{r}$ along the direction $\mathbf{n}$ from $(-\infty)$. Then Eq. (3) is the exponential series including $\tilde{S}(g_\nu;\mathbf{r})$; it is constructed by the scheme (4) with $\tau(\omega,\mathbf{r})$ in spite of $\varkappa(\omega)$. However, if we consider purely molecular atmosphere, the solution of the problem for $A^{(0)}$ is the series

$$A^{(0)} = \sum_\nu b_\nu\, e^{-\int_0^\infty S(g_\nu;-R\mathbf{n})dR}.$$

It should clearly be treated as an approximation resulting from the condition (14). This is a cost of elimination of the term $\varkappa$ selective in frequency, from the transfer equation in the case of an inhomogeneous medium because there is essentially no alternative to version (8).

At the same time, this approximation, that we were forced to accept, can be corrected by purely heuristic replacement of $S(g_\nu;\,\mathbf{r})$ by $\mathbf{n}\,\mathrm{grad}\,\tilde{S}(g_\nu;\mathbf{r})$. Actually:

$$S(g_\nu;\mathbf{r}) = \mathbf{n}\ \mathrm{grad}\int\limits_0^\infty \tilde{S}(g;\mathbf{r}-R\mathbf{n})dR$$

then the equation for $I_\nu^{(0)}$ also can be reduced to $\mathbf{n}\,\mathrm{grad}\,I_\nu^{(0)} = -\,(\mathbf{n}'\,\mathrm{grad}\,\tilde{S}\,)\,I_\nu^{(0)}$ with the exact solution. However, there is little sense in such an obvious complication of calculations, keeping in mind good approximating capabilities of the discussed approximation (see Ref. 6).

A particular solution is

$$I'(\mathbf{r}',\mathbf{n}';\omega) = \int d\mathbf{r}d\mathbf{n}\ \eta(\mathbf{r},\mathbf{n};\omega)\,G_\omega(\mathbf{r}\mathbf{n}|\mathbf{r}'\mathbf{n}') \qquad (15)$$

with the Green's function $G_\omega(\mathbf{r}\mathbf{n}|\mathbf{r}'\mathbf{n}')$ being a solution to the problem

$$-\mathbf{n}\ \mathrm{grad}\,G_\omega(\mathbf{r}\mathbf{n}|\mathbf{r}'\mathbf{n}') = -\big(\sigma(\mathbf{r},\omega)+\varkappa(\mathbf{r},\omega)\big)\,G_\omega(\mathbf{r}\mathbf{n}|\mathbf{r}'\mathbf{n}') + $$
$$+\int d\mathbf{n}''\ \varphi(\mathbf{r},\mathbf{n},\mathbf{n}'';\omega)\,G_\omega(\mathbf{r}\mathbf{n}''|\mathbf{r}'\mathbf{n}') + \delta(\mathbf{r}-\mathbf{r}')\,\delta(\mathbf{n}-\mathbf{n}') \qquad (16)$$

under zero boundary conditions. Equations (15) and (16) are proved in a rather standard way from the mathematical point of view. Equation (1) is multiplied by G_ω, while Eq. (16) is multiplied by I, thus the obtained equations are subtracted, and the difference is integrated over $\mathbf{r}$ and $\mathbf{n}$; the integration limits are the medium volume and the sphere of a unit radius. Obviously, first terms in the right-hand side disappear. Then

$$\int d\mathbf{n}d\mathbf{n}''\left\{\varphi(\mathbf{n},\mathbf{n}'')\,I(\mathbf{n}'')\,G_\omega(\mathbf{n}) - \varphi(\mathbf{n},\mathbf{n}'')\,G_\omega(\mathbf{n}'')\,I(\mathbf{n})\right\} = 0.$$

It is sufficient to make permutation of integration variables $\mathbf{n} \rightleftarrows \mathbf{n}''$ in some term and to take into account the circumstance that the argument of φ is actually $\mathbf{n}-\mathbf{n}''$:

$$\int d\mathbf{r}\ \left\{G_\omega(\mathbf{r}|\mathbf{r}')\ \mathbf{n}\ \mathrm{grad}\,I(\mathbf{r}) + I(\mathbf{r})\ \mathbf{n}\ \mathrm{grad}\,G_\omega(\mathbf{r}|\mathbf{r}'')\right\} = 0.$$

Actually, the expression under consideration is reduced to

$$\mathbf{n}\int d\mathbf{r}\ \mathrm{grad}(I(\mathbf{r})\,G_\omega(\mathbf{r}|\mathbf{r}'))$$

with the following transition to integration over the medium surface, where $G = 0$. The program of elimination of the selective $\varkappa(\omega)$ from Eqs. (2), (15), and (16) and the scenarios of this action remain, in fact, the same. Of course, some details change a little bit. Let us comment just these details.

The function $\psi_\omega(\mathbf{r},\mathbf{n},l|\mathbf{r}'\mathbf{n}')$ is introduced which satisfies the equation

$$-\mathbf{n}\ \mathrm{grad}\,\psi_\omega(\mathbf{r}\ \mathbf{n}\ l|\mathbf{r}'\mathbf{n}') + \frac{\partial\psi_\omega}{\partial l} = -(\sigma+\varkappa)\,\psi_\omega + $$
$$+\int d\mathbf{n}''\,\varphi(\mathbf{r},\mathbf{n},\mathbf{n}'';\omega)\,\psi_\omega(\mathbf{r}\mathbf{n}''l|\mathbf{r}'\mathbf{n}') + \delta(\mathbf{r}-\mathbf{r}')\,\delta(\mathbf{n}-\mathbf{n}')\,2\delta(l) \qquad (17)$$

with the conditions $\psi_\omega(\mathbf{r}\ \mathbf{n}\ 0|\mathbf{r}'\mathbf{n}') = \psi_\omega(\mathbf{r}\ \mathbf{n}\ \infty|\mathbf{r}'\mathbf{n}') = 0$. We can immediately see that $G_\omega = \int\limits_0^\infty \psi_\omega\,dl$ (with the use of the condition that $\int\limits_0^\infty 2\delta(l)\,dl = 1$). The analog of Eq. (8) is

$$\psi_\omega(\mathbf{r}\ \mathbf{n}\ l|\mathbf{r}'\mathbf{n}') = e^{-\varkappa(\mathbf{r},\omega)l}\ \Phi(\mathbf{r}\ \mathbf{n}\ l|\mathbf{r}'\mathbf{n}'). \qquad (18)$$

The substitution of Eq. (18) into Eq. (17) with the mandatory reference to the condition (14) leads to the equation

$$-\mathbf{n}\,\mathrm{grad}\,\Phi + \frac{\partial\Phi}{\partial l} = -\,\sigma\Phi\int d\mathbf{n}''\varphi\Phi + \delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{n}-\mathbf{n}')2\delta(l). \qquad (19)$$

This equation is actually independent of ω (see also the comments to Eq. (9)). When deriving

Eq. (19), we used usual relationship $\delta(l)\exp(\varkappa l) = \delta(l)$. The corollary of Eqs. (2), (15), and (18) is the equation

$$A'(\mathbf{r}',\mathbf{n}') = \int \mathrm{d}\mathbf{r}\mathrm{d}\mathbf{n} \int_0^\infty \mathrm{d}l\ \Phi(\mathbf{r}\ \mathbf{n}\ l|\mathbf{r}'\mathbf{n}') \times$$

$$\times \frac{1}{\Delta\omega} \int_{\omega'}^{\omega''} \eta(\mathbf{r},\omega)\ \mathrm{e}^{-\varkappa(\omega,\mathbf{r})l}\ \mathrm{d}\omega. \qquad (20)$$

Equation (20) includes the parameter

$$H(\mathbf{r},l) = \frac{1}{\Delta\omega} \int_{\omega'}^{\omega''} \eta(\mathbf{r},\omega)\ \mathrm{e}^{-\varkappa(\omega,\mathbf{r})l}\ \mathrm{d}\omega. \qquad (21)$$

In the theory of radiation transfer this parameter is often called the source function. Application of the idea (3) and (4) to the equation of the type (21) with some $f(\omega)$ in spite of η gives the exponential series

$$\frac{1}{\Delta\omega} \int_{\omega'}^{\omega''} f(\omega)\ \mathrm{e}^{-\varkappa l}\ \mathrm{d}\omega = \sum_\nu b_\nu\ \mathrm{e}^{-\gamma(g_\nu)l}\ D, \qquad (22)$$

where $\gamma(g)$ is the monotonic function inverse to

$$g(\gamma) = \frac{1}{\Delta\omega} \int \mathrm{d}\omega\, U(\omega), \quad U(\omega) = \frac{f}{\frac{1}{\Delta\omega}\int_{\omega'}^{\omega''} f(\omega)\ \mathrm{d}\omega} \equiv \frac{f}{D},$$

$$\varkappa(\omega) \le \gamma, \ \ \omega \in [\omega', \omega'']. \qquad (23)$$

Then, $\eta = \eta_1 + \eta_2$, where η_1 and η_2 are the coefficients of emission of the aerosol and molecular gas. Of course, $\eta_1 = B(\omega, \Theta)q$ with the Planck function B (for the temperature Θ) and the aerosol coefficient of absorption q. The latter, as earlier scattering characteristics, is independent of ω within $\Delta\omega$. Similar equation is often written for η_2 as well. However, there are situations[7–9] where local thermodynamic equilibrium is violated and $\eta_2 = \varkappa B\rho$ with the corresponding factor ρ. The substitution of η_1 in Eq. (20) (A'_1 is the term corresponding to η_1 in A') and notes on q result in the procedure (22) and (23) with $f = B$ and $\varkappa(\omega) \to \varkappa(\omega, \mathbf{r})$. The latter also means that $\gamma(g) \to \gamma(g, \mathbf{r})$. The parameter

$$D \to \Omega(\mathbf{r}) = \frac{1}{\Delta\omega} \int_{\omega'}^{\omega''} B(\omega, \Theta(\mathbf{r}))\ \mathrm{d}\omega.$$

In these designations

$$A'_1(\mathbf{r}',\ \mathbf{n}') = \sum_\nu b_\nu \int \mathrm{d}\mathbf{r}\ \mathrm{d}\mathbf{n}\ \Omega(\mathbf{r})\ q(\mathbf{r})\ \Phi_\nu(\mathbf{r}\mathbf{n}|\,\mathbf{r}'\mathbf{n}'),$$

$$\Phi_\nu(\mathbf{r}\mathbf{n}|\,\mathbf{r}'\mathbf{n}') = \int_0^\infty \mathrm{d}l\ \Phi(\mathbf{r}\mathbf{n}l|\,\mathbf{r}'\mathbf{n}')\ \mathrm{e}^{-\gamma(g_\nu;\mathbf{r})l}. \qquad (24)$$

The equation for Φ_ν follows from Eq. (19), integrated according to Eq. (24), and taking into account that the condition (14) is naturally transformed in the approximation for $\gamma(g;\ \mathbf{r})$. Equation (12) was derived in the similar way. Now we have

$$-\mathbf{n}\ \mathrm{grad}\,\Phi_\nu = -\big(\sigma + \gamma(g_\nu;\mathbf{r})\big)\Phi_\nu +$$

$$+ \int \mathrm{d}\mathbf{n}''\varphi\,\Phi_\nu + \delta(\mathbf{r} - \mathbf{r}')\ \delta(\mathbf{n} - \mathbf{n}'), \qquad (25)$$

because

$$2\int_0^\infty \delta(l)\ \mathrm{e}^{-\varkappa l}\ \mathrm{d}l = 1.$$

In discussion of Eqs. (15) and (16) in the “reverse order,” we see that Eq. (25) is the Green’s function for Eq. (1) in which $\varkappa \to \gamma(g_\nu;\ \mathbf{r})$ and $\eta \to \Omega(\mathbf{r}|q|\mathbf{r})$. If we denote its particular solution as $I_\nu^{(1)\prime}$, then the corollary of Eq. (24) is $A'_1 = \sum_\nu b_\nu\ I_\nu^{(1)\prime}$. Similar reasoning can be applied to A'_2 (the term corresponding to η_2 in Eq. (20)). It should only be noted that the place of q is occupied by the correspondingly selective $\varkappa$, and it certainly is in the integrand of Eq. (23). Therefore, we need to differentiate Eq. (21) (with $\eta \to \eta_2$) with respect to l in order to return to the already used procedure. Although it is rather rarely needed to pay attention to violation of the local thermodynamic equilibrium, for our consideration, to be general, we take $B \to B\rho$, what leads to different γ' and Ω'. So, $A'_2 = \sum_\nu b_\nu\ I_\nu^{(2)\prime}$, $I_\nu^{(2)\prime}$ is a particular solution of Eq. (1), if

$$\varkappa \to \gamma'(g_\nu;\mathbf{r}), \quad \eta \to \Omega'(\mathbf{r})\ \gamma'(g_\nu;\mathbf{r}).$$

3. List of formulas used in calculations

Let us write down the final formulas (and the corresponding equations) for the parameter given by Eq. (2). The designations introduced above are left without any comments:

$$A = \sum_\nu b_\nu\ A_\nu(\mathbf{r}, \mathbf{n}),$$

$$A_\nu = A_\nu^{(0)} + A_\nu^{(1)} + A_\nu^{(2)};$$

$$\mathbf{n}\ \mathrm{grad}\,A_\nu^{(0)} = -\big(\sigma + S(g_\nu;\mathbf{r})\big)\ A_\nu^{(0)} +$$

$$+ \int \mathrm{d}\mathbf{n}'\ \varphi(\mathbf{r},\mathbf{n},\mathbf{n}')\ A_\nu^{(0)}(\mathbf{r},\mathbf{n}');$$

$S(g_\nu;\ \mathbf{r})$ is the function inverse to

$$g(S;\mathbf{r}) = \frac{1}{\Delta\omega}\int \mathrm{d}\omega;$$

$$\varkappa(\omega,\mathbf{r}) \le S,\ \omega \in [\omega', \omega''];$$

$$\mathbf{n}\ \mathrm{grad}\,A_\nu^{(1)} = -\big(\sigma + \gamma(g_\nu;\mathbf{r})\big)\ A_\nu^{(1)} +$$

$$+ \int \mathrm{d}\mathbf{n}'\ \varphi(\mathbf{r},\mathbf{n},\mathbf{n}')\ A_\nu^{(1)}(\mathbf{r},\mathbf{n}') + \Omega(\mathbf{r})\ q;$$

$\gamma(g;\ \mathbf{r})$ is the function inverse to

$$g(\gamma;\mathbf{r}) = \frac{1}{\Delta\omega}\int \mathrm{d}\omega\ U(\omega), \quad U = \frac{B}{\Omega};$$
$$\varkappa(\omega,\mathbf{r}) \le \gamma, \quad \omega \in [\omega',\omega''];$$
$$\mathbf{n}\ \mathrm{grad}\ A_\nu^{(2)} = -\left(\sigma + \gamma'(g_\nu;\mathbf{r})\right) A_\nu^{(2)} +$$
$$+ \int \mathrm{d}\mathbf{n}'\ \varphi(\mathbf{r},\mathbf{n},\mathbf{n}')\ A_\nu^{(2)}(\mathbf{r},\mathbf{n}') + \Omega'(\mathbf{r})\ \gamma'(g_\nu;\mathbf{r}),$$
$$\Omega' = \frac{1}{\Delta\omega}\int_{\omega'}^{\omega''} B(\omega)\ \rho(\omega)\ \mathrm{d}\omega\ ;$$

γ' is the function inverse to

$$g(\gamma';\mathbf{r}) = \frac{1}{\Delta\omega}\int U'(\omega)\ \mathrm{d}\omega\ , \quad U' = \frac{B\ \rho}{\Omega'},$$
$$\varkappa(\omega,\mathbf{r}) \le \gamma', \quad \omega \in [\omega',\omega''].$$

References

1. R. Goody, R. West, L. Chen, and D. Grisp, J. Quant. Spectrosc. Radiat. Transfer **42**, 539 (1989).
2. S.D. Tvorogov, Atmos. Oceanic Opt. **7**, No. 3, 165–171 (1994).
3. S.D. Tvorogov, Atmos. Oceanic Opt. **10**, Nos. 4–5, 249–254 (1997).
4. H.C. Van de Hulst and W.M. Irvin, Meteor. Soc. Roy. Sci. Liege, Ser. 5–7, No. 1, 78 (1963).
5. B.V. Sobolev, *Light Scattering in the Atmospheres of Planets* (Nauka, Moscow, 1972), 335 pp.
6. S.D. Tvorogov, L.I. Nesmelova, and O.B. Rodimova, Atmos. Oceanic Opt. **9**, No. 3, 239–242 (1996).
7. R. Goody, *Atmospheric Radiation* [Russian translation] (Mir, Moscow, 1966), 522 pp.
8. S.D. Tvorogov, Atmos. Oceanic Opt. **5**, No. 9, 647–656 (1992).
9. L.I. Nesmelova, O.B. Rodimova, and S.D. Tvorogov, *Spectral Line Profiles and Intermolecular Interaction* (Nauka, Novosibirsk, 1986), 215 pp.