STOCHASTIC ATMOSPHERIC RADIATIVE TRANSFER

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A radiation treatment of the broken cloud problem is described, based upon a class of stochastic models of the equation of radiative transfer, which considers the clouds and clear sky as a two component random mixture. These models, recently introduced in the kinetic theory literature, allow for both Markovian and non–Markovian statistics as well as spatial variations of the cloudiness. Numerical results are given which compare different models of stochastic radiative transfer, and which point out the importance of treating the broken cloud problem as a stochastic process. It is also shown that an integral Markovian model proposed within the atmospheric radiation community by Titov is equivalent to one of our differential models.

1. INTRODUCTION

It is generally accepted that cloud–radiation interaction is an essential component in the determination of earth climate, and that an accurate treatment of the broken cloud problem in General Circulation Models (GCMs) is a key element in the prediction of climate changes. We refer the reader to the papers of Stephens,[20] Ramanathan et al.,[17] Stephens et al.,[21] and the references therein. The need for a statistical description in which the clouds and clear sky are treated as a two component stochastic mixture has also been recognized for some time (Titov,[23] Stephens et al.[21]).

In this paper we present some recent models introduced in the kinetic theory literature concerning particle and radiation transport in stochastic media, that can easily and naturally be applied to this atmospheric radiative transfer problem. The kinetic theory applications of this stochastic transport formalism are many and include radiative transfer in Rayleigh–Taylor unstable inertially confined fusion pellets, neutron transport in boiling water reactors, gamma and neutron flow through concrete shields, and light transport through murky water and sooty air. For a list of published papers in the kinetic theory literature, we refer the reader to the book by Pomraning[15] and the more recent papers by Malvagi and Pomraning,[10] Vanderhaegen et al.,[27] and Su and Pomraning.[22]

The relevance of these models to the atmospheric radiative transfer have also been discussed in a recent paper by Malvagi et al.[11] Here we want to supplement that discussion and introduce these ideas to a wider audience within the climate community. In particular we discuss an approach which we consider quite promising for future implementation in GCMs, where the size of the spatial numerical cell does not allow for the resolution of individual clouds, even if the description (size, shape, and location) of such clouds were known. This approach, based upon a Markovian model that has been modified to account for arbitrary (non–Markovian) cloud size and spacing distributions, is applicable to an arbitrary binary mixture and has been generalized to a mixture of more than two components. Further, it requires relatively few input parameters, possesses several exact limits, has proven to be robust and relatively accurate far from those limits, and has the form of integro–differential equations that are convenient for analysis, and for which an extensive body of numerical solution methods is readily available. Finally, it allows for arbitrary variations of the cloudiness, both vertically and laterally.

In the next section, the treatment of the broken cloud problem as a binary mixture is described, and two classes of statistics, Markovian and renewal, are discussed. The following section presents the theory of stochastic radiative transfer and shows how this theory leads to an exact set of integro–differential equations which present a closure problem. Two closures proposed for Markovian statistics are then presented in Section 4. Section 5 briefly discusses how those Markovian closures can be modified in order to account for non–Markovian statistics. In Section 6 some numerical results are presented, which complement the results previously reported in Malvagi et al.[11] An appendix is also included, where we show that in its low order and Markovian form our approach is equivalent to a Markovian model introduced in the atmospheric science community by Titov[23] and coworkers.[29] The Titov formalism involves two coupled integral equations, and we show that these equations can be reduced to a standard differential form. This differential form is in fact identical to the low–order kinetic theory model in the special case when the clear sky is treated as completely transparent (the case treated by Titov).

2. BROKEN CLOUDS AS A BINARY MIXTURE

In real cloudiness inhomogeneities can be found at many scale lengths (Lovejoy et al.[8]). Nonetheless, due to the largely differing optical properties of liquid water and water vapour and to the fact that the macroinhomogeneities of the cloud field are on a scale which is large compared to the mean free path of photons in clouds, one can consider clouds and clear sky as a binary mixture of immiscible fluids, each with distinct physical properties. On the typical scale of a GCM spatial grid (i.e., several thousand square kilometers), the geometry of the cloud field (cloud locations, size, and spacings) often appears to be random, thus generating a radiation field which is stochastic in nature, and whose average properties one wants to determine. A statistical formulation then seems to be the best way to approach the problem.

In our treatment, we consider a model of the cloud field in which clouds and clear sky are viewed, for the

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purpose of evaluating the radiative transfer within an atmospheric layer, as a random mixture of two components labelled by an index $i = 0$ (for clear sky) and $i = 1$ (for clouds). The first class of statistics we consider are Markovian statistics, described by the equation

$$\mathrm{Prob}\,(i \to j) = ds \,/\, \lambda_i(s)\ , \quad j \neq i\ , \tag{2.1}$$

where s is a spatial coordinate along the direction of sight $\mathbf{\Omega}$, and Prob $(i \to j)$ is the (differential) probability that point $s + \mathrm{d}s$ is in component j, given that point s is in component i. Here the $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ are the Markovian transition probabilities prescribed by the cloud field, and completely describe the clouds–clear sky mixture. The probabilities $p_i(\mathbf{r})$ of finding component i at position $\mathbf{r}$ are related to the $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ by the Chapman–Kolmogorov equations given by

$$\mathbf{\Omega}\cdot\nabla p_i = p_j/\lambda_j - p_i/\lambda_i\ , \quad i = 0,\ 1\ , \quad j \neq i\ . \tag{2.2}$$

The dependence of the λ_i on $\mathbf{\Omega}$ must be such that the p_i as determined by Eq. (2.2) do not depend on $\mathbf{\Omega}$. In the following we will sometimes emphasize the special case of homogeneous (but anisotropic) statistics, for which the λ_i depend on the direction $\mathbf{\Omega}$, but not the space point $\mathbf{r}$. The probabilities p_i have the simple interpretation of being the volume fractions of the two components, and for homogeneous statistics they are related to the λ_i by the expression

$$p_i = \lambda_i/(\lambda_0 + \lambda_1)\ . \tag{2.3}$$

From Eq. (2.3) we see that in order for the p_i to be independent of $\mathbf{\Omega}$, λ_0 and λ_1 must have the same dependence on $\mathbf{\Omega}$. The correlation length λ_c associated with this homogeneous Markovian mixture is given by[14]

$$1/\lambda_c = 1/\lambda_0 + 1/\lambda_1\ . \tag{2.4}$$

A more general class of statistics that have been considered are renewal statistics.[25,7] Here we will only consider the homogeneous case, where the statistics are entirely described by the spatially independent chord length distribution functions $f_i(s)$, such that $f_i(s)\mathrm{d}s$ is the probability that component i has a chord length along the line of sight $\mathbf{\Omega}$ between s and $s + \mathrm{d}s$. The average chord length in component i, which we denote by λ_i, is then given by

$$\lambda_i = \int_0^\infty s\, f_i(s)\,\mathrm{d}s\ , \tag{2.5}$$

and the volume fraction p_i are still related to the average chord lengths λ_i by Eq. (2.3). Homogeneous Markovian statistics correspond to the special case of having the alternating paths of clouds and clear sky populate, along any direction $\mathbf{\Omega}$, two exponential distributions with mean chord lengths $\lambda_i(\mathbf{\Omega})$, i.e.,

$$f_i(s) = 1/\lambda_i\, e^{-s/\lambda_i}\ . \tag{2.6}$$

In this homogeneous case Eq. (2.1) follows from Eq. (2.6), and the two definitions of λ_i are consistent.

Returning to inhomogeneous Markovian statistics, the $\mathbf{\Omega}$ dependence of $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ can be used to introduce directionally dependent cloud sizes and spacings into the stochastic formalism. As an example, for an atmospheric layer with a volume fraction p_{cloud} occupied by clouds of average vertical (along the z axis) dimension H and average horizontal dimension D, we could write, recalling that the subscripts 0 and 1 refer to clear sky and clouds, respectively,

$$p_1 = p_{\mathrm{cloud}}\ , \quad p_0 = 1 - p_{\mathrm{cloud}}\ , \tag{2.7}$$

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_{\mathrm{cloud}}} = \left(\frac{\mu^2}{H^2} + \frac{1-\mu^2}{D^2}\right)^{1/2}, \quad \frac{1}{\lambda_0} = \left(\frac{p_{\mathrm{cloud}}}{1 - p_{\mathrm{cloud}}}\right)\frac{1}{\lambda_1}\ , \tag{2.8}$$

where μ is the cosine of the angle between the z axis and the direction $\mathbf{\Omega}$. Equation (2.8), which follows by characterizing the clouds as ellipses, accounts for clouds with an arbitrary, but constant within the layer, aspect ratio $\gamma = D/H$. For clouds with aspect ratio $\gamma = 1$ the transition probabilities λ_i became independent of $\mathbf{\Omega}$ (isotropic statistics).

In general, our Markovian model of the cloud field is completely described by two independent parameters [for instance, p_{cloud} and $\lambda_{\mathrm{cloud}}(\mathbf{\Omega})$]. These are the statistical quantities that will need to be specified in order to determine the radiative transfer in the layer.

3. STOCHASTIC RADIATIVE TRANSFER

The radiative transfer equation we will be concerned with is written

$$\mathbf{\Omega}\cdot\nabla I + \sigma I = \sigma_s \int_{4\pi} f(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, I(\mathbf{\Omega}')\,\mathrm{d}\Omega' + S\ . \tag{3.1}$$

The dependent variable in Eq. (3.1) is the specific intensity of radiation $I(\mathbf{r}, \mathbf{\Omega})$, with $\mathbf{r}$ and $\mathbf{\Omega}$ denoting the spatial and angular (photon flight direction) variables, respectively. The quantity $\sigma(\mathbf{r})$ is the macroscopic total cross section (extinction coefficient), $\sigma_s(\mathbf{r})$ is the macroscopic scattering cross section, $f(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ is the single scatter angular redistribution function normalized according to

$$\int_{4\pi} f(\mathbf{\Omega}\cdot\mathbf{\Omega}')\,\mathrm{d}\Omega' = 2\pi \int_{-1}^{1} f(\xi)\,\mathrm{d}\xi = 1\ , \tag{3.2}$$

and $S(\mathbf{r}, \mathbf{\Omega})$ denotes any emission source of photons. If one assumes local thermodynamic equilibrium for the matter, then $S = \sigma_a B$, where B is the Planck function and σ_a is the macroscopic absorption cross section corrected for induced emission. We have assumed no time dependence and coherent (no energy exchange) scattering in Eq. (3.1), but these simplifications are not necessary for the essentials of the models of stochastic transport we will consider. Thus Eq. (3.1) is a time–independent, monochromatic (gray) transfer equation, and there is no need to display the independent frequency variable which is simply a parameter.

To treat the case of a binary statistical mixture, the quantities σ, σ_s, f, and S in Eq. (3.1) are considered as discrete random variables, each of which assumes, at any $\mathbf{r}$, one of two sets of values characteristic of the two components constituting the mixture, namely, the clouds and the clear sky. We denote these two sets by σ_i, σ_{si}, f_i, and S_i, where $i = 0$ for clear sky and $i = 1$ for clouds. That is, as a photon traverses the mixture along any path, it encounters alternating segments of clouds and clear sky, each of which has known deterministic values of σ, σ_s, f, and S. The stochastic nature of the problem enters through the statistics of the cloud field, i.e., through the statistical knowledge as to whether a cloud or clear sky is present at point **r**. Since σ, σ_s, f, and S in Eq. (3.1) are (two–state, discrete) random variables, the solution of Eq. (3.1) for I is stochastic, and we let $\langle I \rangle$ denote the ensemble–averaged intensity. (In the following, the notation $\langle g \rangle$ will always denote the average of the quantity g over all physical realizations of the statistics.) The goal in any statistical model of cloud–radiation interaction is to obtain a relatively simple and accurate set of equations for $\langle I \rangle$. It may also be of interest to have a model for the higher moments of the stochastic radiation field, such as the variance.

In order to derive equations for $\langle I \rangle$, we introduce the characteristic function $\chi_i(\mathbf{r})$, defined, for each physical realization, as

$$\chi_i(\mathbf{r}) = \begin{cases} 1 \text{ , if position } \mathbf{r} \text{ is in component } i, \\ 0 \text{ , otherwise .} \end{cases} \tag{3.3}$$

It is easy to see that

$$\langle \chi_i(\mathbf{r}) \rangle = p_i(\mathbf{r}) \ , \tag{3.4}$$

that is, the ensemble average of the characteristic function is simply the volume fraction of component i. We now multiply Eq. (3.1) by χ_i and rewrite the result as

$$\mathbf{\Omega}\cdot\nabla(\chi_i I) + \sigma_i \chi_i I =$$

$$= \sigma_{si} \int_{4\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}') \chi_i I(\mathbf{\Omega}') \, d\mathbf{\Omega}' + \chi_i S_i + I \, \mathbf{\Omega}\cdot\nabla\chi_i \ , \tag{3.5}$$

where we have taken χ_i under the gradient sign and added the required extra term to the right–hand side. Taking the ensemble average of Eq. (3.5) over all possible physical realizations of the statistics, one obtains

$$\mathbf{\Omega}\cdot\nabla(p_i I_i) + \sigma_i p_i I_i =$$

$$= \sigma_{si} \int_{4\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}') p_i I_i(\mathbf{\Omega}') \, d\mathbf{\Omega}' + p_i S_i + \langle I \, \mathbf{\Omega}\cdot\nabla\chi_i \rangle \ , \tag{3.6}$$

where we have defined

$$p_i I_i = \langle \chi_i I \rangle \ . \tag{3.7}$$

According to Eq. (3.7), $I_i(\mathbf{r}, \mathbf{\Omega})$ can be interpreted as the conditional ensemble average of I, conditioned upon position **r** being in component i. In terms of the I_i, the ensemble averaged intensity $\langle I \rangle$ is given by

$$\langle I \rangle = p_0 I_0 + p_1 I_1 \ . \tag{3.8}$$

We note that Eq. (3.6) can be simply interpreted as describing the equation of transfer within component i, with the last term on the right–hand side providing a coupling between the two components. It has been shown (Adams et al.[1]) that this coupling term can be rewritten in a more physically meaningful way as

$$\langle I \, \mathbf{\Omega}\cdot\nabla\chi_i \rangle = p_j \overline{I}_j/\lambda_j - p_i \overline{I}_i/\lambda_i \ , \tag{3.9}$$

where $\overline{I}_i$ denotes a new conditional average of I, conditioned upon position **r** being an interface between component i and component j, with component i to the left of the interface (the vector $\mathbf{\Omega}$ points from left to right). For inhomogeneous Markovian statistics, the $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ in Eq. (3.9) are simply the Markov transition lengths as defined by Eq. (2.1). For homogeneous non–Markovian statistics, $\lambda_i(\mathbf{\Omega})$ has the physical interpretation of the mean chord length in component i. In the general case of inhomogeneous non–Markovian statistics, $\lambda_i(\mathbf{r}, \mathbf{\Omega})$ is well defined mathematically in terms of the statistics, and while a precise physical interpretation is not apparent, λ_i in this case can be qualitatively interpreted as a characteristic chord length in component i in direction $\mathbf{\Omega}$. In all cases, the ratio $p_i ds/\lambda_i$ represents the probability of crossing an interface between component i and component j when moving from **r** to $\mathbf{r} + \mathbf{\Omega}\, ds$.

Using Eq. (3.9) in Eq. (3.6) we obtain

$$\mathbf{\Omega}\cdot\nabla(p_i I_i)+\sigma_i p_i I_i = \sigma_{si}\int_{4\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}') p_i I_i(\mathbf{\Omega}') \, d\mathbf{\Omega}' +$$

$$+ p_i S_i+p_j \overline{I}_j/\lambda_j - p_j \overline{I}_i/\lambda_i \ . \tag{3.10}$$

Equation (3.10) represents an exact set of two equations for the four unknowns I_i and $\overline{I}_i$, valid for arbitrary statistics, but underdetermined, since we have more unknowns than equations. At this point, either more equations, or closures that relates the $\overline{I}_i$ to the I_i, are needed to supplement Eq. (3.10).

4. TWO MARKOVIAN CLOSURES

The simplest closure proposed for Eq. (3.10) is given by the relation

$$\overline{I}_i = I_i \ . \tag{4.1}$$

The closed set of equations resulting from using Eq. (4.1) in Eq. (3.10) can be considered as a low–order model, and corresponds to the physical assumption that not only the underlying geometry, but the radiation field itself can be described as a Markovian process. Once the solutions for the I_i are obtained from this set, the ensemble averaged intensity $\langle I \rangle$ can be found from Eq. (3.8).

The closure given by Eq. (4.1) is known to be exact for inhomogeneous Markovian statistics in the absence of photon scattering ($\sigma_{si} = 0$), for which the Markovian assumption is satisfied. With scattering in the underlying transport problem, Eq. (4.1) is an approximation, but the resulting model for stochastic transport has been shown to be robust and accurate for Markovian statistics (Adams et al.[1]). For non–Markovian statistics, the closure given by Eq. (4.1) is an approximation in all cases, even in the absence of scattering. The issue of how to treat non–Markovian statistics is addressed in next section. The model expressed by Eqs. (3.10) and (4.1) has been derived by several authors in alternative ways: using the method of smoothing (Pomraning,[17] Levermore et al.[6]), utilizing the Liouville master equation (Vanderhaegen[24]), using reactor noise techniques (Sahni[18]), and making the assumption that the particle trajectories are uncorrelated (Sahni[19]). Of particular interest to the atmospheric sciences community has been the work of Titov and coauthors,[27] in which they considered the non–scattering problem and the problem of clouds in a vacuum (Titov[23]). In their approach they first formally integrate the equation of transfer along the z axis, take ensemble averages, and then close their moment equations by making the assumption that the process is Markovian. Although their final result is in the form of integral equations for the ensemble averaged intensity, we show in the Appendix that their equations are equivalent to Eqs. (3.10) and (4.1), once we specialize them to their problem.

The Markovian assumption that underlies Eq. (4.1) also allows one to obtain equations for the higher moments of the intensity. If we define the moments $I_i^{(n)}$ according to

$$p_i\, I_i^{(n)}(\mathbf{r},\, \mathbf{\Omega}) = \langle \chi_i(\mathbf{r})\, I^{(n)}(\mathbf{r},\, \mathbf{\Omega})\rangle, \qquad (4.2)$$

one can derive the equations (Boffi et al.[3])

$$\mathbf{\Omega}\cdot\nabla\left(p_i\, I_i^{(n)}\right) + n\, \sigma_i\, p_i\, I_i^{(n)} =$$

$$= n \int_{4\pi} \sigma_{si}(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, p_i\, I_i^{(1)}(\mathbf{\Omega}')\, I_i^{(n-1)}(\mathbf{\Omega})\, \mathrm{d}\mathbf{\Omega}' +$$

$$+ n\, p_i\, S_i\, I_i^{(n-1)} + p_j\, I_j^{(n)}/\lambda_j - p_i\, I_i^{(n)}/\lambda_i\, . \qquad (4.3)$$

Again, Eq. (4.3) is exact for the non–scattering case. In the presence of scattering it is only an approximation whose accuracy has not been tested, except for $n = 1$ and planar geometry, in which case Eq. (4.3) reduces to Eq. (3.10) closed by Eq. (4.1). From Eq. (4.3) it is also clear that the equations for the moments can be solved in succession.

Returning to Eqs. (3.10) and (4.1), it is clear that all quantities can in principle depend upon all three spatial coordinates x, y, and z. In the restricted problem of three–dimensional clouds embedded in a planar layer, we make the assumption that the statistics and the boundary conditions in the layer are independent of x and y (in particular, this restricts the clouds to a common characteristic dimension along both x and y). If we further make the assumption that the physical properties of the clouds and of the clear sky do not change, on average, along the horizontal directions within a spatial cell, then the solution for the ensemble averages I_i and $\overline{I}_i$ will be independent of x and y, although they will still depend upon the azimuthal angle φ which, along with μ, defines $\mathbf{\Omega}$. However, it is only these ensemble averages in Eq. (3.10) which depend upon φ. This allows us to integrate Eq. (3.10) over φ to obtain the planar equation

$$\mu \frac{\partial}{\partial z}\left[\, p_i(z)\, \psi_i(z,\, \mu)\right] + \sigma_i(z)\, p_i(z)\, \psi_i(z,\, \mu) =$$

$$= \sigma_{si}(z) \int_{-1}^{1} g_i(z,\, \mu,\, \mu')\, p_i(z)\, \psi_i(z,\, \mu')\, \mathrm{d}\mu' + p_i(z)\, S_i(z,\, \mu) +$$

$$+ \frac{p_j(z)\, \overline{\psi}_j(z,\, \mu)}{\lambda_j(z,\, \mu)} - \frac{p_i(z)\, \overline{\psi}_i(z,\, \mu)}{\lambda_i(z,\, \mu)}\, , \qquad (4.4)$$

where

$$\psi_i(\mu) = \int_0^{2\pi} I_i(\mu,\, \varphi)\, \mathrm{d}\varphi\, , \qquad (4.5)$$

with a similar relationship between $\overline{\psi}_i(\mathrm{m})$ and $\overline{I}_i(\mu,\, \varphi)$. The redistribution function g_i is given by

$$g_i(\mu,\, \mu') = \int_0^{2\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, \mathrm{d}\varphi\, , \qquad (4.6)$$

and clearly has the normalization

$$\int_{-1}^{1} g_i(\mu,\, \mu')\, \mathrm{d}\mu = 1\, . \qquad (4.7)$$

Equation (4.4) still allows for altitude dependence of the cross section, and altitude and zenith angle dependence of the cloud size and spacing. However, all horizontal effects of cloud–cloud and cloud–sky interactions are taken into account by the coupling term on the right–hand side, and any explicit dependence on x and y has disappeared; it has been "averaged out" in a rigorous, nonapproximate way by the ensemble averaging. In other words, Eq. (4.4) is a one–dimensional model that rigorously accounts for the three–dimensional geometry of the clouds under the assumed translational invariance of the cross sections and the λ_i. If for some reason one is interested in the φ dependence of the ensemble averaged intensities, one can expand the intensities in Eq. (3.10) in a Fourier series in φ, the result being an uncoupled set of equations for each Fourier component (Case and Zweifel[4]). Equation (4.4) is, in fact, the equation for the $n = 0$ cosine mode. The low–order closure for Eq. (4.4) equivalent to Eq. (4.1) is then obtained by setting

$$\overline{\psi}_i = \psi_i. \qquad (4.8)$$

It has recently been suggested (Pomraning[16]) that in planar geometry the simple closure given by Eq. (4.8) can be replaced with an approximate set of two coupled transfer equations for the $\overline{\psi}_i$. These equations are given by, for $i = 0,1$ and $j \neq i$,

$$\mu \frac{\partial}{\partial z}\left(p_i\, \overline{\psi}_i\right) + \sigma_i\, p_i\, \overline{\psi}_i = \int_{-1}^{0} \sigma_{si}(\mu,\, \mu')\, p_i\, \overline{\psi}_j(\mu')\, \mathrm{d}\mu' +$$

$$+ \int_0^1 \sigma_{si}(\mu,\, \mu') p_i\, \overline{\psi}_i(\mu')\mathrm{d}\mu' + \frac{p_j\, \overline{\psi}_j}{\lambda_j} - \frac{p_i\, \overline{\psi}_i}{\lambda_i} + p_i S_i,\ \mu > 0; \qquad (4.9)$$

$$\mu \frac{\partial}{\partial z}\left(p_i\, \overline{\psi}_i\right) + \sigma_i\, p_i\, \overline{\psi}_i = \int_{-1}^{0} \sigma_{si}(\mu,\, \mu')\, p_i\, \overline{\psi}_i(\mu')\, \mathrm{d}\mu' +$$

$$+ \int_0^1 \sigma_{si}(\mu,\, \mu') p_i\, \overline{\psi}_j(\mu')\mathrm{d}\mu' + \frac{p_j\, \overline{\psi}_j}{\lambda_j} - \frac{p_i\, \overline{\psi}_i}{\lambda_i} + p_i S_i\, ,\ \mu < 0. \qquad (4.10)$$

Thus, a higher order model for the $\psi_i(z, \mu)$ consists of Eq. (4.4) coupled with Eqs. (4.9) and (4.10). Both the low order and the higher order models are tested numerically in Section 6.

With regard to boundary conditions, we assume that the incoming intensity on the surface of the system is specified and nonstochastic. Then for each physical realization of the statistical mixture, the boundary condition on Eq. (3.1) is taken as

$$I(\mathbf{r}_s, \mathbf{\Omega}) = F(\mathbf{r}_s, \mathbf{\Omega}) \, , \quad \mathbf{n}\cdot\mathbf{\Omega} < 0 \, , \tag{4.11}$$

where $\mathbf{n}$ is a normal outward pointing unit vector at a surface point $\mathbf{r}_s$, and F is the specified boundary data. Equation (4.11) implies that all conditional ensemble averaged intensities satisfy the same boundary condition. In particular, we have

$$I_i(\mathbf{r}_s, \mathbf{\Omega}) = \overline{I}_i(\mathbf{r}_s, \mathbf{\Omega}) = F(\mathbf{r}_s, \mathbf{\Omega}) \, , \quad \mathbf{n}\cdot\mathbf{\Omega} < 0 \, . \tag{4.12}$$

The azimuthally integrated intensities contained in Eq. (4.4) then satisfy

$$\psi_i(z_s, \mu) = \overline{\psi}_i(z_s, \mu) = G(z_s, \mu) \, , \quad \mathbf{n}\cdot\mathbf{\Omega} < 0 \, , \tag{4.13}$$

where z_s is a surface point of the planar system and

$$G(z_s, \mu) = \int\limits_0^{2\pi} F(\mathbf{r}_s, \mathbf{\Omega}) \, \mathrm{d}\varphi \, . \tag{4.14}$$

5. A TREATMENT OF NON–MARKOVIAN STATISTICS

The item we discuss in this section is the treatment of non–Markovian statistics. The models discussed in the last section were both derived for Markovian statistics, i.e., when along any line of sight the alternating segments of clouds and clear sky are exponentially distributed [see Eq. (2.6)]. The common feature of these models is that they manifest themselves in integro–differential equations. For non–Markovian statistics, i.e., for more general distribution functions $f_i(s)$, a statistical treatment of radiative transfer yields integral equations, arising from the theory of alternating renewal processes (Vanderhaegen,[25] Levermore et al.[7]). This formalism, exact in the non–scattering case, can be generalized to include, in an approximate way, the scattering interaction (Pomraning[14]). The accuracy of this approximation has not been fully tested.

For homogeneous statistics, these integral equations are of convolution type, and are readily solved by the Laplace transform. Levermore et al.[7] have shown that certain characteristics, namely, the deep–in behavior and the mean distance to collision, of the non–Markovian solution can be captured by the Markovian models by introducing an effective correlation length λ_{eff}, which plays the role of λ_c as given by Eq. (2.4). This effective correlation length is given by

$$\lambda_{\text{eff}} = q \, \lambda_c \, , \tag{5.1}$$

with λ_c given by Eq. (2.4) and

$$q = \frac{1}{\sigma_0}\left[\frac{1}{\tilde{Q}_0(\sigma_0)} - \frac{1}{\lambda_0}\right] + \frac{1}{\sigma_1}\left[\frac{1}{\tilde{Q}_1(\sigma_1)} - \frac{1}{\lambda_1}\right] - 1 \, . \tag{5.2}$$

Here λ_i is the mean chord length in component i as given by Eq. (2.5) and $Q_i(s)$ is the probability that the chord length in component i exceeds s, and is related to the chord length distribution function $f_i(s)$ by

$$Q_i(s) = \int\limits_s^{\infty} f_i(s') \, \mathrm{d}s' \, , \quad i = 0, 1 \, , \tag{5.3}$$

$\tilde{Q}_i(\sigma_i)$ is the Laplace transform of $Q_i(s)$ evaluated at a transform variable σ_i, i.e.,

$$\tilde{Q}_i(\sigma_i) = \int\limits_0^{\infty} \mathrm{e}^{-\sigma_i s} \, Q_i(s) \, \mathrm{d}s \, . \tag{5.4}$$

It has been shown by Levermore et al.[7] that $q \ge 0$ for all chord lengths distributions. For Markovian statistics corresponding to Eq. (2.6), we have $q = 1$. In the special case in which the clear sky is treated as vacuum ($\sigma_0 = 0$), a simple limiting process gives

$$q = \frac{1}{\sigma_1}\left[\frac{1}{\tilde{Q}_1(\sigma_1)} - \frac{1}{\lambda_1}\right] + \frac{V_0}{2\lambda_0^2} - \frac{1}{2} \, . \tag{5.5}$$

Here λ_0 is the mean spacing between clouds, and V_0 is the corresponding variance of the clear sky chord length distribution function.

Now, Eqs. (2.3) and (2.4) imply

$$\lambda_i = \lambda_c / p_j \, , \quad j \ne i \, . \tag{5.6}$$

Thus replacing the correlation length λ_c by $q\lambda_c$ implies that the λ_i must be replaced by $q\lambda_i$, since the p_i are taken as given, independent of the statistics. In summary, an approximate treatment for non–Markovian statistics is to use any of the Markovian models previously discussed, but with the first of Eq. (2.8) replaced by

$$\frac{1}{\lambda_1} = \frac{1}{\lambda_{\text{cloud}}} = q \left(\frac{\mu^2}{H^2} + \frac{1-\mu^2}{D^2}\right)^{1/2} \, , \tag{5.7}$$

where q is given by Eq. (5.2). This approximate method of treating non–Markovian mixing statistics has been tested numerically by Levermore et al.[7] and was shown to predict exact results, found by solving the renewal equations, quite well.

6. NUMERICAL RESULTS

In this section we present some numerical results obtained by solving the stochastic radiative transfer in an atmospheric layer as described by Eq. (4.4), supplemented with the low–order closure (4.8) (which we will refer to as Model 1), and the more complex closure given by Eqs. (4.9) and (4.10) (which we will refer to as Model 2). For some test problems we can compare our results to exact solutions. We also compare those two models to a simple fractional cloud approximation.

The first problem we consider (Problem 1) is the case of a layer of thickness L populated with alternating layers of clouds and clear sky (which for our purposes we treat as a vacuum), with no internal sources of radiation. In the notation of Section 2, this corresponds to considering clouds with an unbounded aspect ratio ($\gamma = \infty$ or $D = \infty$).

While this problem has limited applicability to stratiform clouds, the main reason for considering it is that an explicit exact solution to this special stochastic problem is available for truly one–dimensional (or rod) geometry (Vanderhaegen and Deutsch,[26] Pomraning,[13] Stephens et al.[21]). This is mathematically equivalent to a two–stream approximation for the full planar problem, with the quadrature angles chosen at $\mu = \pm 1$. Our purpose is then to use this problem as a test of the accuracy of Models 1 and 2. In particular, it has been argued (Sahni[19]) that the closure characterizing Model 1 [see Eq. (4.1)] is the least accurate in this case of one–dimensional geometry. The presumption then is that this test case can provide a qualitative but reliable estimate of the accuracy of Model 1, extendable to problems for which no exact solutions are available.

We write the two–stream approximation to Eqs. (4.4) and (4.8) for a source free medium and homogeneous statistics as

$$\pm \frac{d\psi_i^{\pm}}{dz} + \sigma_i \psi_i^{\pm} = \frac{\sigma_{si}}{2}\left(\psi_i^{+} + \psi_i^{-}\right) + \frac{1}{\lambda_i}\left(\psi_j^{\pm} + \psi_i^{\pm}\right), \qquad (6.1)$$

where $\psi_i^+(z)$ and $\psi_i^-(z)$ are the intensities in the $+z$ and $-z$ directions, respectively. Here $\lambda_1 = \lambda_{\text{cloud}} = qH$ is the effective average vertical size of the clouds, and λ_0 is given by the second of Eqs. (2.8). We note that for any cloud size distribution which allows the possibility of infinite size clouds (such as an exponential, i.e., Markovian, distribution which corresponds to $q = 1$), one would have to choose the atmospheric layer thickness L as infinite so that this layer would, for all realizations, completely contain all clouds. This points out one of the difficulties in using a Markovian model to describe an atmospheric layer in which all physical realizations completely contain the clouds in a layer of finite thickness L. Rather, one should use a cloud size distribution which involves some maximum allowable cloud size, with this maximum size being less than the layer thickness which contains all of the clouds. This in turn points out the importance of having a stochastic formalism available which allows for arbitrary cloud size distributions. In fact q, defined by Eq. (5.2), is the ingredient in the formalism described here which allows one to treat arbitrary cloud size distributions.

In writing Eq. (6.1) we have treated the scattering as isotropic. Although the differential scattering cross section is very forwardly peaked for clouds, for thick systems (such as clouds) a cross section of the Henyey–Greenstein or Mie scattering type can be adequately taken into account by using an effective cross section defined as

$$\sigma_{s,\text{eff}} = \sigma_s(1 - \bar{\mu}), \qquad (6.2)$$

where $\bar{m}$ is the average cosine of the scattering angle. We then interprete σ_{si} in Eq. (6.1) as meaning $\sigma_{si,\text{eff}}$. For Eq. (6.1) we use the boundary condition

$$\psi_i^+(0) = 1, \quad \psi_i^-(L) = 0. \qquad (6.3)$$

In this two–stream approximation and in view of Eq. (6.3), the transmission probability T and reflection probability R are defined as

$$T = \frac{\langle\psi^+(L)\rangle}{\langle\psi^+(0)\rangle} = \langle\psi^+(L)\rangle, \quad R = \frac{\langle\psi^-(0)\rangle}{\langle\psi^+(0)\rangle} = \langle\psi^-(0)\rangle, \qquad (6.4)$$

where

$$\langle\psi^{\pm}\rangle = p_0\psi_0^{\pm} + p_1\psi_1^{\pm}. \qquad (6.5)$$

Analogously, the two–stream approximation to Model 2 can be written as

$$\pm \frac{d\psi_i^{\pm}}{dz} + \sigma_i \psi_i^{\pm} = \frac{\sigma_{si}}{2}\left(\psi_i^{+} + \psi_i^{-}\right) + \frac{1}{\lambda_i}\left(\bar{\psi}_j^{\pm} + \bar{\psi}_i^{\pm}\right), \qquad (6.6)$$

$$\pm \frac{d\bar{\psi}_i^{\pm}}{dz} + \sigma_i \bar{\psi}_i^{\pm} = \frac{\sigma_{si}}{2}\left(\bar{\psi}_i^{\pm} + \bar{\psi}_j^{\mp}\right) + \frac{1}{\lambda_i}\left(\bar{\psi}_j^{\pm} + \bar{\psi}_i^{\pm}\right). \qquad (6.7)$$

As boundary conditions on Eqs. (6.6) and (6.7) we use

$$\psi_i^+(0) = \bar{\psi}_i^+(0) = 1, \quad \psi_i^-(L) = \bar{\psi}_i^-(L) = 0. \qquad (6.8)$$

The transmission and reflection are again given by Eq. (6.4).

For this layered geometry we consider two sets of cross sections representative of longwave and shortwave calculations in cumulus clouds (Welch et al.[28]), respectively. The first set (which we will refer to as Case 1, and corresponds to purely absorbing clouds) is given by

$$\sigma_1 = 30 \text{ km}^{-1}, \quad \sigma_0 = \sigma_{s0} = \sigma_{s1} = 0, \qquad (6.9)$$

where all the cross sections are taken as constant along the vertical direction. The second set (which we will refer to as Case 2, and corresponds to purely scattering clouds) is given by

$$\sigma_1 = \sigma_{s1} = 10 \text{ km}^{-1}, \quad \sigma_0 = \sigma_{s0} = 0. \qquad (6.10)$$

In both Case 1 and Case 2 we take the total thickness of the layer to be the unit length, i.e.,

$$L = 1 \text{ km}. \qquad (6.11)$$

Figure 1 shows T as a function of the effective cloud thickness λ_{cloud} for Case 1, as computed according to Models 1 and 2. Since these curves extend to $\lambda_{\text{cloud}} > L = 1$, this implies that certain realizations of the statistics correspond to the layer completely occupied by clouds. We have considered three values of the cloud volume fraction $p_{\text{cloud}}=p=0.1$, 0.5, and 0.9. Models 1 and 2 gives the same result, which is exact for a Markovian cloud field. For Case 1 (purely absorbing clouds) we obviously have $R = 0$ and $A = 1 - T$, where A is the absorption probability for the layer.

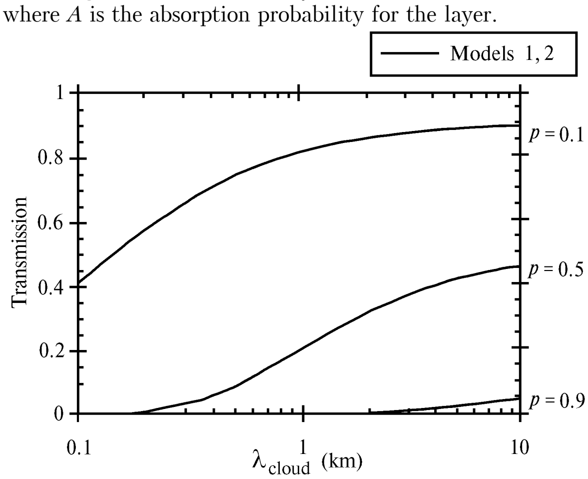

FIG. 1. Transmission versus effective cloud layer thickness for Problem 1 (parallel layers and two–stream approximation), values of the cross sections and total layer thickness corresponding to Case 1 (pure absorber), and three values of the cloud volume fraction p.

Figure 2 shows T vs. λ_{cloud} for Case 2. Here Models 1 and 2 give different results. The thick line indicates the exact solution computed according to Pomraning[13] and for the case of an exponential distribution. We see that Model 2 gives a result that is very close to the exact solution. Model 1, while not as accurate as Model 2, still provides a useful approximation, within about 10%, to the exact solution. This is in accordance with the results reported by Adams et al.[1] and Titov.[23] We also note the interesting result that Model 1 always overestimates the transmission. A physical explanation can be argued in view of the fact that in Model 1 interface averages are approximated with volumetric averages. This has the effect of skewing the Model 1 results towards the solution of a problem in which the clouds and clear sky are completely decoupled, thus overestimating the transmission. In other words, for any problem of clouds in vacuum, Model 1 provides an upper bound for T. Model 2, while more accurate than Model 1, does not share this property, and in fact it sometimes overestimates and sometimes underestimates T. This is also in accord with the findings of Pomraning[16] and Malvagi and Pomraning.[10] For Case 2 (purely scattering clouds) we have $R = 1 - T$ and $A = 0$.

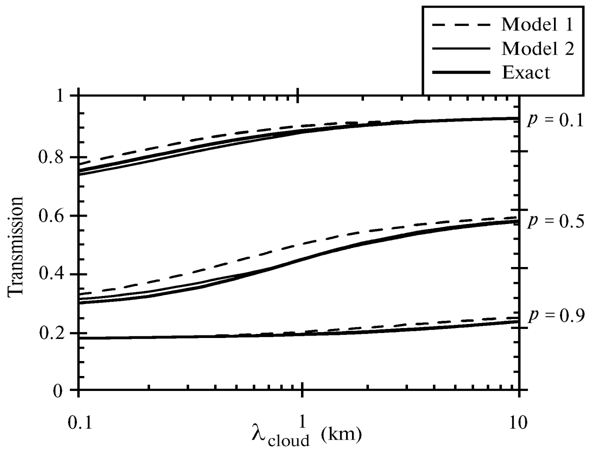

FIG. 2. Same as Figure 1, but with values of the cross sections corresponding to Case 2 (pure scatterer).

The second problem we consider (Problem 2) is the more general problem of finite size clouds ($D < \infty$) imbedded in a layer of constant thickness L. Model 1 is now written, for homogeneous statistics and after dividing Eq. (4.4) by p_i as

$$\mu \frac{\partial \psi_i}{\partial z} + \sigma_i \psi_i = \frac{\sigma_{si}}{2} \int_{-1}^{1} \psi_i(\mu)\, d\mu + \frac{1}{\lambda_i} (\psi_j - \psi_i) , \qquad (6.12)$$

where the $\lambda_i(\mu)$ are given by Eq. (5.7) and the second of Eqs. (2.8). We consider the generic boundary conditions on Eq. (6.12) given by

$$\psi_i(0) = G(\mu) , \ \ \mu > 0 ; \ \ \psi_i(L) = 0 , \ \ \mu < 0 . \qquad (6.13)$$

The transmission and reflection probabilities of the layer are now defined as

$$T = \frac{\int_0^1 \mu \langle\psi_i(L, \mu)\rangle\, d\mu}{\int_0^1 \mu F(\mu)\, d\mu} , \ R = \frac{\int_0^1 \mu \langle\psi_i(0, -\mu)\rangle\, d\mu}{\int_0^1 \mu F(\mu)\, d\mu} . \qquad (6.14)$$

Model 2 can now be written as

$$\mu \frac{\partial \psi_i}{\partial z} + \sigma_i \psi_i = \frac{\sigma_i}{2} \int_{-1}^{1} \psi_i(\mu)\, d\mu + \frac{1}{\lambda_i} (\bar{\psi}_j - \bar{\psi}_i) , \qquad (6.15)$$

$$\mu \frac{\partial \bar{\psi}_i}{\partial z} + \sigma_i \bar{\psi}_i =$$

$$= \frac{\sigma_i}{2} \left[\int_{-1}^{0} \bar{\psi}_j(\mu)\, d\mu + \int_{0}^{1} \bar{\psi}_i(\mu)\, d\mu \right] + \frac{1}{\lambda_i} (\bar{\psi}_j - \bar{\psi}_i) , \ \mu > 0, \qquad (6.16)$$

$$\mu \frac{\partial \bar{\psi}_i}{\partial z} + \sigma_i \bar{\psi}_i =$$

$$= \frac{\sigma_i}{2} \left[\int_{-1}^{0} \bar{\psi}_i(\mu)\, d\mu + \int_{0}^{1} \bar{\psi}_j(\mu)\, d\mu \right] + \frac{1}{\lambda_i} (\bar{\psi}_j - \bar{\psi}_i) , \ \mu < 0. \qquad (6.17)$$

For Eqs. (6.15) through (6.17) we impose the boundary conditions

$$\psi_i(0) = \bar{\psi}_i(0) = G(\mu) , \ \mu > 0 ; \ \psi_i(L) = \bar{\psi}_i(L) = 0, \ \mu < 0. \qquad (6.18)$$

Reflection and transmission are still defined by Eq. (6.14).

For the purpose of comparison, we also consider a fractional cloud model based on the solution of the two uncoupled equations

$$\mu \frac{\partial \tilde{\psi}_i}{\partial z} + \sigma_i \tilde{\psi}_i = \frac{\sigma_{si}}{2} \int_{-1}^{1} \tilde{\psi}_i\, d\mu , \qquad (6.19)$$

with boundary conditions

$$\tilde{\psi}_i(0) = G(\mu) , \ \ \mu > 0 ; \ \ \tilde{\psi}_i(L) = 0 , \ \ \mu < 0 . \qquad (6.20)$$

An approximation to the average intensity is then computed according to the weighting formula

$$\langle I(z, \mu)\rangle = (1 - f_c)\, \tilde{I}_0(z, \mu) + f_c\, \tilde{I}_1(z, \mu) , \qquad (6.21)$$

where $f_c(\mu_0)$ is the probability of finding a cloud along a line of sight $\mathbf{\Omega}$ forming with the vertical an angle whose cosine is μ_0. This probability is simply one minus the probability of finding clear sky at each point along the line of sight. This latter probability is the product of p_0, the probability of finding clear sky at the top of the layer, and $\hat{Q}_0(L/\mu_0)$, the conditional probability (conditioned upon the top of the layer being clear sky) of finding a clear sky chord length which exceeds the distance L/μ_0 through this layer along the line of sight. This conditional probability is discussed by Pomraning[14] and Su and Pomraning.[22] For homogeneous Markovian statistics we have

$$f_c(\mu_0) = 1 - p_0\hat{Q}_0(L/\mu_0) = 1 - (1 - p_1) \times \exp\left[-\left(\frac{p_1}{1-p_1}\right)\frac{L}{\mu_0\lambda_1(\mu_0)}\right] \qquad (6.22)$$

We refer to the model expressed by Eqs. (6.19) through (6.22) as the Fractional Cloud Model. Transmission and reflection are still defined by Eqs. (6.14).

For the values of the cross sections and the slab thickness we use the same two sets previously called Case 1 and Case 2. For Case 1 (longwave radiation) we choose isotropic incidence as the boundary condition, i.e.,

$$G(\mu) = 1 . \qquad (6.23)$$

For Case 2 (shortwave radiation), the selected boundary condition is a beam incident at an angle θ_0, i.e.,

$$G(\mu) = \delta(\mu - \mu_0) , \qquad (6.24)$$

where μ_0 is the cosine of the angle θ_0. The transfer equations are solved numerically using diamond differences in space, the discrete ordinate method with 16 angles, and a power iteration on the scattering source (Duderstadt and Martin,[5] Bell and Glasstone[2]). First we consider the case of isotropic statistics [i.e., $\lambda_i \neq \lambda_i(\mu)$]. This corresponds to considering clouds with aspect ratio equal to unity, i.e., $H = D$.

Figure 3 shows T vs. p_{cloud} for the value $\lambda_{\text{cloud}} = 0.5$ and values of the physical parameters corresponding to Case 1, as computed according to Models 1 and 2 and the Fractional Cloud Model. To compute the fractional cloud parameter f_c, we have used exponentially distributed clouds and selected the zenith line of sight [this corresponds to setting $\mu_0 = 1$ in Eq. (6.22)]. Both Models 1 and 2 provide the same solution to the stochastic problem, which is exact for exponential distributions. The simple Fractional Cloud Model overestimates the transmission, but due to the high value of the extinction coefficient ($\sigma_1 = 30$) the error never exceeds 10% of the incident radiation. Figures 4 through 6 show T vs. p_{cloud} for the same problem, but with the values of the physical parameters corresponding to Case 2 and the three angles of incidence $\theta_0 = 0$, 30 and 60 degrees. To compute the fractional cloud parameter f_c, the angle of the incident beam is now chosen as the line of sight. Here no exact solution is available, and we take the solution of Model 2 as the best available approximation. Based on this assumption, Model 1 gives a fairly accurate approximation and, as noted before, consistently overestimates the transmission. The Fractional Cloud Model in this case gives substantially larger errors, performing the worst for intermediate values of p_{cloud}. The behavior of the three models shows little dependence on the angle of incidence.

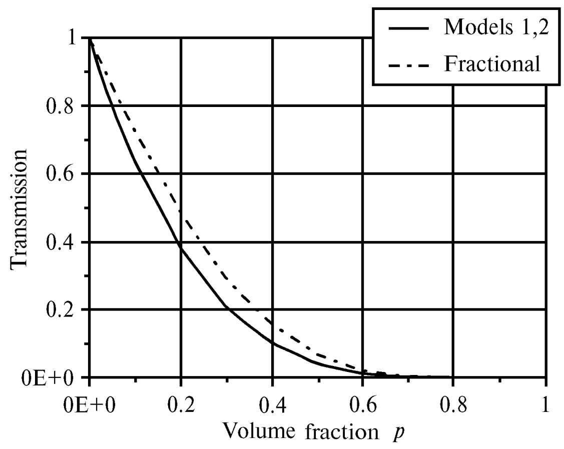

FIG. 3. Transmission versus cloud volume fraction for Problem 2 (finite size clouds imbedded in a layer of thickness L), values of the cross sections and total layer thickness corresponding to Case 1, cloud average size $H = 0.5$ km, isotropic statistics (cloud aspect ratio $\gamma = D/H = 1$) and isotropic incidence.

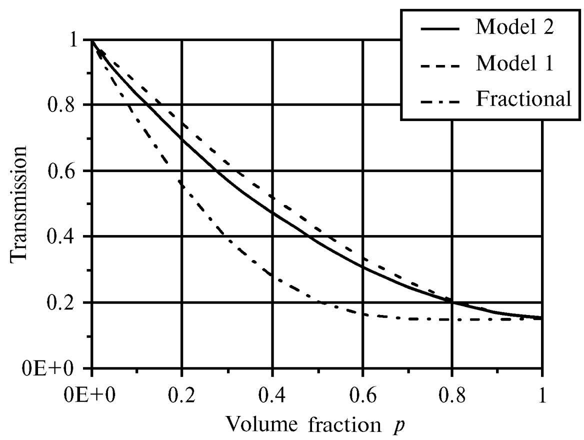

FIG. 4. Same as Figure 3, but with values of the cross sections corresponding to Case 2 and a beam incident upon the layer at an angle $\theta_0 = 0°$.

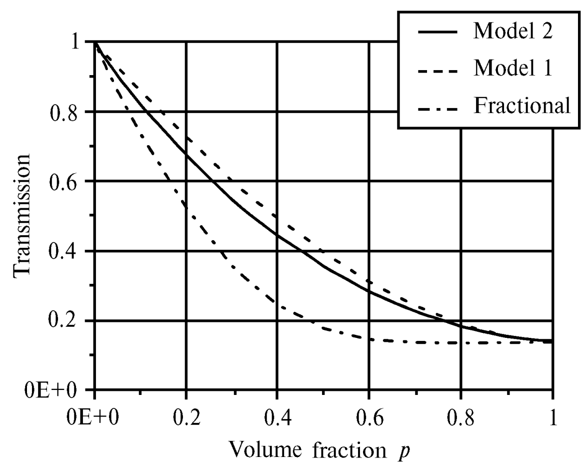

FIG. 5. Same as Figure 4, but with $\theta_0 = 30°$.

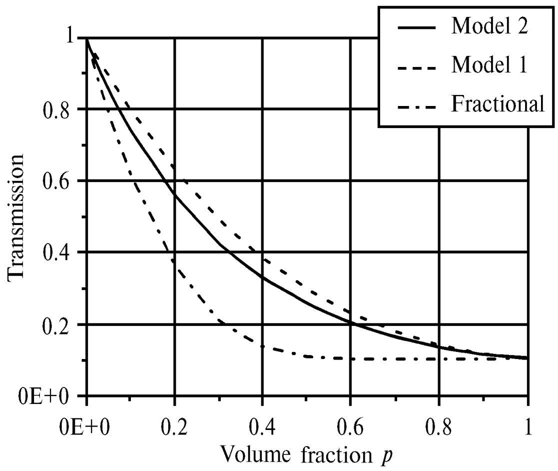

FIG. 6. Same as Figure 4, but with $\theta_0 = 60°$.

Finally, we consider the effects of the cloud aspect ratio on the predictions of our models. Figure 7 shows T vs. p_{cloud} for values of the physical parameters corresponding to Case 1, as computed according to Model 2 with values of the aspect ratio $\gamma = 0.1$ ($D = 0.5$ and $H = 5$), $\gamma = 1$ ($D = 0.5$ and $H = 0.5$), and $\gamma = 10$ ($D = 5$ and $H = 0.5$). Figures 8 through 10 show T vs. p_{cloud} for the same problem, but with the values of the physical parameters corresponding to Case 2 and the three angles of incidence $\theta_0 = 0$, 30 and 60 degrees. We see that the value of the transmission of the layer does depend, although not dramatically, on the aspect ratio of the clouds, which is thus an important parameter in determining heating rates. Intercomparisons of the three models for various values of the aspect ratio, not reported here, show the same trends seen in Figs. 3 through 6.

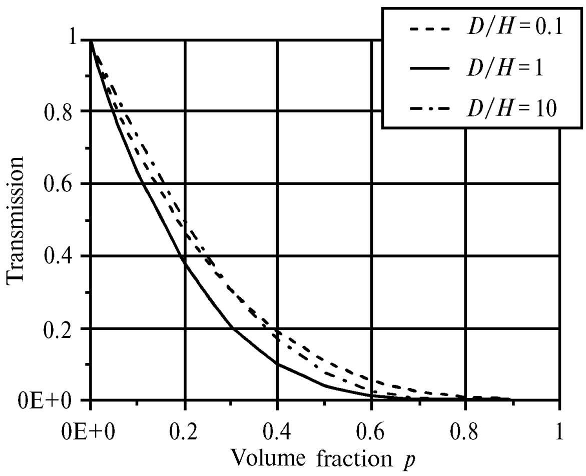

FIG. 7. Transmission versus cloud volume fraction for Problem 2, values of the cross sections and total layer thickness corresponding to Case 1, three values of the aspect ratio γ*, and isotropic incidence upon the layer. All curves are computed according to Model 2.*

7. SUMMARY AND CONCLUSIONS

In this paper we have presented what we consider a promising approach for the treatment of radiative transfer through an atmospheric layer populated with randomly distributed clouds. This approach treats the radiative transfer problem as a stochastic process, and provides predictions for the average radiative intensity once the statistical description of the cloud field is given. This approach is entirely based on the properties of the equation of transfer itself, and there is no need to use approximate empirical relations.

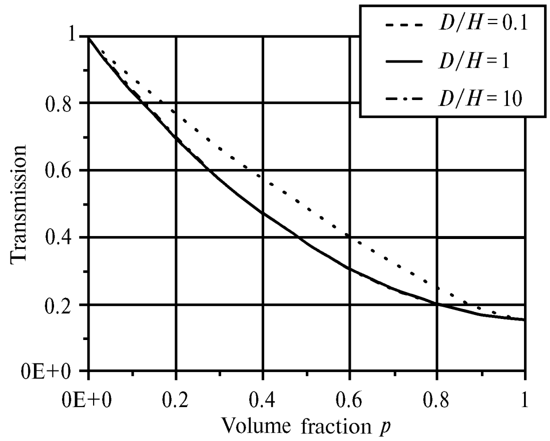

FIG. 8. Same as Figure 7, but with values of the cross sections corresponding to Case 2 and a beam incident upon the layer at an angle $\theta_0 = 0°$.

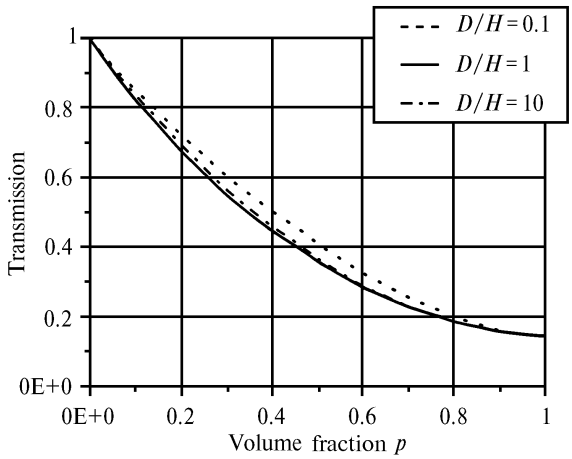

FIG. 9. Same as Figure 8, but with $\theta_0 = 30°$.

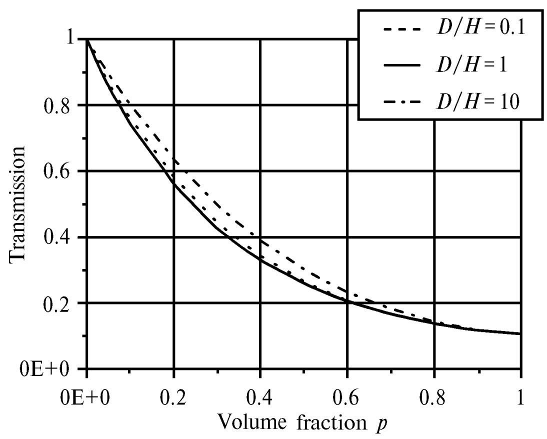

FIG. 10. Same as Figure 8, but with $\theta_0 = 60°$.

In dealing with the stochastic equation of transfer, one is confronted with a problem of closure, since the balance equation for the volumetric averages involves interface averages. We have discussed two possible closures. The simplest closure produces a set of two coupled integro–differential equations [Eq. (3.10) and (4.1)], which we referred to as Model 1. This model is exact for purely absorbing media with Markovian statistics, but approximate when scattering is present and/or if the statistics are non–Markovian. This model is also shown in an appendix to this paper to be equivalent to the Markovian model of Titov and coauthors involving integral equations, in the particular case considered by those authors. A more sophisticated closure for planar geometry, Eqs. (4.4), (4.9), and (4.10), produces a set of four coupled integro–differential equations, which we referred to as Model 2. This second model, while still approximate when scattering is present, is shown to give very accurate predictions in the cases when exact solutions are available. A simple model for the higher moments of the radiation field has also been proposed [Eq. (4.3)], although its accuracy has not been thoroughly tested. The models of stochastic radiative transfer discussed here apply equally well to all wavelengths. In particular, they treat the incoming short wavelengths and the re–emitted long wavelengths by the same formalism. It should be emphasized that both of these models are formulated for non–Markovian statistics (non–exponential distributions). This is an important issue, since experimental characterizations of cloud fields often employ power law distributions.

Both models account for non–homogeneous statistics, and in particular allow for vertical variations of the statistical characteristics of the cloud field. Also, these models, while presented here for a mixture of two components (clouds and clear sky), can be easily generalized to a mixture of an arbitrary number of components (Malvagi and Pomraning[9]). For instance, clear sky and several kinds of clouds with different physical properties can be treated simultaneously in a single set of coupled equations. Combining these two features, inhomogeneous statistics and an arbitrary number of components, one could in principle solve for the radiative intensity in the entire vertical extension of the single cell atmospheric column at once, without the need to subdivide it into relatively thin layers (except for numerical discretization purposes).

Numerical results for a number of test problems clearly indicate that the discrepancy between the predictions of the stochastic transfer model and a specific fractional cloud model can be quite significant. These results thus underscore the need for an approach to the problem of transfer in partially cloudy atmospheres that is more realistic than the ones currently in use. Any existing numerical algorithms for the solution of the equation of transfer, such as the popular two–stream (diffusive) model, apply directly to our stochastic method. Thus, it would be relatively straightforward to incorporate our method into existing GCM radiative transfer treatments. We also note in this regard that the variation of cloud characteristics and fractional cloud cover are easily incorporated. The various parameters that enter are simply allowed to be spatially (both vertically and laterally) dependent.

The main issue at this stage of development is the availability of those cloud field parameters necessary to implement this stochastic radiative transfer technique. In all instances, one requires the radiative properties, namely the total and scattering cross section and the scattering phase functions for pure clouds and clear atmosphere. In the simplest model of Markovian statistics, one requires in addition the mean cloud size and the cloud volume fraction (or the mean cloud spacing). The theory allows these lengths to be direction dependent. For non–Markovian statistics, one requires the distributions of both the cloud size and the cloud spacing, although numerical tests seem to indicate that, to a good approximation, only the mean and the variance of these distributions affect the radiative transfer (Levermore et al.[7]).

The advantage of the stochastic approach is that it can accurately calculate the average radiative heating rates through a broken cloud layer without requiring an exact description of the cloud geometry. The methods discussed here provide the average solution for this radiative transfer problem that depends on macroscopic properties of each component in the broken cloud layer and the statistical description (size and distribution) of the clouds within the layer. Some observational data of this type are already available and more will become available in the future. As GCMs become more sophisticated in their modeling of clouds, it is expected that they will be able to provide the type of cloud data required by a stochastic treatment of radiative transfer. Our hope is that this approach would significantly improve our understanding of cloud–radiation feedback mechanisms and our ability to predict climate changes.

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APPENDIX – THE TITOV MODEL REFORMULATED

We now consider the model introduced by Titov[23] within the atmospheric radiation context, and show that its integral equation formalism is equivalent to the low order model described by Eq. (3.10) with the closure (4.1). The case considered by Titov assumed negligible emission ($S_i = 0$), and treated the clear sky as completely transparent ($\sigma_0 = \sigma_{s0} = 0$). To simplify the notation somewhat, we introduce the spatial variable s along the direction $\mathbf{\Omega}$ (so that $\mathbf{\Omega}\cdot\nabla = d/ds$), and define the collision operators C_i as

$$C_i(p_i I_i) = \sigma_i p_i I_i - \sigma_{si}\int_{4\pi} f_i(\mathbf{\Omega}\cdot\mathbf{\Omega}')\, p_i I_i(\mathbf{\Omega}')\, \mathrm{d}\Omega' . \qquad (A.1)$$

Then the Titov model, derived under the Markovian assumption for both the mixing and transport processes, is given by the two coupled integral equations

$$\langle I(s)\rangle + \int_0^s C_1\, p_1(s')\, I_1(s')\, ds' = F ; \qquad (A.2)$$

$$p_1(s)\, I_1(s) + \int_0^s P_{11}(s', s)\, C_1\, p_1(s')\, I_1(s')\, \mathrm{d}s' = p_1 F . \qquad (A.3)$$

Here the surface of the system is taken as $s = 0$, and F denotes the incoming intensity at that point. The remaining notation is the same as introduced earlier, with $P_{ij}(s', s)$ being defined as the (Markovian) conditional probability that position s is in mixture component j, given that position s' is in component i. These quantities satisfy the (forward form) Chapman–Kolmogorov equations given by

$$\partial P_{ii}/\partial s = P_{ij}/\lambda_j - P_{ii}/\lambda_i , \quad j \neq i ; \qquad (A.4)$$

$$\partial P_{ij}/\partial s = P_{ii}/\lambda_i - P_{ij}/\lambda_j , \quad j \neq i , \qquad (A.5)$$

with boundary conditions

$$P_{ii}(s', s') = 1 ; \qquad (A.6)$$

$$P_{ij}(s', s') = 0 , \quad j \neq i . \qquad (A.7)$$

It is clear from their definition that the $P_{ij}(s', s)$ satisfy the constraint

$$P_{ii} + P_{ij} = 1 , \quad j \neq i . \qquad (A.8)$$

We now want to cast the integral Titov model given by Eqs. (A.2) and (A.3) into an equivalent differential form. First we subtract Eq. (A.3) from Eq. (A.2) to find, recalling Eq. (3.8),

$$p_0(s) I_0(s) + \int_0^s P_{10}(s', s)\, C_1\, p_1(s')\, I_1(s')\, \mathrm{d}s' = p_0(s) F, \qquad (A.9)$$

where we have used $p_0 + p_1 = 1$ and [see Eq. (A.8)] $P_{11} + P_{10} = 1$. If we differentiate Eqs. (A.2) and (A.3) we find

$$\frac{\mathrm{d}\langle I(s)\rangle}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) = 0 ; \qquad (A.10)$$

$$\frac{\mathrm{d}\left[p_1(s)\, I_1(s)\right]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) + \int_0^s \frac{\partial P_{11}(s', s)}{\partial s}\, C_1\, p_1(s')\, I_1(s')\, \mathrm{d}s' = \frac{\mathrm{d}p_1(s)}{\mathrm{d}s} F , \qquad (A.11)$$

where we have made use of Eq. (A.6). We now use the first of the Chapman–Kolmogorov equations given by Eq. (A.4) with $i = 1$ and $j = 0$ to rewrite Eq. (A.11) as

$$\frac{\mathrm{d}\left[p_1(s)\, I_1(s)\right]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) - \frac{1}{\lambda_1(s)}\int_0^s P_{11}(s', s)\, C_1\, p_1(s')\, I_1(s')\, \mathrm{d}s' + \frac{1}{\lambda_0(s)}\int_0^s P_{10}(s', s)\, C_1\, p_1(s')\, I_1(s')\, \mathrm{d}s' = \frac{\mathrm{d}p_1(s)}{\mathrm{d}s} F. \qquad (A.12)$$

Equations (A.3) and (A.9) can now be used to eliminate the integral terms in Eq. (A.12) and obtain, upon collecting terms,

$$\frac{\mathrm{d}\left[p_1(s)\, I_1(s)\right]}{\mathrm{d}s} + C_1\, p_1(s)\, I_1(s) = \frac{p_0(s)}{\lambda_0(s)} I_0(s) - \frac{p_1(s)}{\lambda_1(s)} I_1(s) + \left[\frac{\mathrm{d}p_1(s)}{\mathrm{d}s} + \frac{p_1(s)}{\lambda_1(s)} - \frac{p_0(s)}{\lambda_0(s)}\right] F. \qquad (A.13)$$

We now recall the Chapman–Kolmogorov differential equation for the $p_i(s)$ as given by Eq. (2.2). We then see that the coefficient of F in Eq. (A.13) is zero and we have

$$\frac{\mathrm{d}\left[p_1(s)\, I_1(s)\right]}{\mathrm{d}s} + C_1 p_1(s) I_1(s) = \frac{p_0(s)}{\lambda_0(s)} I_0(s) - \frac{p_1(s)}{\lambda_1(s)} I_1(s). \qquad (A.14)$$

The final algebraic manipulation is to subtract Eq. (A.14) from Eq. (A.10). The result is

$$\frac{\mathrm{d}\left[p_0(s)\,I_0(s)\right]}{\mathrm{d}s} = \frac{p_1(s)}{\lambda_1(s)}\,I_1(s) - \frac{p_0(s)}{\lambda_0(s)}\,I_0(s)\ . \qquad \text{(A.15)}$$

Equations (A.14) and (A.15) are the final results of our analysis, and are entirely equivalent in content to the integral equations of Titov given by Eqs. (A.2) and (A.3) once they are supplemented with the identity

$$\langle I(s)\rangle = p_0(s)\,I_0(s) + p_1(s)\,I_1(s) \qquad \text{(A.16)}$$

and the initial conditions

$$I_0(0) = I_1(0) = F\ . \qquad \text{(A.17)}$$

We write these two equations in a more explicit form by using $d/ds = \mathbf{\Omega}\cdot\nabla$ and the definition of the operator C_1 given by Eq. (A.1). We then have

$$\mathbf{\Omega}\cdot\nabla\,(p_0\,I_0) = p_1\,I_1/\lambda_1 - p_0\,I_0/\lambda_0\ , \qquad \text{(A.18)}$$

$$\mathbf{\Omega}\cdot\nabla\,(p_1\,I_1) + \sigma_1\,p_1\,I_1 =$$

$$= \sigma_{s1}\int_{4\pi} f_1(\mathbf{\Omega}\cdot\mathbf{\Omega}')\,p_1 I_1(\mathbf{\Omega}')\,\mathrm{d}\mathbf{\Omega}' + p_0\,I_0/\lambda_0 - p_1\,I_1/\lambda_1. \qquad \text{(A.19)}$$

In this form, it is clear that the Titov integral model is a special case of the Markovian low order model. This special case is the case corresponding to no emission ($S_i = 0$), no interaction between the radiation and the clear sky ($\sigma_0 = \sigma_{s0} = 0$), and Markovian statistics. Thus the rudimentary differential model effectively generalizes the Titov model. It is clear that the generalization to include emission and photon–sky interaction can be easily incorporated into the Titov integral equations as well.